On gaugeability for generalized Feynman-Kac functionals and its applications

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This talk is based on the following papers:

- **Kim-K, Analytic characterizations of gaugeability for generalized Feynman-Kac functionals, 2013, to appear in TAMS.**
- **Kim-K, General analytic characterization of gaugeability for Feynman-Kac functionals, preprint, 2013.**
- **Kim-K, On a stability of heat kernel estimates under Feynman-Kac perturbations for symmetric Markov processes, 2015, in preparation. (cf. Kim-K (2014) for α-stable process).**
- **Kim-Kurniawaty-K, Analytic characterizations of gaugeability for generalized Feynman-Kac functionals and its application, 2015, in preparation.**

1 Gauge function (History)

Dirichlet Problem: D: **bdd regular domain in** $\mathbb{R}^d$, $f \in C(\partial D)$.

$$\begin{cases} \Delta u &= 0 \text{ in } D \\ u &= f \text{ on } \partial D \end{cases}$$

$\Longrightarrow u(x) = \mathbf{E}_x[f(X_{\tau_D})]$. **(Kakutani (1944))**

V: **Borel**

$$\begin{cases} \Delta u + Vu &= 0 \text{ in } D \\ u &= f \text{ on } \partial D \end{cases}$$

$\Longrightarrow u(x) = \mathbf{E}_x\left[\exp\left[\int_0^{\tau_D} V(X_s)ds\right] f(X_{\tau_D})\right]$?

For this, we need

$$g_{D,V}(x) := \mathbf{E}_x\left[\exp\left[\int_0^{\tau_D} V(X_s)ds\right]\right] < \infty \text{ for all } x \in D.$$

We call $g_{D,V}$ gauge function for (D,V).

Thm 1.1 (Gauge Theorem, Chung-Rao (1981))

Suppose $|D| < \infty$ and $\|V\|_\infty < \infty$. Then TFAE:

(1) $g_{D,V}(x) < \infty$ for some $x \in D$ ($g_{D,V} \not\equiv \infty$).

(2) $\sup_{x\in D} g_{D,V}(x) < \infty$.

We call (D,V) gaugeable if (1) or (2) holds.

Aizenmann-Simon (1982): Study of K_d, K_d^{loc}.

Zhao (1983): $d \geq 3$, D: bdd, $V \in K_d^{loc}$.

Chung (1988): $d \geq 3$, $1_D V \in K_d \cap L^1(D)$,

$$d \geq 1,\ |D| < \infty,\ 1_D V \in K_d.$$

Chung-Rao (1988): general Markov, but bdd D.

Zhao (1989): $V \in K_d$, $|D| < \infty \Longrightarrow$

$$\sup_{x \in D} \int_{D \setminus K} G_D(x,y)|V(y)|dy \to 0 \text{ as cpt } K \uparrow D.$$

Zhao (1992): $D = \mathbb{R}^d$, $d \geq 3$. $V \in K_d^\infty$.

Sturm (1991). **Glover-Rao-Song** (1993) : GT for $V =$ "Δu".

Chen-Song AOP(2002), PTRF(2003).

Chen (2002). **Takeda** (2002). **Chen** (2003). **Getoor** (2004).

Taekda-Uemura (2004).

2 Framework

$(\mathcal{E}, \mathcal{F})$: regular Dirichlet form on $L^2(E; \mathfrak{m})$.

$\mathrm{X} = (\Omega, X_t, \mathrm{P}_x, \zeta)$: $\mathfrak{m}$-symmetric transient Hunt process on E satisfying the following conditions :

(I) : X is irreducible.

(AC) : $P_t(x, dy) \ll m(dy)$ for all $t > 0$ and $x \in E$.

(AC) follows the (resolvent) strong Feller (RSF) property:

(RSF) : $R_\alpha f \in C_b(E)$ for any $f \in \mathcal{B}(E)_b$ and $\forall \alpha > 0$.

But we do not assume (RSF) in our <u>main</u> results!

