Escape rate of symmetric Markov processes

Yuichi Shiozawa
(Okayama University, Japan)

German-Japanese Conference on
Stochastic Analysis and Applications

Tohoku University
September, 2015

1. Introduction

Joint work with Jian Wang (Fujian Normal Univ./Kyoto Univ.)

Main interest in this talk:

global path properties of symmetric Markov processes

conservativeness/transience/recurrence

∘ Quantitative characterizations

($\Rightarrow$ upper/lower rate functions)

Purpose: to establish 0-1 laws for upper/lower rate functions

▷ $(\{B_t\}_{t\geq 0}, P)$: Brownian motion on $\mathbb{R}^d$, $B_0 = 0$ a.s.

Kolmogorov's test (e.g., see Itô-McKean)

▷ $R(t) = \sqrt{t}g(t)$ $\quad (g(t) \nearrow \infty$ as $t \to \infty)$

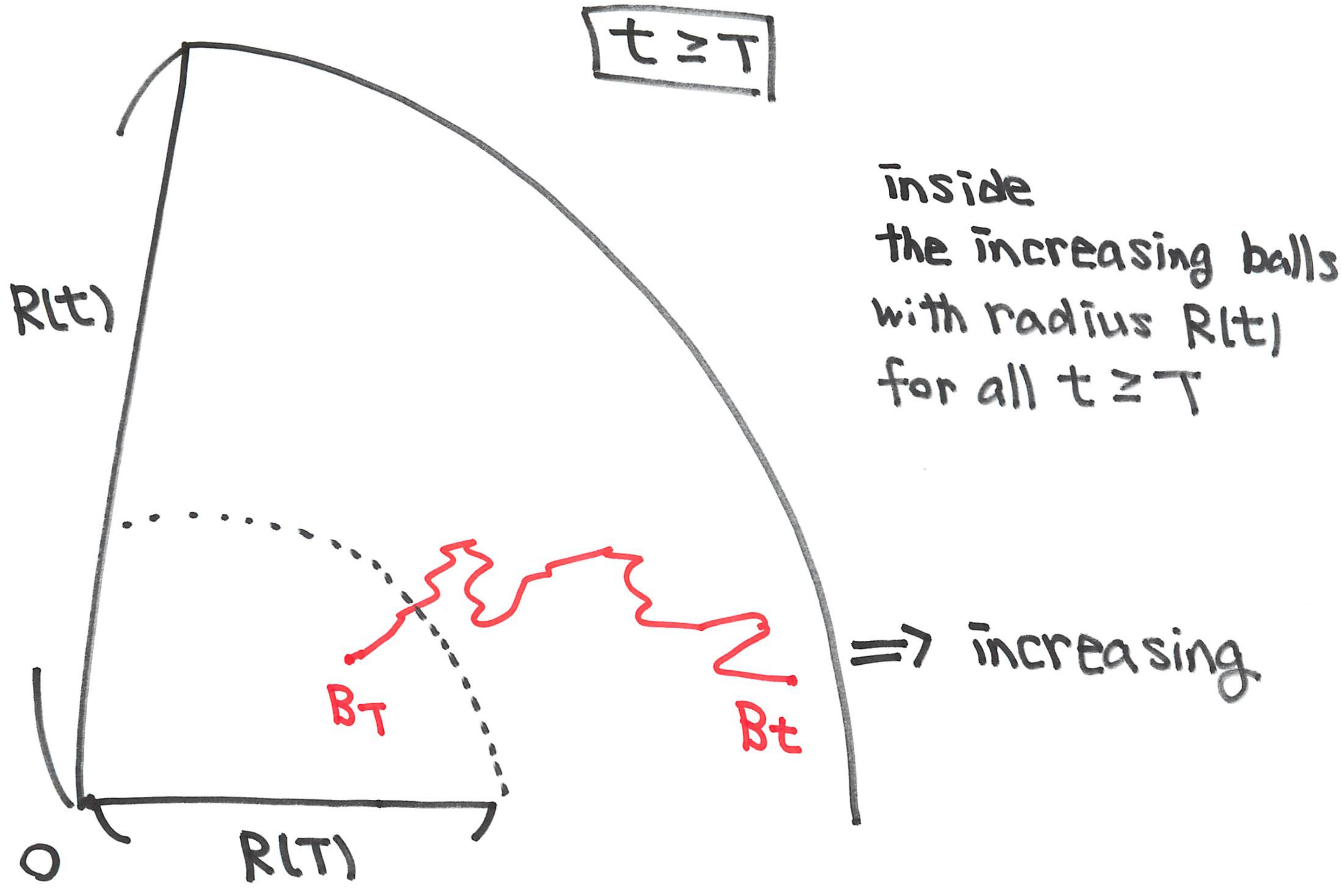
$$\text{(U)} \quad \int_{\cdot}^{\infty} g(t)^d \exp\left(-\frac{g(t)^2}{2}\right) \frac{\mathrm{d}t}{t} < \infty \ (\text{or} = \infty)$$

