Quantum field theory by Gibbs measures on càdlàg path space

Fumio Hiroshima
joint work with Takeru Hidaka

German-Japanese conference on Stochastic Analysis and Applications,
Sendai, Japan

2015/8/30-9/4

Introduction

Finite volume Gibbs measure P_T

- $V(\cdot)$ scalar potential and $W(\cdot,\cdot)$ pair potential

$$P_T(\cdot) = \frac{1}{Z_T}\int dx \mathbb{E}^x\left[\cdot\ e^{-\int_{-T}^T V(B_t)dt} e^{\int_{-T}^T dt \int_{-T}^T ds W(B_t - B_s, t-s)}\right]$$

Gibbs measure P_∞

- $\lim_T P_T = P_\infty$ in some sense
- Ground state φ_g of a QFT model H,

$$i.e., \quad H\varphi_g = \inf \sigma(H)\varphi_g,$$

and an observable $O \Longrightarrow (\varphi_g, O\varphi_g) = \mathbb{E}_{P_\infty}[F_O]$

Feynman-Kac type formulae

dD Schrödinger op.

- $h = -\frac{1}{2}\Delta + V$

$$(f, e^{-Th}g) = \int dx \mathbb{E}^x \left[\bar{f}(B_0) g(B_T) e^{-\int_0^T V(B_s) ds} \right]$$

- $h(a) = \frac{1}{2}(-i\nabla - a)^2 + V$

$$(f, e^{-Th(a)}g)$$
$$= \int dx \mathbb{E}^x \left[\bar{f}(B_0) g(B_T) e^{-\int_0^T V(B_s) ds - i \int_0^T a(B_s) \circ dB_s} \right]$$

Relativistic Schrödinger op.

- $\mathbb{E}[e^{-T_t u}] = e^{-t(\sqrt{2u+m^2}-m)}$, T_t: subordinator
- $h_R = \sqrt{-\Delta + m^2} - m$

$$(f, e^{-th_R} g) = \mathbb{E}[(f, e^{-T_t(-\frac{1}{2}\Delta)} g)] = \int dx \mathbb{E}\mathbb{E}^x \left[\bar{f}(B_0) g(B_{T_t})\right]$$

- $h_R(a, V) = \sqrt{(-i\nabla - a)^2 + m^2} - m + V$

$$(f, e^{-th_R(a,V)} g)$$
$$= \int dx \mathbb{E}\mathbb{E}^x \left[\bar{f}(B_0) g(B_{T_t}) e^{-\int_0^t V(B_{T_s}) ds - i\int_0^{T_t} a(B_s) \circ dB_s}\right]$$

Schrödinger op. with spin

- 2×2 Pauli matrices $\sigma = (\sigma_1, \sigma_2, \sigma_3)$
- $h_S(a,V) = \frac{1}{2}(-i\nabla - a)^2 + V - \frac{1}{2}\sigma \cdot b$

Thm. [Angelis,Rinaldi,Serva(91), FH-Ichinose-Lőrinczi(08)]

$$(f, e^{-th_S(a,V)}g) = e^t \sum_{\alpha=1,2} \int dx \mathbb{E}^x \mathbb{E}^\alpha \left[\overline{f(B_0, \theta_{N_0})} g(B_t, \theta_{N_t}) e^Z \right]$$

$$Z = -\int_0^t \left(V(B_s) - \frac{1}{2} b_3(B_s) \theta_{N_s} \right) ds - i \int_0^t a(B_s) \circ dB_s + \int_0^{t+} W(B_s, -\theta_{N_{s-}}) dN_s$$

- $W(x, -\theta) = \log \left[\frac{1}{2}(b_1(x) + i(-\theta) b_2(x)) \right]$
- $\theta_{N_t} = (-1)^{N_t}$ and N_t denotes a Poisson process.