Under (AC), we have resolvent kernel $R_\alpha(x,y)$ and Green kernel $R(x,y)$ and set for non-negative measure ν

$R_\alpha\nu(x) := \int_E R_\alpha(x,y)\nu(\mathrm{d}y)$, $R\nu(x) := \int_E R(x,y)\nu(\mathrm{d}y)$.

$Q_t f(x) := \mathbf{E}_x[e_A(t)f(X_t)]$: Feynman-Kac semigroup

$e_A(t) := \exp(A_t)$,

$A_t := A_t^{u,\mu,F} := N_t^u + A_t^\mu + A_t^F$.

$\mu = \mu_1 - \mu_2 \in S_1(X) - S_1(X)$

: signed smooth measure in the strict sense.

$S_1(X) \ni \nu \leftrightarrow A_t^\nu \in \mathrm{A}_{c,1}^+$: Revuz correspondence, i.e.

$$\mathbf{E}_{g\nu}\left[\int_0^\infty e^{-\alpha t} f(X_s)\mathrm{d}s\right] = \mathbf{E}_{fm}\left[\int_0^\infty e^{-\alpha t} g(X_s)\mathrm{d}A_s^\nu\right].$$

F : a sym bdd ft. on $E\times E$ with $F(x,\partial)=0$, $F\in J_1(X) \overset{\text{def}}{\Leftrightarrow}$
$N(|F|)\mu_H\in S_1(X)$,

(N,H): Lévy system, $N(|F|)(x):=\int_E|F(x,y)|N(x,dy)$,

$A_t^F:=\sum_{0<s\le t}F(X_{s-},X_s)$.

N_t^u: CAF of 0-energy part for $u\in\mathcal{F}_e\cap C_\infty(E)$ with $\mu_{\langle u\rangle}\in S_D^1(X)$:

for all $t\in[0,\infty[$

$u(X_t)-u(X_0)=M_t^u+N_t^u$ $\quad$ $\mathbf{P}_x$ -a.s. for all $x\in E$.

Schrödinger operator $\mathcal{H}:=$ "$\mathcal{L}+\mathcal{L}u+\mu+\mu_H\mathrm{F}$", where

$$\mathrm{F}f(x):=\int_{E_\partial}(e^{F(x,y)}-1)f(y)N(x,\mathrm{d}y).$$

It is known that

$$\mathcal{Q}(f,g)=(-\mathcal{H}f,g)_{\mathfrak{m}}\Leftrightarrow Q_tf(x)=\mathbf{E}_x[e_A(t)f(X_t)].$$

3 Kato class, Green-tight Kato class

$$\nu \in S^1_K(X) \overset{\text{def}}{\iff} \nu \in S_1(X), \lim_{\alpha\to\infty} \sup_{x\in E} R_\alpha\nu(x) = 0.$$

$$\nu \in S^1_{EK}(X) \overset{\text{def}}{\iff} \nu \in S_1(X), \lim_{\alpha\to\infty} \sup_{x\in E} R_\alpha\nu(x) < 1.$$

$$\nu \in S^1_D(X) \overset{\text{def}}{\iff} \nu \in S_1(X), \sup_{x\in E} R_\alpha\nu(x) < \infty \ \exists/\forall\alpha > 0.$$

$\nu \in S^1_{D_0}(X)$ **(Green-bounded)**

$$\overset{\text{def}}{\iff} \nu \in S_1(X), \sup_{x\in E} R\nu(x) < \infty.$$

$\nu \in S^1_{K_\infty}(X)$ **(Zhao's Green-tight Kato class)**

$\overset{\text{def}}{\iff} \nu \in S^1_K(X)$ **&** $\forall\varepsilon > 0, \exists K(\subset E)$ **: compact set s.t.**

$$\sup_{x\in E} R1_{K^c}\nu(x) < \varepsilon.$$

$\nu \in S^1_{K_1}(X)$ **(Zhao's semi-Green-tight extended Kato class)**

$\overset{\text{def}}{\iff} \nu \in S^1_{EK}(X)$ **&** $\exists K(\subset E)$ **: cpt set s.t.** $\sup_{x\in E} R1_{K^c}\nu(x) < 1$.