$$\Longrightarrow P\left(\exists T > 0 \text{ s.t. } |B_t| \leq R(t) \text{ for all } t \geq T\right) = 1 \ (\text{or } 0)$$

• $R(t)$: upper rate function for the one-probability case

Example. $R(t) = \sqrt{(2+\varepsilon)t \log\log t}$

$$\boxed{\text{(U)} \iff \varepsilon > 0}$$


t ≥ T
inside
the increasing balls
with radius R(t)
for all t ≥ T
R(t)
B_T
B_t
⟹ increasing
O
R(T)

Dvoretzky-Erdös' test ('51)

Assume $d \geq 3$ ($\iff$ transient)

▷ $r(t) = \sqrt{t}h(t)$ **(**$h(t) \searrow 0$ **as** $t \to \infty$**)**

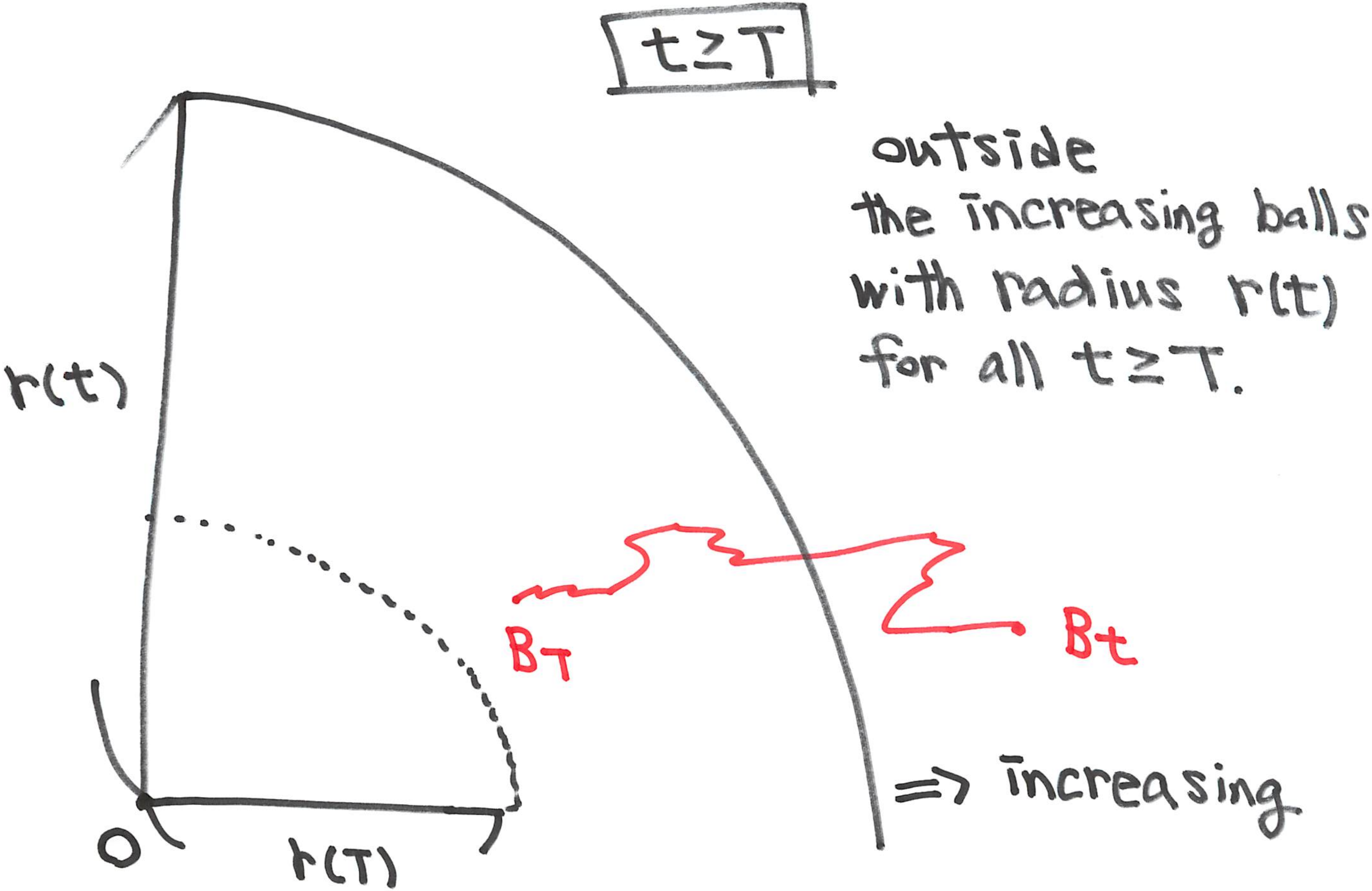
$$(\mathrm{L}_1) \quad \int_{\cdot}^{\infty} h(t)^{d-2} \frac{\mathrm{d}t}{t} < \infty \ (\text{or} = \infty)$$