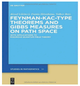
Relativistic Schrödinger op. with spin

- $h_{RS}(a,V) = \sqrt{(-i\nabla - a)^2 - \sigma \cdot b + m^2} - m + V$

Thm. [Angelis,Rinaldi,Serva(91), FH-Ichinose-Lőrinczi(08)]

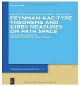
$$(f, e^{-th_{RS}(a,V)} g) = e^{T_t} \sum_{\alpha=1,2} \int dx \mathbb{E}\mathbb{E}^x \mathbb{E}^\alpha \left[\overline{f(B_0, \theta_{N_0})} g(B_{T_t}, \theta_{N_{T_t}}) e^Z \right]$$

$$Z = -\int_0^t V(B_{T_s}) ds - \frac{1}{2} \int_0^{T_t} b_3(B_s) \theta_{N_s} ds - i \int_0^{T_t} a(B_s) \circ dB_s + \int_0^{T_t+} W(B_s, -\theta_{N_{s-}}) dN_s$$

See 

Ψ: Bernstein function

$$\Psi\left(\frac{1}{2}(-i\nabla - a)^2 - \frac{1}{2}\sigma\cdot b\right) + V$$

- Extension of spin
- See the 2nd edition of 

- Schrödinger op. +QFT
- $H(0) = h_p \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{H_f}(\nu)$
- $H(\alpha) = H(0) + \alpha H_{\mathrm{I}}$
- **$H(\alpha) \Longrightarrow \exists$ground state?$\exists$Gibbs measure?**

- Nelson model **(particle+scalar field)**

$$\left(-\frac{1}{2m}\Delta_x + V\right) \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{H_f}(\nu) + \alpha\phi(x)$$

- Spin-boson model **(spin+scalar field)**

$$\sigma_z \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{H_f}(\nu) + \alpha\sigma_x \otimes \phi(f)$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- $T_A = -i\nabla_x \otimes 1\!\!1 - \alpha A(x)$
- PF model **(particle+radiation field)**

$$\frac{1}{2m}T_A^2 + V \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{H_f}(\nu)$$

- SRPF model **(my talk)**

$$\sqrt{T_A^2 + m^2} - m + V \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{H_f}(\nu) \quad m \geq 0$$

References

- **Self-adjointness** Arai (81,83), FH (00,02), Loss-Miyao-Spohn (07), Haslar-Herbst (08)
- **Existence of GS** Bach-Fröhlich-Sigal (97,99), Arai-Hirokawa (97), Spohn (99), Gérard (00), Griesemer-Lieb-Loss (01), Lieb-Loss (02), Arai(01), Bruneau-Dereziński (04), Hirokawa-FH-Spohn (05), Miyao-Spohn (08), Gérard-FH-Suzuki-Panatti (09), Könenberg-Matte-Stockmeyer (11)+many papers, Hidaka-FH (14)
- **Enhanced binding** FH-Spohn (02), Catto-Hainzl (02), Arai-Kawano (02), Chen-Vogalter-Vugalter (03), H-Sasaki (08,14)

References

- **Absence of GS** Arai-Hirokawa-FH (99), Chen(01), Derezinski-Gérard(04), Lorinczi-Minlos-Spohn(02), Hirokawa (06), Haslar-Herbst (06) Gérard-FH-Suzuki-Panatti (10)
- **Multiplicity of GS** Bach-Fröhlich-Sigal (97), Arai-Hirokawa (97), FH (00,02), FH-Spohn (01), Bach-Fröhlich-Pizzo (05)
- **Asymptotic field** Spohn(97), Derezinski-Gérard(99,04), Gérard (02), Fröhlich-Griesemer-Schlein (01, 02)

Reference to my talk

- Betz,FH,Lorinczi,Minlos,Spohn, RMP(01) **(Nelson)**
- Betz and Spohn PTRF(05) **(Nelson)**
- Betz and FH,IDAQP(05) **(PF)**
- FH, J.Funct.Anal.(07) **(translation inv. PF)**
- FH and Lorinczi, J.Funct.Anal.(08) **(PF with spin)**
- Hirokawa, FH and Lorinczi, Math.Z (14) **(SB)**
- FH, Adv in Math.(14) **(SRPF)**
- Hidaka and FH, to appear in Publ RIMS **(SRPF)**

SRPF model

(Particle part)

- $h_p = \sqrt{-\Delta + m^2} - m + V$ on $L^2(\mathbb{R}^d)$

  (1) $\lim_{|x|\to\infty} V(x) = \infty$

  (2) $\|Vf\| \le a\|\sqrt{-\Delta + m^2} f\| + b\|f\|$, $a < 1$

- Particle mass $m \ge 0$.
- **Massless case** $m = 0 \Longrightarrow h = \sqrt{-\Delta} + V$