$\nu \in S^1_{CK_\infty}(X)$ **(Chen's Green-tight measures of Kato class)**

$\overset{\text{def}}{\iff} \nu \in S_1(X)$, $\forall\varepsilon > 0$, $\exists K(\subset E)$ **: Borel set s.t.** $\nu(K) < \infty$

& $\exists\delta > 0$ **s.t.** $\forall$ **mea'ble set** $B \subset K$ **with** $\nu(B) < \delta$,

$$\sup_{x\in E} R1_{B\cup K^c}\nu(x) < \varepsilon.$$

$\nu \in S^1_{CK_1}(X)$ **(Chen's semi-Green-tight measures of extended Kato class)**

$\overset{\text{def}}{\iff} \nu \in S_1(X)$, $\exists K(\subset E)$ **: Borel set s.t.** $\nu(K) < \infty$ **&** $\exists\delta > 0$

s.t. $\forall$ **mea'ble set** $B \subset K$ **with** $\nu(B) < \delta$,

$$\sup_{x\in E} R1_{B\cup K^c}\nu(x) < 1.$$

$\nu \in S^1_{NK_\infty}(X)$ (Natural Green-tight measures of Kato class)

$\overset{\text{def}}{\iff} \nu \in S^1_{D_0}(X)$, $\forall \varepsilon > 0$, $\exists K(\subset E)$: Borel set & $\exists \delta > 0$ s.t. $\forall$ mea'ble set $B \subset K$ with $C^\nu(B) < \delta$,

$$\sup_{x \in E} \mathbf{E}_x[A^\nu_{\tau_{K^c \cup B}}] < \varepsilon. \quad (\text{Rem:} \quad \mathbf{E}_x[A^\nu_{\tau_{K^c \cup B}}] \le R1_{K^c \cup B}\nu(x) \quad)$$

$\nu \in S^1_{NK_1}(X)$ (Natural semi-Green-tight meas. of extended Kato class)

$\overset{\text{def}}{\iff} \nu \in S^1_{D_0}(X)$, $\exists K(\subset E)$: Borel set & $\exists \delta > 0$ s.t. $\forall$ mea'ble set $B \subset K$ with $C^\nu(B) < \delta$,

$$\sup_{x \in E} \mathbf{E}_x[A^\nu_{\tau_{K^c \cup B}}] < 1.$$

Here C^ν is the 1-weighted capacity w.r.t. the time changed process $(\check{\mathrm{X}}, \nu)$ defined by $(\check{\mathrm{X}}, \nu) := (\Omega, X_{\tau_t}, \mathbf{P}_x)$, where $\tau_t := \inf\{s > 0 \mid A^\nu_s > t\}$ is the right continuous inverse of A^ν_t.

Lem 3.1 (Chen (2002), Kim-K (2013) & Kim-Kurniawaty-K (2015))

(1) $S^1_{CK_\infty}(\mathrm{X}) \subset S^1_{CK_1}(\mathrm{X}) \subset S^1_{D_0}(\mathrm{X})$.

(2) $S^1_{CK_\infty}(\mathrm{X}) \subset S^1_{K_\infty}(\mathrm{X}) \subset S^1_{K}(\mathrm{X})$ & $S^1_{CK_1}(\mathrm{X}) \subset S^1_{K_1}(\mathrm{X}) \subset S^1_{EK}(\mathrm{X})$.

(3) $S^1_{CK_\infty}(\mathrm{X}) \subset S^1_{NK_\infty}(\mathrm{X}) \subset S^1_{K}(\mathrm{X})$ & $S^1_{CK_1}(\mathrm{X}) \subset S^1_{NK_1}(\mathrm{X}) \subset S^1_{EK}(\mathrm{X})$.

(4) $S^1_{CK_\infty}(\mathrm{X}) = S^1_{NK_\infty}(\mathrm{X})$ under **(RSF)**.