$\Longrightarrow P\,(\exists T > 0$ **s.t.** $|B_t| \geq r(t)$ **for all** $t \geq T) = 1\ (\text{or } 0)$

• $r(t)$**: lower rate function for the one-probability case**

Example: $r(t) = \dfrac{\sqrt{t}}{(\log t)^{\frac{1+\varepsilon}{d-2}}}$

$$\boxed{(\mathrm{L}_1) \iff \varepsilon > 0}$$


t≥T
outside
the increasing balls
with radius r(t)
for all t≥T.
r(t)
B_T
B_t
O
r(T)
⇒ increasing

Spitzer's test ('58)

Assume $d = 2$ **(** $\iff$ **recurrent and can not hit any point)**

▷ $r(t) = \sqrt{t}h(t)$ **(**$h(t) \searrow 0$ **as** $t \to \infty$**)**

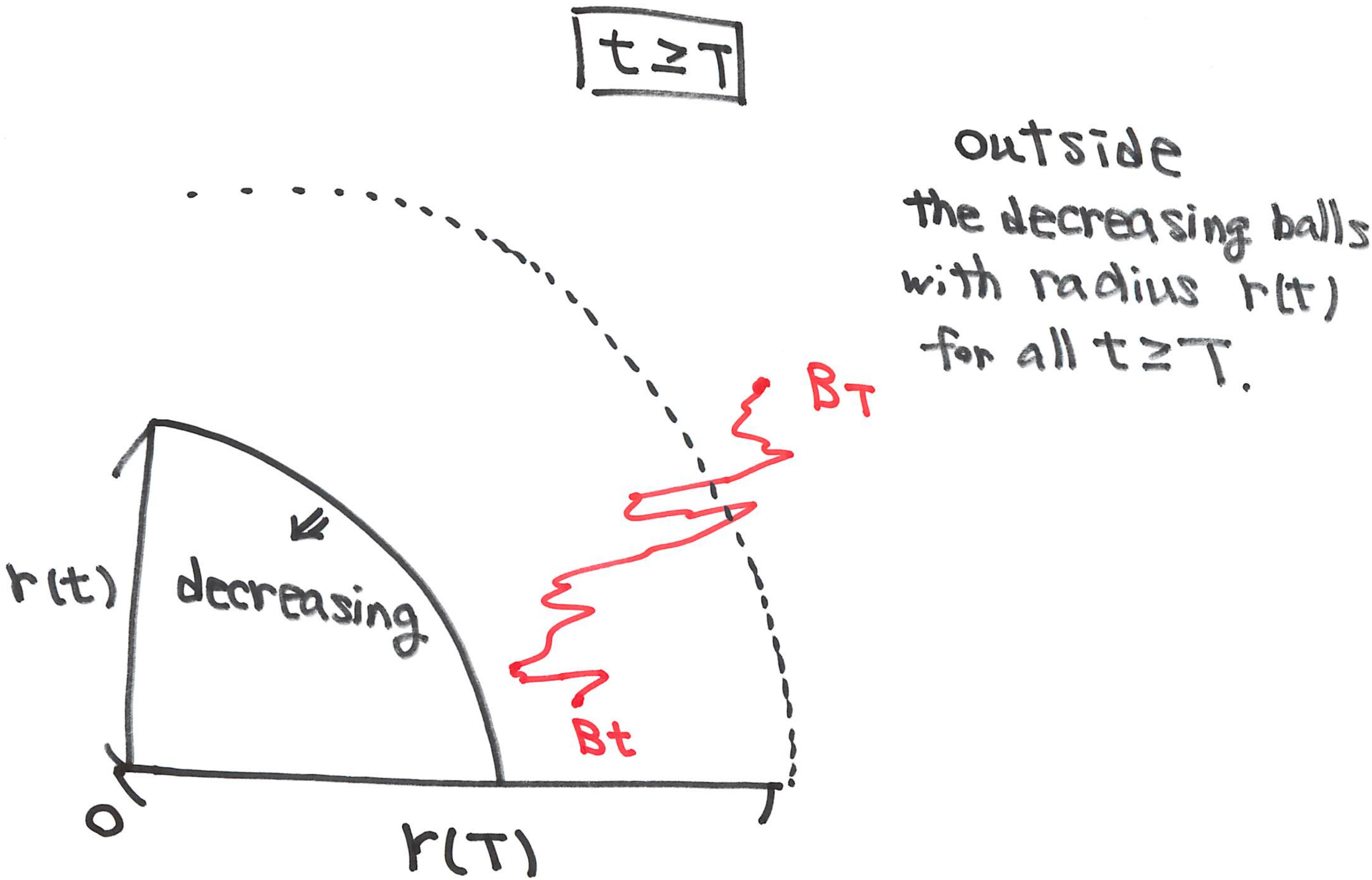
$$(\mathrm{L}_2) \quad \int_{\cdot}^{\infty} \frac{\mathrm{d}t}{t|\log h(t)|} < \infty \ (\text{or} = \infty)$$

