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Lévy-type processes

Existence and estimates of moments

Since Lévy processes are homogeneous both in time and space, they are not well suited for modeling time dependent random phenomena. Dropping the assumption of homogeneity leads to Lévy-type processes: They behave locally like Lévy processes, but the characteristic Lévy triplet may depend on the current position. Typical examples are solutions of (Lévy-driven) SDEs, affine processes and stable-like processes. A Lévy-type process can be characterized by its symbol

$$q(x,\xi) = -i\,b(x)\xi + \frac{1}{2}\xi Q(x)\xi + \int (1 - e^{i\,y\xi} + i\,y\xi \mathbb{1}_{|y|\leq 1})\,N(x,dy), \qquad x \in \mathbb{R}^d, \xi \in \mathbb{R}^d.$$

Here, we extend some results on moments from Lévy to Lévy-type processes. For simplicity we assume that $(X_t)_{t\geq 0}$ has bounded coefficients.

Existence of moments

Time independence

Let $f \geq 0$ be a submultiplicative function.

- Backward in time: If $\mathbb{E}^x f(X_t) < \infty$ for some $t > 0$, then $\mathbb{E}^x f(X_s)$ for all $s \leq t$:

$$\mathbb{E}^x f(X_t) < \infty \iff \sup_{s\leq t} \mathbb{E}^x f(X_s) < \infty.$$

- Forward in time: If $\sup_{x\in\mathbb{R}^d} \mathbb{E}^x f(X_t) < \infty$ for some $t > 0$, then $\mathbb{E}^x f(X_s) < \infty$ for all $s > 0$. In fact,

$$\exists t: \sup_{x\in\mathbb{R}^d} \mathbb{E}^x f(X_t) < \infty \iff \forall t: \sup_{s\leq t}\sup_{x\in\mathbb{R}^d} \mathbb{E}^x f(X_s) < \infty.$$

Sufficient conditions

- If $\sup_{x\in\mathbb{R}^d} \int_{|y|\geq 1} f(y)\,N(x,dy) < \infty$ for some submultiplicative function $f \in C^2$, $f \geq 0$, then

$$\sup_{x\in\mathbb{R}^d}\sup_{s\leq t} \mathbb{E}^x f(X_s - x) < \infty \quad \text{for all } t \geq 0.$$

- $(d=1)$ If $q(x,\cdot)$ is $2n$ times differentiable at $\xi = 0$ for all $x \in \mathbb{R}$ and $|\partial_\xi^{2k} q(x,0)| \leq C_k(1+|x|^{2k})$ for all $k \leq n$, then

$$\sup_{x\in K} \int |y|^{2n}\,N(x,dy) < \infty \quad \text{and} \quad \sup_{x\in K} \mathbb{E}^x(|X_t - x|^{2n}) < \infty$$

for any compact set $K \subseteq \mathbb{R}$.

Estimates of fractional moments

Bounded variation/martingale type

Set $M_\alpha := \sup_{x\in\mathbb{R}^d} \int |y|^\alpha\,N(x,dy)$. If

- ... q is of "bounded-variation-type", i. e.

$$q(x,\xi) = \int (1 - e^{iy\xi})\,N(x,dy),$$

and $M_\alpha < \infty$ for some $\alpha \in (0,1]$, **or**

- ... q is of "martingale-type", i. e.

$$q(x,\xi) = \int (1 - e^{iy\xi} + iy\xi)\,N(x,dy),$$

and $M_\alpha < \infty$ for some $\alpha \geq 1$,

then for any $t \leq 1$ and $\kappa \in [0,\alpha]$

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}^x\Big(\sup_{s\leq t} |X_s - x|^\kappa\Big) \leq ct^{\kappa/\alpha}.$$

Blumenthal–Getoor indices

- If q satisfies the growth condition

$$\sup_{|y-x|\leq|\xi|^{-1}} \sup_{|\eta|\leq|\xi|} |q(y,\eta)| \leq C|\xi|^\alpha$$

for all $|\xi| \geq 1$, then

$$\mathbb{E}^x\Big(\sup_{s\leq t} |X_s - x|^\kappa\Big) \leq ct^{\min\{\kappa/\alpha, 1\}}$$

for $t \leq 1$ and $\kappa \in [0,\beta)$, $\kappa \neq \alpha$; here β denotes the generalized Blumenthal–Getoor index at 0.

- Similar results hold for
  - Lévy-type processes with unbounded coefficients,
  - large times t.

Reference Kühn, F.: Existence and estimates of moments for Lévy-type processes. ArXiv: 1507.07907.

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