(5) $S^1_{NK_\infty}(\mathrm{X}) = S^1_{CK_\infty}(\mathrm{X}) = S^1_{NK_\infty}(\mathrm{X})$ under **(RSF)**+ Feller property.

(1)–(2) is proved in Chen (2002). (3)–(4) is proved in Kim-K (2013).

(5) is proved in Kim-Kurniawaty-K (2015).

Rem 3.1 (Advantage of $S^1_{NK_1}(\mathrm{X})$)

$S^1_{CK_1}(\mathrm{X})$ may not be stable under Girsanov transform.

3.1 Gaugeability: Previous Result in Kim-K (2013)

Thm 3.1 (Kim-K (2013)) Suppose $\mu_1 \in S^1_{NK_1}(X)$, $\mu_{\langle u \rangle} + N(F_1)\mu_H \in S^1_{NK_\infty}(X)$ **and** $\mu_2 + N(F_2)\mu_H \in S^1_{D_0}(X)$. **Then TFAE:**

(1) (X, A) **is gaugeable**, **i.e.,** $\sup_{x \in E} \mathbf{E}_x\left[e_A(\zeta)\right] < \infty$.

(2) $\lambda^{\mathcal{Q}}(\bar{\mu}_1) := \inf\left\{\mathcal{Q}(f,f) \mid f \in \mathcal{F} \cap C_0(E), \int_E f^2 \mathrm{d}\bar{\mu}_1 = 1\right\} > 0$,

where $\bar{\mu}_1 := N(V)\mu_H + \mu_1 + \frac{1}{2}\mu^c_{\langle u \rangle}$, $\bar{\mu}_2 := N(F_2)\mu_H + \mu_2$,

$$V := (G_u - F_u) + F_1 (\Rightarrow V \geq 0),$$

$$G_u := e^{F_u} - 1, F_u(x,y) := F(x,y) + u(x) - u(y).$$

3.2 Gaugeability: New Result in Kim-Kurniawaty-K (2015)

Thm 3.2 (Kim-Kurniawaty-K (2015)) Suppose that $\mu_1 + N(e^{F_1} - 1)\mu_H \in S^1_{NK_1}(X)$, $\mu_{\langle u\rangle} \in S^1_{NK_\infty}(X)$ and $\mu_2 + N(F_2)\mu_H \in S^1_{D_0}(X)$ hold.

Then TFAE:

(1) (X, A) is gaugeable, i.e., $\sup_{x \in E} \mathbf{E}_x [e_A(\zeta)] < \infty$.

(2) $\lambda^{\mathcal{Q}}(\bar{\mu}_1) := \inf\left\{ \mathcal{Q}(f, f) \,|\, f \in \mathcal{F} \cap C_0(E), \int_E f^2 \mathrm{d}\bar{\mu}_1 = 1 \right\} > 0$,

3.3 Idea of the proof of Thm 3.2

In proving Thm 3.1, we use the following MF Z_t:

$$Z_t := Y_t^1 e^{-A_t^{F_2}},$$

where $Y_t^1 := \mathrm{Exp}(M_t)$ is the stochastic exponential of M_t, the sol. of SDE: $Y_t^1 = 1 + \int_0^t Y_{s-}^1 \mathrm{d}M_s$ and $M_t := M_t^{G_1^u} + M_t^{-u,c}$, and $M_t^{G_1^u}$ is the purely discontinuous MAF satisfying $\Delta M_t^{G_1^u} := G_1^u(X_{t-}, X_t) = e^{F_1^u(X_{t-},X_t)} - 1$ defined by

$$M_t^{G_1^u} := M_t^{-u,j} + M_t^{-u,\kappa} + \sum_{s\le t}(G_1^u - F_1^u + F_1)(X_{s-}, X_s)$$
$$- \int_0^t N(G_1^u - F_1^u + F_1)(X_s)\mathrm{d}H_s.$$