$\Longrightarrow P\,(\exists T > 0$ **s.t.** $|B_t| \geq r(t)$ **for all** $t \geq T) = 1 \ (\text{or } 0)$

- $r(t)$: **lower rate function for the one-probability case**

Example: $r(t) = \dfrac{1}{t^{(\log\log t)^{1+\varepsilon}}}$

$$\boxed{(\mathrm{L}_2) \iff \varepsilon > 0}$$


t ≥ T
outside
the decreasing balls
with radius r(t)
for all t ≥ T.
B_T
B_t
decreasing
r(t)
0
r(T)

∗ **Symmetric diffusion processes**

◦ **Upper: Takeda ('89), Grigor'yan-Hsu ('09), Hsu-Qin ('10), Ouyang ('13)**

◦ **Lower: Ichihara ('79), Bendikov-Saloff-Coste ('05)**

◦ **Both: Grigor'yan-Kelbert ('98), Grigor'yan ('99)**

Characterizations in terms of

- **heat kernel estimates**
- **volume/coefficient growth rates**

* **Symmetric stable processes**
  - **Upper: Khintchine ('38)**
  - **Lower: Takeuchi ('64), Takeuchi-S. Watanabe ('64), Hendricks ('70), Khoshnevisan ('97)**

---

* **Symmetric Markov processes with no killing inside**

Volume/coefficient growth rates

  - **Upper: Huang ('14), Huang-S. ('14), S. ('15+)**
  - **Lower: S. ('15+) [transient case]**

Purpose: rate functions under heat kernel estimates

Consequence/Advantage

- **0-1 laws for symm. jump processes with no scaling property**
- **no use of the condition "$d(o,x) \in \mathcal{F}_{\rm loc}$"**

($\Longrightarrow$ "symm. α-stable-like processes with $\alpha \geq 2$")

Related work: P. Kim-Kumagai-J. Wang ('15, arXiv)

LIL for $\sup_{0<s\leq t} d(X_s, X_0)$ and local times

2. Results

▷ (X, d): loc. compact, separable and connected metric space

▷ m: positive Radon measure on X with full support

▷ $(\mathcal{E}, \mathcal{F})$: regular Dirichlet form on $L^2(X; m)$

▷ $\mathbb{M} = (\{X_t\}_{t \geq 0}, \{P_x\}_{x \in X})$: m-symmetric Hunt process generated by $(\mathcal{E}, \mathcal{F})$

Assumption.