SRPF model

(Field part)

- Boson Fock space
  $L^2(Q,d\mu)=\overline{\mathrm{LH}\{:\prod_{j=1}^n\phi(f_j):, f_j\in L^2_{\mathbb{R}}(\mathbb{R}^d)\}}$
- $\phi(f)$ Gaussian r.v.$\mathbb{E}_\mu[e^{z\phi(f)}]=e^{z^2/4\|f\|^2}$
- Dispersion relation $\omega(k)=\sqrt{|k|^2+\nu^2}$, $\nu\geq 0$
- Free field Hamiltonian $\mathrm{H_f}(\nu)$

$$e^{-t\mathrm{H_f}(\nu)}:\prod_{j=1}^n\phi(f_j):=:\prod_{j=1}^n\phi(e^{-t\omega}f_j):\Longrightarrow$$

$$\mathrm{H_f}(\nu):\prod_{j=1}^n\phi(f_j):=\sum_j:\phi(f_1)\cdots\phi(\omega f_j)\cdots\phi(f_n):$$

Quantized radiation field and total system

Radiation field $A(x)=(A_1(x),A_2(x),A_3(x))$.

- Gaussian r.v. $A_\alpha(x)$ (zero mean)

$$\mathbb{E}_\mu[A_\alpha(x)A_\beta(y)] = \frac{1}{2}\int \frac{|\hat{\varphi}(k)|^2}{\omega(k)} e^{-ik(x-y)}(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{|k|^2})dk$$

- $\hat{\varphi}$ is UV cutoff and $\hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^d)$
- Physics "$\hat{\varphi} \Longrightarrow \mathbb{1}$"

SRPF model

Total Hamiltonian

- State space (Hilbert sp) $\mathscr{H} = L^2(\mathbb{R}^d) \otimes L^2(Q)$
- Zero coupling op. $H(0) = h_p \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{H_f}(\nu)$
- Minimal coupling
  $-i\nabla_x \otimes 1\!\!1 \Longrightarrow T_A = -i\nabla_x \otimes 1\!\!1 - \alpha A(x)$, $\alpha \in \mathbb{R}$.
- SRPF Hamiltonian

$$H = \sqrt{T_A^2 + m^2} - m + V \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{H_f}(\nu) \quad m \geq 0$$

H has three parameters $\alpha \in \mathbb{R}$, $m \geq 0$ and $\nu \geq 0$. Singular case is $m = 0 = \nu$.

Gibbs measures for H_{PF}

Let $0 \le f \in L^2(\mathbb{R}^d)$ be fixed and $1\!\!1 \in L^2(Q)$. Then

$$
\begin{aligned}
&(f \otimes 1\!\!1, e^{-2TH_{PF}} f \otimes 1\!\!1) \\
=&\mathbb{E}\left[e^{-\int_{-T}^{T} V(B_s)ds} (1\!\!1, e^{-i\alpha \sum_\mu A_\mu(K_T^\mu)} 1\!\!1)_{L^2(Q)} \right] \\
=&\mathbb{E}\left[e^{-\int_{-T}^{T} V(B_s)ds} e^{-(\alpha^2/4)\|\sum_\mu K_T^\mu\|^2} \right] \\
=&\mathbb{E}\left[e^{-\int_{-T}^{T} V(B_s)ds - \frac{\alpha^2}{2}\sum_{\mu,\nu}\int_{-T}^{T} dB_s^\mu \int_{-T}^{T} dB_r^\nu W_{\mu\nu}} \right]
\end{aligned}
$$

- $\mathbb{E}[\cdots] = \int_{\mathbb{R}^d} dx \mathbb{E}^x[f(B_{-T}) f(B_T) \cdots]$
- Pair interaction $W_{\mu\nu} = W_{\mu\nu}(B_t - B_s, t-s)$
- $W_{\mu\nu}(X,t) = \frac{1}{2}\int \frac{\hat{\varphi}(k)^2}{\omega(k)} e^{-ikX} e^{-|t|\omega(k)} (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}) dk$