In proving Thm 3.2, we use the following MF U_t:

$$U_t := \mathrm{Exp}(M_t) := e^{M_t - \frac{1}{2}\langle M^c \rangle_t} \prod_{s \le t} (1 + \Delta M_s) e^{-\Delta M_s},$$

with $\Delta M_t := M_t - M_{t-}$, the sol. of SDE: $U_t = 1 + \int_0^t U_{s-} \mathrm{d}M_s$, $M_t := M_t^{e^U - 1} + M_t^{-u,c}$, and $M_t^{e^U-1}$ is the purely discontinuous MAF satisfying $\Delta M_t^{e^U-1} := (e^U - 1)(X_{t-}, X_t) = e^{u(X_{t-}) - u(X_t)} - 1$ defined by

$$M_t^{e^U-1} := M_t^{-u,j} + M_t^{-u,\kappa} + \sum_{s \le t} (e^U - U - 1)(X_{s-}, X_s) - \int_0^t N(e^U - U - 1)(X_s) \mathrm{d}H_s.$$

with $U(x, y) := u(x) - u(y)$.

The transformed process $\mathrm{U} := (\Omega, X_t, \mathsf{P}_x^U)$ defined by

$$P_t^U f(x) := \mathsf{E}_x^U[f(X_t)] := \mathsf{E}_x[U_t f(X_t)] \quad f \in \mathcal{B}_b(E).$$

Then P_t^U is $e^{-2u}\mathfrak{m}$-symmetric (Chen-Zhang (2002)). Let $(\mathcal{E}^U, \mathcal{F}^U)$ be the associated Dirichlet form on $L^2(E; e^{-2u}\mathfrak{m})$. Then $(\mathcal{E}^U, \mathcal{F}^U)$ has the following expression: $\mathcal{F} = \mathcal{F}^U$ and

$$\mathcal{E}^U(f,f) = \frac{1}{2}\int_E e^{-2u(x)}\mu^c_{\langle f\rangle}(\mathrm{d}x) + \int_{E\times E}(f(x)-f(y))^2 e^{-u(x)-u(y)}J(\mathrm{d}x\mathrm{d}y)$$
$$+\int_E f(x)^2 e^{-u(x)}\kappa(\mathrm{d}x).$$

for any $f \in \mathcal{F}^U$. Moreover, for $\bar{\nu}_1 := \mu_1 + N(e^U - U - 1)\mu_H + \frac{1}{2}\mu^c_{\langle u\rangle}$, $\bar{\nu}_2 := \mu_2$ and $\bar{\nu} := \bar{\nu}_1 - \bar{\nu}_2$,

$$\mathcal{Q}(f,g) = \mathcal{E}^U(fe^u, ge^u) - \int_E fg\,\mathrm{d}\bar{\nu} - \int_{E\times E} f\otimes g(e^F - 1)N(\cdot, \mathrm{d}\cdot)\mathrm{d}\mu_H.$$

1st Step: It suffices to prove the case $\mathfrak{m}(E) < \infty$, $\mathfrak{m} \in S^1_{D_0}(\mathrm{X})$ in view of time change method, because the gauge function $g_A(x) := \mathbf{E}_x[e_A(\zeta)] = \mathbf{E}_x[e_{A_{\tau^{g\mathfrak{m}}}}(\int_0^\zeta g(X_s)\mathrm{d}s)]$ is invariant under time change by PCAF $A_t^{g\mathfrak{m}} := \int_0^t g(X_s)\mathrm{d}s$ with strictly positive g satisfying $g\mathfrak{m} \in S^1_{D_0}(\mathrm{X})$, where $\tau_t^{g\mathfrak{m}}$ is the r.c. inverse of $A_t^{g\mathfrak{m}}$.

2nd Step: The proof for the case $u = 0$, $\mathfrak{m}(E) < \infty$, $\mathfrak{m} \in S^1_{D_0}(\mathrm{X})$ can be done along the same method as in Chen (2002) with a suitable modification under our conditions:

$\mu_1 + N(e^{F_1} - 1)\mu_H \in S^1_{NK_1}(X)$, $\mu_2 + N(F_2)\mu_H \in S^1_{D_0}(X)$.