$\exists p(t,x,y)$: **nonneg. symm. kernel on** $(0,\infty)\times X\times X$ **s.t.**

- $P_x(X_t \in \mathrm{d}y) = p(t,x,y)\, m(\mathrm{d}y)$
- $p(t+s,x,y) = \int_X p(t,x,z)p(s,z,y)\, m(\mathrm{d}z)$

Moreover, $\exists$**strictly inc. positive functions** V **and** ϕ **s.t.**

$$p(t,x,y) \asymp \frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(d(x,y))\phi(d(x,y))}$$

$$c_1 \left(\frac{R}{r}\right)^{d_1} \le \frac{V(R)}{V(r)} \le c_2 \left(\frac{R}{r}\right)^{d_2} \quad (0<r<R<\infty)$$

$$c_3 \left(\frac{R}{r}\right)^{d_3} \le \frac{\phi(R)}{\phi(r)} \le c_4 \left(\frac{R}{r}\right)^{d_4} \quad (0<r<R<\infty)$$

Remark. Under Assumption, $m(B_x(r)) \asymp V(r)$ **[KKW ('15)]**

Theorem 1 (with J. Wang).

- $d_1 > d_4$ **(**$\Rightarrow$ **transient)**
- $B_x(r)$ **is relatively compact for any** $x \in X$ **and** $r > 0$

$\triangleright$ $\varphi(t) = \phi^{-1}(t)g(t)$ **(**$g(t) \searrow 0$ **as** $t \to \infty$**)**

$$\int_1^\infty \frac{V(\varphi(t))}{\phi(\varphi(t))V(\phi^{-1}(t))}\,\mathrm{d}t < \infty \ (\text{or } = \infty)$$

$\Longrightarrow$ **for all** $x \in X$**,**

$$P_x\left(\exists T > 0 \text{ s.t. } d(X_t, x) \geq \varphi(t) \text{ for all } t \geq T\right) = 1 \ (\text{or } 0)$$

Theorem 2 (with J. Wang).

• $d_1 = d_2 = d_3 = d_4$ (⇒ **recurrent and can not hit points**)

▷ $\varphi(t) = \phi^{-1}(t)g(t)$ ($g(t) \searrow 0$ **as** $t \to \infty$)

$$\int_1^\infty \frac{1}{t|\log g(t)|}\,\mathrm{d}t < \infty \text{ (or } = \infty)$$

$\Longrightarrow$ **for all** $x \in X$,

$$P_x\left(\exists T > 0 \text{ s.t. } d(X_t, x) \ge \varphi(t) \text{ for all } t \ge T\right) = 1 \text{ (or } 0)$$

Example.

Assume that $\exists \alpha > 0$ **and** $\beta > 0$ **s.t.**

$$(*) \quad p(t,x,y) \asymp \frac{1}{t^{\alpha/\beta}} \wedge \frac{t}{d(x,y)^{\alpha+\beta}}$$

- $V(r) \asymp r^{\alpha}$ $(\Rightarrow d_1 = d_2 = \alpha)$
- $\phi(r) \asymp r^{\beta}$ $(\Rightarrow d_3 = d_4 = \beta)$

$$\mathcal{E}(u,u) = \iint_{X\times X\setminus \mathrm{diag}} (u(x)-u(y))^2 J(x,y) m(\mathrm{d}x) m(\mathrm{d}y)$$

$$\mathcal{F} = \left\{ u \in L^2(X;m) \mid \mathcal{E}(u,u) < \infty \right\}$$

Remark.

(i) $(*)$ **is valid if** $\exists\alpha > 0$ **and** $\beta \in (0,2)$ **s.t.**

$$m(B_x(r)) \asymp r^{\alpha}, \quad J(x,y) \asymp \frac{1}{d(x,y)^{\alpha+\beta}}$$

[Chen-Kumagai ('03, '08)]

(ii) $\beta \geq 2$ **for some subordinated diffusion processes**

[Kumagai ('03), Bogdan-Stós-Sztonyk ('03)]

$*$ **Ôkura ('02): subordinated symmetric Markov processes**

$\bullet\ V(r) \asymp r^{\alpha}$, $\phi(r) \asymp r^{\beta}$

(i) $\alpha > \beta$ **(transient):**

$$\int_1^{\infty} \frac{V(\varphi(t))}{\phi(\varphi(t))V(\phi^{-1}(t))}\,\mathrm{d}t \asymp \int_1^{\infty} \frac{\varphi(t)^{\alpha-\beta}}{t^{\alpha/\beta}}\,\mathrm{d}t$$

$\triangleright\ \varphi(t) = \dfrac{t^{\frac{1}{\beta}}}{(\log t)^{\frac{1+\varepsilon}{\alpha-\beta}}}$

$\varepsilon > 0$ $(\varepsilon \leq 0) \Longrightarrow$

$$P_x\left(\exists T > 0 \text{ s.t. } d(X_t, x) \geq \varphi(t) \text{ for all } t \geq T\right) = 1 \; (= 0)$$