Ground state and Gibbs measures

Prob. meas. μ_T on $(C(\mathbb{R}), \mathscr{B})$ is defined by

$$A \mapsto \mu_T(A) = \frac{1}{Z_T}\mathbb{E}\left[\mathbb{1}_A e^{S_T}\right]$$

$$S_T = -\int_{-T}^{T} V(B_s)ds - \frac{\alpha^2}{2}\sum_{\mu,\nu}\int_{-T}^{T} dB_s^\mu \int_{-T}^{T} dB_r^\nu W_{\mu\nu}$$

Thm. (Betz+FH (05)) For $\forall\alpha \in \mathbb{R}, \forall\nu \geq 0$, $\{\mu_T\}$ is tight on $(C(\mathbb{R}), \mathscr{B})$

Cor. For $\forall\alpha \in \mathbb{R}, \forall\nu \geq 0$, $\exists \lim_T \mu_T$.

Gibbs measure for SRPF model

Let H be the SRPF Hamiltonian.

$$(f\otimes \mathbb{1}, e^{-2tH} f\otimes \mathbb{1}) = \mathbb{E}[e^{S_t}]$$

$$S_t = -\int_{-t}^{t} V(B_{T_s})ds - \frac{\alpha^2}{2}\sum_{\mu,\nu}\int_{-T_t}^{T_t} dB_s^{\mu}\int_{-T_t}^{T_t} dB_r^{\nu} W_{\mu\nu}$$

- $\int_{-t}^{t} V(B_{T_s})ds \stackrel{def}{=} \int_{-t}^{0} V(B_{-T_{-s}})ds + \int_{0}^{t} V(B_{T_s})ds$
- $\not\int dx\mathbb{E}^x[\cdots] = \int dx\mathbb{E}^x[f(B_{-T_t})f(B_{T_t})\cdots]$
- $W_{\mu\nu} = W_{\mu\nu}(B_t - B_s, T_t^* - T_s^*)$
- $T_s^* = \inf\{t | T_t = s\}$

μ_t is defined by

$$\mathscr{B}_B \times \mathscr{B}_T \ni A \mapsto \mu_t(A) = \frac{1}{Z_t}\mathbb{E}\left[\mathbb{1}_A e^{S_t}\right]$$

Local Weak Convergence

- Let $X_t = B_{T_t}(t \geq 0)$ and $X_{-t} = B_{-T_t}(-t < 0)$.
- $\mathscr{F}_t = \sigma(X_s; s \in [-t,t])$
- $\mathscr{G} = \cup_{0 \leq s}\mathscr{F}_s$

Thm.FH(14) Suppose that H has the unique ground state. Then $\exists \mu_\infty$ on $(D(\mathbb{R}), \sigma(\mathscr{G}))$ such that for each $A \in \mathscr{F}_s$, $\lim_t \mu_t(A) = \mu_\infty(A)$

Proof:

- Let $A \in \mathscr{F}_s$. Show for $t > s$

$$\mu_t(A) = e^{2Es} \int dx \mathbb{E}^x \left[1\!\!1_A \cdot \left(\frac{\phi_{t-s}(X_{-s})}{\|\phi_t\|}, \mathrm{J}_s \frac{\phi_{t-s}(X_s)}{\|\phi_t\|} \right)_{L^2(Q)} \right]$$

- $\phi_t = e^{-t(H-E)} f \otimes 1\!\!1$, $E = \inf \sigma(H)$
- J_s is a bounded operator on $\mathscr{H}$
- **If** $\exists \varphi_g \Longrightarrow \frac{\phi_{t-s}}{\|\phi_t\|} \stackrel{t\to\infty}{\to} e^{s(H-E)} \frac{(f \otimes 1\!\!1, \varphi_g)}{|(f \otimes 1\!\!1, \varphi_g)|} \varphi_g = \varphi_g$

$$\mu_t(A) \stackrel{t\to\infty}{\to} e^{2Es} \int dx \mathbb{E}^x [1\!\!1_A (\varphi_g(X_{-s}), \mathrm{J}_s \varphi_g(X_s))] = \mu_\infty(A)$$

Ground states of SRPF model

Particle mass m and boson mass v

particle mass ＼ boson mass	$v > 0$	$v = 0$
$m > 0$	OK	OK
$m = 0$	OK	**my Talk**

Figure: Ground state

Existence of the ground state and resonances

Figure: $\sigma(H(0))$ for $v > 0$

Figure: $\sigma(H(0))$ for $v = 0$

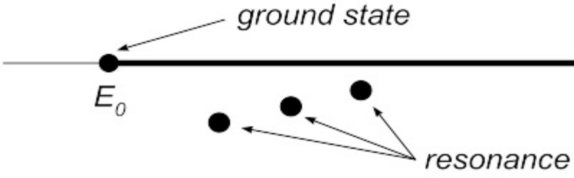


Figure: $\sigma(H)$ for $v = 0$

Main results

Let $\forall\alpha \in \mathbb{R}, \forall m, \nu \geq 0$.