Thm 3.3 (Gauge Theorem under $u = 0$) Under the above conditions,

$$\sup_{x\in E} \mathbf{E}_x[e_A(\zeta)] < \infty \Leftrightarrow \mathbf{E}_x[e_A(\zeta)] < \infty \quad \exists x \in E.$$

3rd Step: Consider the Girsanov transformed process U by U_t and its subprocess $\mathrm{U}^{(\beta)}$ killed by $e^{-\beta t}$. We prove for small β,

$$e^{-2u}(\eta_1+N(e^{F_1}-1)\mu_H)\in S^1_{NK_1}(U^{(\beta)}),\ e^{-2u}(\eta_2+N(F_2)\mu_H)\in S^1_{D_0}(U^{(\beta)}).$$

Here $\eta_i := \beta\mathfrak{m}+\bar{\nu}_i$ $(i=1,2)$ and $\bar{\nu}_1 := \mu_1+N(e^U-U-1)\mu_H+\frac{1}{2}\mu^c_{\langle u\rangle}$ and $\bar{\nu}_2 := \mu_2$. Then we can apply **2nd Step** so that

$$\lambda^{\mathcal{Q}}(\bar{\mu}_1)>0 \overset{\text{Takeda-Uemura (2004)}}{\Longleftrightarrow} \lambda^{\mathcal{Q}}(\mathfrak{m})>0 \Leftrightarrow \sup_{x\in E}\mathbf{E}_x^{U^{(\beta)}}[e^{\beta\zeta+A_\zeta^{\bar{\nu}}+A_\zeta^F}]<\infty.$$

$$\Rightarrow \sup_{x\in E}\mathbf{E}_x^{U}[e^{A_\zeta^{\bar{\nu}}+A_\zeta^F}]<\infty \Leftrightarrow \sup_{x\in E}\mathbf{E}_x[e_A(\zeta)]<\infty.$$

because $\mathbf{E}_x^U\left[e^{A_\zeta^{\bar{\nu}}+A_\zeta^F}\right]=e^{u(x)}\mathbf{E}_x\left[e^{-u(X_{\zeta-})}e_A(\zeta)\right]$ and

$$\mathbf{E}_x^{U^{(\beta)}}[e^{\beta\zeta+A_\zeta^{\bar{\nu}}+A_\zeta^F}]=\beta\mathbf{E}_x^U\left[\int_0^\zeta e^{A_s^{\bar{\nu}}+A_s^F}\mathrm{d}s\right]+\mathbf{E}_x^U[e^{A_\zeta^{\bar{\nu}}+A_\zeta^F}].$$

4th Step: We need to prove the converse

$$\sup_{x\in E} \mathbf{E}_x^U[e^{A_\zeta^{\bar\nu}+A_\zeta^F}] < \infty \Rightarrow \sup_{x\in E} \mathbf{E}_x^{U^{(\beta)}}[e^{\beta\zeta+A_\zeta^{\bar\nu}+A_\zeta^F}] < \infty.$$

It suffices to prove

$$\mathbf{E}_x^U\left[\int_0^\zeta e^{A_s^{\bar\nu}+A_s^F}\mathrm{d}s\right] < \infty \text{ for } \exists x\in E.$$

This is equivalent to

$$\sup_{x\in E} \mathbf{E}_x^U[e^{A_\zeta^{\bar\nu}+A_\zeta^F}] < \infty$$

under $e^{-2u}(\bar\nu_1 + N(e^{F_1}-1)\mu_H) \in S^1_{NK_1}(\mathrm{U})$ and $e^{-2u}(\bar\nu_2 + N(F_2)\mu_H) \in S^1_{D_0}(\mathrm{U})$.