- $V(r) \asymp r^{\alpha}$, $\phi(r) \asymp r^{\beta}$

(ii) $\alpha = \beta$ **(recurrent and can not hit any point):**

$$\triangleright\ g(t) = \frac{1}{\phi^{-1}(t)t^{(\log\log t)^{1+\varepsilon}}} \quad (\varepsilon > -1)$$

$$\left(\Rightarrow \varphi(t) = \frac{1}{t^{(\log\log t)^{1+\varepsilon}}}\right)$$

$$\int_1^{\infty} \frac{1}{t|\log g(t)|}\,\mathrm{d}t \asymp \int_1^{\infty} \frac{1}{t(\log t)(\log\log t)^{1+\varepsilon}}\,\mathrm{d}t$$

$\varepsilon > 0$ $(-1 < \varepsilon \le 0) \Longrightarrow$

$$P_x\,(\exists T > 0 \text{ s.t. } d(X_t, x) \ge \varphi(t) \text{ for all } t \ge T) = 1\ (= 0)$$

(iii) Upper rate functions:

▷ $\varphi(t)$: increasing function

$$(**) \quad \int_1^\infty \frac{1}{\varphi(t)^\beta}\,\mathrm{d}t < \infty \ (= \infty) \implies$$

$$P_x\left(\exists T > 0 \text{ s.t. } d(X_t, x) \le \varphi(t) \text{ for all } t \ge T\right) = 1 \ (= 0)$$

▷ $\varphi(t) = t^{\frac{1}{\beta}}(\log t)^{\frac{1+\varepsilon}{\beta}}$

$$\boxed{(**) \iff \varepsilon > 0}$$

Example [S. ('15+)] Assume that

- $d_0(x) = d(o,x) \in \mathcal{F}_{\mathrm{loc}}$ (*$o \in X$*: **fixed point**)
- $B_x(r)$ **is relatively compact for any** $x \in X$ **and** $r > 0$

If $\exists\ \alpha > 0$, $\beta \in (0,2)$ **s.t.**

$$m(B_x(r)) \lesssim r^{\alpha}, \quad J(x,y) \lesssim \frac{1}{d(x,y)^{\alpha+\beta}}$$

$\Longrightarrow \forall \varepsilon > 0$, **for q.e.** $x \in X$,

$$P_x\left(\exists T > 0 \text{ s.t. } d(x, X_t) \le t^{\frac{1}{\beta}} (\log t)^{1+\frac{1+\varepsilon}{\beta}} \text{ for all } t \ge T\right) = 1$$

3. Sketch of the proof for Theorem 1.

$\triangleright$ $Q(x,r,t) = P_x\left(\exists s > t \text{ s.t. } d(X_s, x_0) \le r\right)$

