Random walks on hyperbolic groups: entropy and speed

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- Discrete potential theory in hyperbolic case.
- A description of harmonic functions
- Entropy criterion
- Quantitative question

Random walks on a Cayley graph

- A group $\Gamma = \langle S \rangle$, $S = S^{-1}$ a finite set of generators:
- A Cayley graph Γ: $x \sim y \Leftrightarrow x^{-1}y \in S$.
- μ: a probability measure on Γ (step)

The Green function:

$$G(x, y) = \sum p_n(x, y) = \sum \mu^{*n}(x^{-1}y) < \infty.$$

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Martin boundary: a compactification s.t.

$$\frac{G(x,\cdot)}{G(y,\cdot)} \text{ extends } \Gamma \cup \partial_M \Gamma \text{ continuously for any } x, y$$

$$\mu^{*n} \to \nu \text{ on } \partial_M \Gamma \text{ weakly (harmonic measure).}$$

- $\partial_M \Gamma$ as a topological space (Martin boundary)
- $(\partial_M \Gamma, \nu)$ as a measure space (Poisson boundary)

μ: finitely supported

- Γ regular tree : $\partial_M \Gamma = \partial_G \Gamma$ (Cantor) (Cartier et al '70)
- Γ hyperbolic : $\partial_M \Gamma = \partial_G \Gamma$ (Geometric boundary)(Ancona '88)

cf. X: Riem. s.c. negatively curved pinched:

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- μ: exponential tail: *uncountable* many $\partial_M \Gamma \not\simeq \partial_G \Gamma$!

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- μ: $H(\mu) < \infty$ & log-moment

$$(\partial_M \Gamma, \nu) \simeq (\partial_G \Gamma, \nu_{hitting}) \qquad \text{(Kaimanovich '00)}$$

Poisson boundary is more tractable.

When $(\partial_M \Gamma, \nu)$ non-trivial ?

$$H(\mu) = -\sum \mu(g) \log \mu(g) \qquad \text{(Shannon entropy)}$$

Theorem.[Kaimanovich-Vershik, '79, '83]

$$H(\mu^{*n}) \geq cn \ (\exists c > 0) \Leftrightarrow (\partial_M \Gamma, \nu) \text{ is non-trivial}$$

$$\Leftrightarrow \exists \text{non-const. bounded } \mu\text{-harmonic fcn on } \Gamma.$$

Example. $H(\mu^{*n}) = o(n) \Rightarrow$ Liouville

$\mathbb{Z}^d$, $\mu =$SRW, $\quad H(\mu^{*n}) = O(\log n^d)$.

Quantitative Question

$$h_\mu \leq l_\mu v$$

where

$$\frac{1}{n} H(\mu^{*n}) \to h_\mu,\ \frac{1}{n}\mathbb{E}d(o, X_n) \to l_\mu, \text{ and } \frac{1}{n}\log \sharp B(n) \to v.$$

"entropy, speed and log-volume growth"

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Easy case: Regular tree, SRW ⇒ "=".

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Theorem.[Gouëzel et al., 14+]
Γ hyperbolic+<u>non-virtually-free</u>: $h_\mu < l_\mu v$

Corollary. *Haus.* and ν are mutually singular.

Theorem.[T, 14+](“Localization”)
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$\forall \varepsilon > 0$, $\exists \Gamma_\varepsilon \subset \Gamma$ such that

$$\mathbb{P}(\forall n \geq 0, X_n \in \Gamma_\varepsilon) \geq 1 - \varepsilon,$$

$$vol(\Gamma_\varepsilon) = \frac{h_\mu}{l_\mu} \leq v$$

Remark. GW trees: Lyons-Pemantle-Peres '95

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Approach to the proof.
Multifractal analysis of the harmonic measure ν:

$$\dim_H \nu = \frac{h_\mu}{l_\mu} \leq \dim_H \partial_G \Gamma = v$$

$$\text{``}=\text{''} \ \text{``}\Rightarrow\text{''} \ Haus. \sim \nu.$$

Dimension spectrum: interpolate between the gap. $\alpha(\theta)$