5th Step: We should prove $e^{-2u}(\bar{\nu}_1+N(e^{F_1}-1)\mu_H)\in S^1_{NK_1}(\mathrm{U})$ and $e^{-2u}(\bar{\nu}_2+N(F_2)\mu_H)\in S^1_{D_0}(\mathrm{U})$. For this, we need to prove the super gauge theorem: For $p\ge 1$, put $F^u_{(p)}:=pF^u$, $G^u_{(p)}:=e^{F^u_{(p)}}-1$ and $V_{(p)}:=G^u_{(p)}-F^u_{(p)}+pF_1$. Define $\bar{\mu}^p:=\bar{\mu}^p_1-\bar{\mu}^p_2$ by

$$\bar{\mu}^p_1:=N(V_{(p)})\mu_H+p\mu_1+\frac{p^2}{2}\mu^c_{\langle u\rangle},\qquad \bar{\mu}^p_2:=N(pF_2)\mu_H+p\mu_2.$$

Set

$$\lambda^{\mathcal{Q}^{(p)}}(\bar{\mu}^p_1):=\inf\left\{\mathcal{Q}^{(p)}(f,f)\;\middle|\; f\in\mathcal{C},\int_E f^2\mathrm{d}\bar{\mu}^p_1=1\right\},$$

where $\mathcal{Q}^{(p)}$ is the quadratic form defined for $pu,p\mu$ and pF as well as $\mathcal{Q}$ is defined for u,μ and F.

Prop 3.1 (Super Gauge Theorem)

Suppose that $\mu_1 + N(e^{F_1} - 1)\mu_H \in S^1_{NK_1}(\mathrm{X})$, $\mu_{\langle u \rangle} \in S^1_{NK_\infty}(\mathrm{X})$ and $\mu_2 + N(F_2)\mu_H \in S^1_{D_0}(\mathrm{X})$ hold. Assume $\lambda^{\mathcal{Q}}(\bar{\mu}_1) > 0$. Then $\exists p_0 > 1$ sufficiently close to 1 s.t. $\lambda^{\mathcal{Q}^{(p)}}(\bar{\mu}_1^p) > 0$ for any $p \in [1, p_0]$, hence $\sup_{x \in E} \mathbf{E}_x[e_A(\zeta)^p] = \sup_{x \in E} \mathbf{E}_x[e_{pA}(\zeta)] < \infty$ for any $p \in [1, p_0]$.

Lem 3.2 Suppose that $\mu_{\langle u \rangle} \in S^1_{NK_\infty}(\mathrm{X})$ holds. Then $\exists p_0 > 1$ sufficiently close to 1 s.t. $\sup_{x \in E} \mathbf{E}_x[U_\zeta^p] < \infty$ for any $p \in [1, p_0]$.

Cor 3.1 $\nu \in S^1_{NK_1}(\mathrm{X})$ implies $e^{-2u}\nu \in S^1_{NK_1}(\mathrm{U})$.

In particular, we have $e^{-2u}(\bar{\nu}_1 + N(e^{F_1} - 1)\mu_H) \in S^1_{NK_1}(\mathrm{U})$ and $e^{-2u}(\bar{\nu}_2 + N(F_2)\mu_H) \in S^1_{D_0}(\mathrm{U})$.

The proof of **Thm** 3.2 is now complete.

3.4 Application: Stability of heat kernel estimates

$\exists p_t(x,y)$ s.t. for $t>0$ and $x \in E$, $y \mapsto p_t(x,y)$ is f.c.

Ass 3.1 Let $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ be another symmetric Dirichlet form on $L^2(E; \widetilde{\mathfrak{m}})$ having no inside killing and $\widetilde{\mathcal{F}} = \mathcal{F}$. Suppose that there exists $C_E > 0$ such that $C_E^{-1}\mathfrak{m} \le \widetilde{\mathfrak{m}} \le C_E \mathfrak{m}$ and for each $i = c, j$

$$C_E^{-1}\mathcal{E}^i(f,f) \le \widetilde{\mathcal{E}}^i(f,f) \le C_E \mathcal{E}^i(f,f) \quad \text{for } f \in \mathcal{F} = \widetilde{\mathcal{F}}.$$