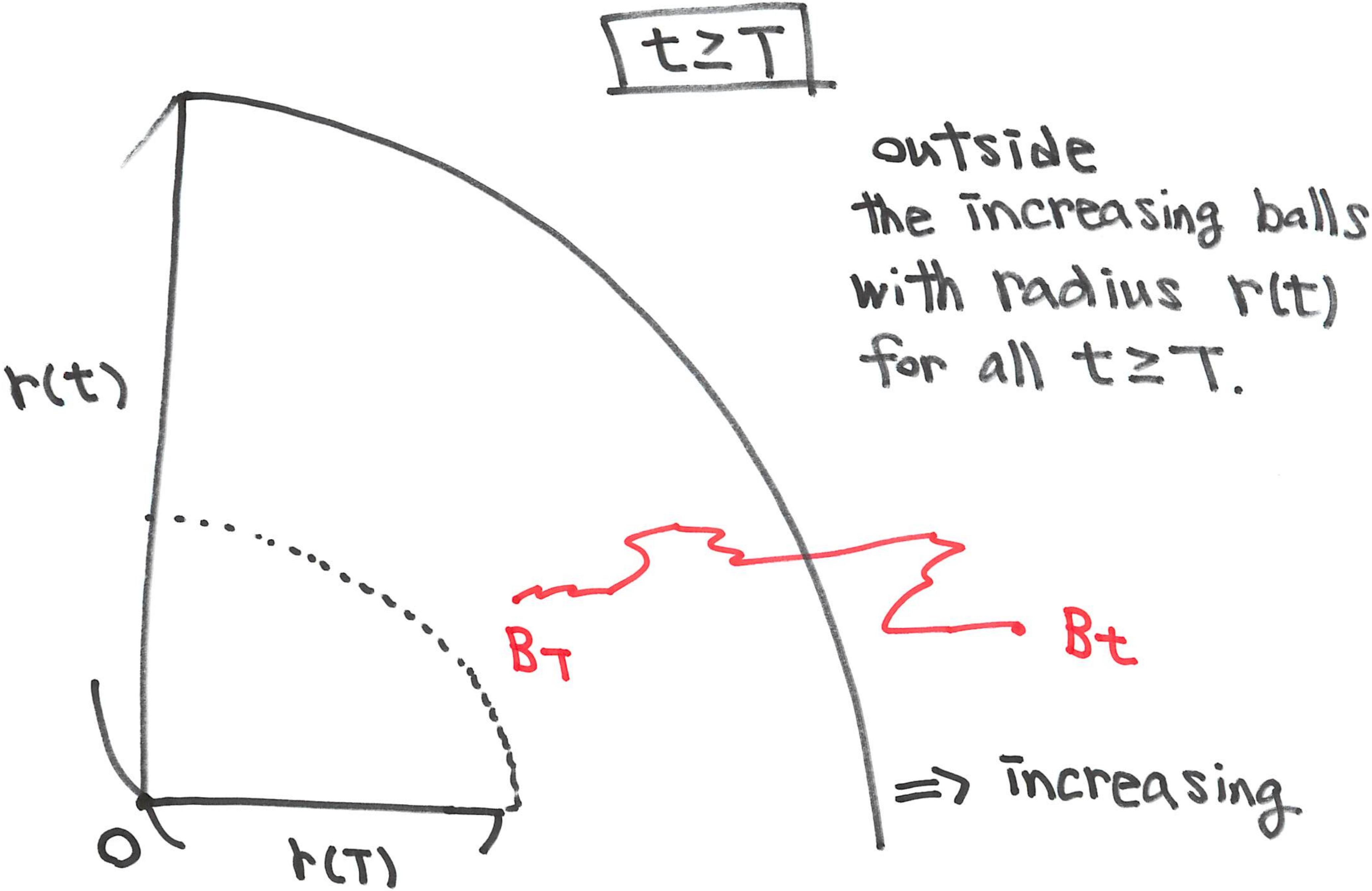
By the Markov property,

$$\begin{aligned} Q(x,r,t) &= E_x\left[P_{X_t}(\exists s > 0 \text{ s.t. } d(X_s, x_0) \le r)\right] \\ &= \int_X p(t,x,y) P_y(\exists s > 0 \text{ s.t. } d(X_s, x_0) \le r)\, \mathrm{d}y \end{aligned}$$

Proposition. $\forall x, x_0 \in X$ with $d(x, x_0) \le r$ and $t \ge \phi(r)$,

$$Q(x,r,t) \asymp \frac{V(r)}{\phi(r)} \frac{t}{V(\phi^{-1}(t))}.$$

Upper: Grigor'yan-Kelbert ('01)/Lower: Capacity estimate


t ≥ T
outside the increasing balls with radius r(t) for all t ≥ T.
r(t)
B_T
B_t
⇒ increasing
O
r(T)

Proof of the one-probability result.

▷ $t_n = 2^n$

▷ $A_n = \{\exists s \in (t_n, t_{n+1}]$ **s.t.** $d(X_s, X_0) \leq \varphi(s)\}$

$\exists c > 1$ **s.t.**

$$A_n \subset \{\exists s > t_n \text{ s.t. } d(X_s, X_0) \leq c\varphi(t_n)\}$$

because for all $n \geq 1$ **and** $s \in (t_n, t_{n+1}]$**,**

$$\varphi(s) = \phi^{-1}(s)g(s) \leq c\phi^{-1}(t_n)g(t_n) = c\varphi(t_n)$$

$$\Longrightarrow P_x(A_n) \leq P_x\left(\exists s > t_n \text{ s.t. } d(X_s, X_0) \leq c\varphi(t_n)\right)$$

$$= Q(x, c\varphi(t_n), t_n)$$

$$\underset{\text{Prop.}}{\asymp} \frac{V(\varphi(t_n))}{\phi(\varphi(t_n))} \frac{t_n}{V(\phi^{-1}(t_n))}$$

$$\left(Q(x, r, t) \asymp \frac{V(r)}{\phi(r)} \frac{t}{V(\phi^{-1}(t))}\right)$$

$$\Longrightarrow P_x(A_n) \le P_x\left(\exists s > t_n \text{ s.t. } d(X_s, X_0) \le c\varphi(t_n)\right)$$