- **Thm1** (Hidaka,FH(14)) Under some condition on $\hat{\varphi}$ and V, H is self-adjoint on $Dom(-\Delta) \cap Dom(\mathrm{H_f}(\nu))$.

- **Thm2** (Hidaka,FH(14)) Let V be confining. Then $\exists$ground state φ_{g} and unique.

- **Cor3** (FH(14)) $\exists\mu_\infty$ on $(D(\mathbb{R}), \sigma(\mathscr{G}))$.

Idea of proof

(Uniqueness of φ_{g})

- e^{-tH} is unitary equivalent to **a positivity improving semigroup**

$$(F, e^{-tH}G) =$$
$$\int dx \mathbb{E}^x[e^{-\int_0^t V(B_{T_s})ds}(F(B_0), J_0^* e^{-iA(K_t)} J_t G(B_{T_t}))] \cong$$
$$\int dx \mathbb{E}^x[e^{-\int_0^t V(B_{T_s})ds}(F(B_0), J_0^* e^{-i\pi(K_t)} J_t G(B_{T_t}))] > 0$$

- $J_0^* e^{-i\pi(K_t)} J_t$ positivity improving!

(Existence of φ_{g})

- (0) $\nu > 0 \Longrightarrow \exists \varphi_{\mathrm{g}}^{\nu}$
- (1) Pull through formula

$$\|N^{1/2}\varphi_{\mathrm{g}}^{\nu}\| \leq c\||x|\varphi_{\mathrm{g}}^{\nu}\|$$

- (2) Spatial decay

$$\|\varphi_{\mathrm{g}}^{\nu}(x)\|_{L^2(Q)} \leq ce^{-|x|} or \leq c(1+|x|^4)^{-1}$$

- (1)+(2) imply that

$$\varphi_{\mathrm{g}}^{\nu} \to \exists \varphi_{\mathrm{g}} \neq 0 \ (\nu \to 0).$$

Application

[Hirokawa,FH,Lorinczi (14)] SB model

- Path integral representation of $X=(\varphi_{\rm g}, e^{-\beta N}\varphi_{\rm g})$
- $X=\lim_T \left(\frac{e^{-TH} f\otimes 1\!\!1}{\|e^{-TH} f\otimes 1\!\!1\|}, e^{-\beta N}\frac{e^{-TH} f\otimes 1\!\!1}{\|e^{-TH} f\otimes 1\!\!1\|}\right)$
- $X=\lim_T \mathbb{E}_{\mu_T}[e^{-(1-e^{-\beta})\int_{-T}^0\int_0^T W}] = \mathbb{E}_{\mu_\infty}[e^{-(1-e^{-\beta})\int_{-\infty}^0\int_0^\infty W}]$
- Analytic continuation $\Longrightarrow$
  $(\varphi_{\rm g}, e^{+\beta N}\varphi_{\rm g}) = \mathbb{E}_{\mu_\infty}[e^{-(1-e^{+\beta})\int_{-\infty}^0\int_0^\infty W}] < \infty$ **for** $\forall\beta>0$

Conclusion and remarks

- For $\forall m, v \geq 0$ and $\forall \alpha \in \mathbb{R}$, **(1)** self-adjointness of H **(2)** ground state $\exists_1 \varphi_{\mathrm{g}}$ **(3)** Gibbs meas. $\exists \mu_\infty$.
- Our results are valid for the case of $m = 0$ and $v = 0$

  $$|-i\nabla_x \otimes 1\!\!1 - \alpha A(x)| + V \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{H_f}(0)$$

- UV renormalization?
  Cf Nelson(64), Gubinelli,FH,Lorinczi(14) for Nelson model