Then $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ admits a heat kernel $\widetilde{p}_t(x,y)$ satisfying that for each $t>0$ and $x \in E$, $y \mapsto \widetilde{p}_t(x,y)$ is finely continuous and there exists $C_p > 0$ and $k \ge 0$ such that

$$C_p^{-1} e^{-kt} p_t(x,y) \le \widetilde{p}_t(x,y) \le C_p e^{kt} p_t(x,y) \quad t>0, \quad x,y \in E.$$

Under what conditions, $p_t^A(x,y)$ on $]0,+\infty[\times E\times E$ also satisfies the following estimate?:

There exists $C_p>0$ and $k\geq 0$ such that

$$C_p^{-1}e^{-kt}p_t(x,y)\leq p_t^A(x,y)\leq C_pe^{kt}p_t(x,y)\ t>0,\ \ x,y\in E. \quad (1)$$

We write $p_t^A(x,y)\asymp_k p_t(x,y)$ for (1).

Thm 3.4 (**Kim-K** (2015)) Assume that **X** is transient. Suppose that **Ass** 3.1 holds with $k=0$. Assume $\mu_1+N(e^{F_1}-1)\mu_H\in S^1_{NK_1}(\mathrm{X})$, $\mu_{\langle u\rangle}\in S^1_{NK_\infty}(\mathrm{X})$ and $\mu_2+N(F_2)\mu_H\in S^1_{D_0}(\mathrm{X})$. Then $\lambda^{\mathcal{Q}}(\bar{\mu}_1)>0$ $\Longrightarrow p_t^A(x,y)\asymp_0 p_t(x,y)$.

Rem 3.2 Takeda (2006), Takeda (2007), Kim-K (2014).

4 Maximum principle of generalized Feynman-Kac semigroups

X: (resolvent) strong Feller transient process

$\mu_1 + N(F_1)\mu_H + \mu_{\langle u \rangle} \in S^1_{K_\infty}(X) = S^1_{CK_\infty}(\mathrm{X})$.

$\mu_2 + N(F_2)\mu_H \in S^1_{D_0}(\mathrm{X})$, $N(F_2)\mu_H \in S^1_K(\mathrm{X})$ and $F_1 \cdot F_2 = 0$.

$\mathcal{SH}^{\mathrm{ub}}(\mathcal{Q}) := \{h \in \mathcal{B}(E) \mid h$ is upper bounded and $h \le P^A_t h$ on $E\}$

and define the maximum principle by

$$(\mathrm{MP}) \qquad \text{If } h \in \mathcal{SH}^{\mathrm{ub}}(\mathcal{Q}), \text{ then } h(x) \le 0 \text{ for all } x \in E.$$

Thm 4.1 Assume that

$$(\mathrm{A}) \qquad \mathbf{E}_x\left[e^{-A^{\mu_2}_\infty - A^{F_2}_\infty} : \zeta = \infty\right] = 0.$$

Then (MP) holds if and only if $\lambda^{\mathcal{Q}}(\bar{\mu}_1) > 0$.

Let us introduce the space $\mathcal{H}^{\mathrm{b}}(\mathcal{Q})$ of (P_t^A)-invariant bounded functions:

$$\mathcal{H}^{\mathrm{b}}(\mathcal{Q}) := \{h \in \mathcal{B}_b(E) \mid h = P_t^A h \text{ on } E\}$$

and define the Liouville property by

$$\text{(L)} \qquad \text{If } h \in \mathcal{H}^{\mathrm{b}}(\mathcal{Q}), \text{ then } h(x) = 0 \text{ for all } x \in E.$$

Then we have the following:

Cor 4.1 Assume (A). Then $\lambda^{\mathcal{Q}}(\bar{\mu}_1) > 0$ implies (L).

Rem 4.1 Takeda (2015) proved Thm 4.1 and Cor 4.1 under $u = 0$ and $F_1 = F_2 = 0$.

Many thanks for your attention!

Vielen Dank für Ihre Aufmerksamkeit!