$$= Q(x, c\varphi(t_n), t_n)$$

$$\underset{\text{Prop.}}{\asymp} \frac{V(\varphi(t_n))}{\phi(\varphi(t_n))} \frac{t_n - t_{n-1}}{V(\phi^{-1}(t_n))}$$

$$\left(Q(x, r, t) \asymp \frac{V(r)}{\phi(r)} \frac{t}{V(\phi^{-1}(t))}\right)$$

$$\sum_{n=1}^{\infty} P_x(A_n) \lesssim \int_1^{\infty} \frac{V(\varphi(s))}{\phi(\varphi(s))V(\phi^{-1}(s))} \mathrm{d}s < \infty$$

Hence $P_x(A_n \text{ i.o.}) = 0$ by the Borel-Cantelli Lemma

Proof of the zero-probability result.

▷ $\theta > 1$ **(large enough)**

▷ $B_n = \{\exists s \in (\theta^n, \theta^{n+1}]$ **s.t.** $d(X_s, X_0) \leq \varphi(s)\}$

$\exists c = c_\theta \in (0,1)$ **s.t.**

$$B_n \supset \left\{\exists s \in (\theta^n, \theta^{n+1}] \text{ s.t. } d(X_s, X_0) \leq c\varphi(\theta^{n+1})\right\} = C_n$$

because for any $s \in [\theta^n, \theta^{n+1}]$**,**

$$c\varphi(\theta^{n+1}) = c\phi^{-1}(\theta^{n+1})g(\theta^{n+1}) \leq \phi^{-1}(s)g(s) = \varphi(s).$$

Hence the proof is complete if $P_x(C_n \text{ i.o.}) = 1$.

∗ **Generalized second Borel-Cantelli lemma**

[Chung-Erdoš ('52), Takeuchi ('64)]

(i) $\sum_{k=1}^{\infty} P_x(C_k) = \infty$ **(ii)** $P_x(C_n \text{ i.o.}) = 0$ **or** 1

(iii) $\exists c > 0$ **s.t. for each fixed** $j \geq 1$,

$$P_x(C_i \cap C_j) \leq c P_x(C_i) P_x(C_j) \quad \text{for all } i \geq j + 2$$

$\Longrightarrow P_x(C_n \text{ i.o.}) = 1$

▷ $C_n = \left\{\exists s \in (\theta^n, \theta^{n+1}] \text{ s.t. } d(X_s, X_0) \leq c\varphi(\theta^{n+1})\right\}$

▷ $\sigma_n = \inf\left\{s \in (\theta^n, \theta^{n+1}] \mid d(X_s, X_0) \leq c\varphi(\theta^{n+1})\right\}$

By the strong Markov property at time σ_j,

$$P_x(C_i \cap C_j) \leq P_x(C_j) \sup_{d(z,x) \leq c\varphi(\theta^{j+1})} P_z(D_{ij,x}) \quad [\text{“}z = X_{\sigma_j}\text{”}]$$

▷ $D_{ij,x} = \{\exists s > \theta^i - \theta^{j+1} \text{ s.t. } d(X_s, x) \leq c\varphi(\theta^{i+1})\}$

$$d(z,x) \leq c\varphi(\theta^{j+1}) \leq \phi^{-1}(\theta^i - \theta^{j+1}) \quad (x \text{ and } z \text{ are “close”})$$

$$\implies \quad p(\theta^i - \theta^{j+1}, z, y) \asymp p(\theta^i - \theta^{j+1}, x, y) \quad (\forall y \in X)$$

$$p(\theta^i - \theta^{j+1}, z, y) \asymp p(\theta^i - \theta^{j+1}, x, y) \quad (\forall y \in X)$$

$$\Longrightarrow P_z(D_{ij,x})$$

$$= E_z[P_{X_{\theta^i - \theta^{j+1}}}(\exists s > 0 \text{ s.t. } d(X_s, x) \leq c\varphi(\theta^{i+1}))]$$

$$\asymp E_x[P_{X_{\theta^i - \theta^{j+1}}}(\exists s > 0 \text{ s.t. } d(X_s, x) \leq c\varphi(\theta^{i+1}))]$$

$$= P_x(D_{ij,x}) \underset{\text{Prop.}}{\lesssim} P_x(C_i)$$

$$\Longrightarrow P_x(C_i \cap C_j) \leq cP_x(C_i)P_x(C_j) \qquad \square$$