Disease spreading models within the framework of two-component configuration spaces

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1 Disease spreading models

1.1 The SIS model (Susceptible-Infected-Susceptible model)

- single species of N individuals, $N \in \mathbb{N}$
- entire population splits into two groups of individuals (susceptible (S) and infected (I)) (compartmental model)
- groups indicate the total number of susceptible (S) and infected (I) individuals
- individuals travel from one compartment to another
- individuals in S are able to get infected by contagious individuals
- $r \in \mathbb{N}$ denotes number of contacts per unit time
- $\beta \in [0,1]$ is the probability of disease transmission per contact
- $\alpha \in [0,1]$ is the recovery rate per capita

Susceptible — infection rate $r\beta\frac{I}{N}$ → Infected; Infected — recovery rate α → Susceptible

- the underlying ode system

$$\frac{dS}{dt} = -r\beta S\frac{I}{N} + \alpha I$$
$$\frac{dI}{dt} = r\beta S\frac{I}{N} - \alpha I$$

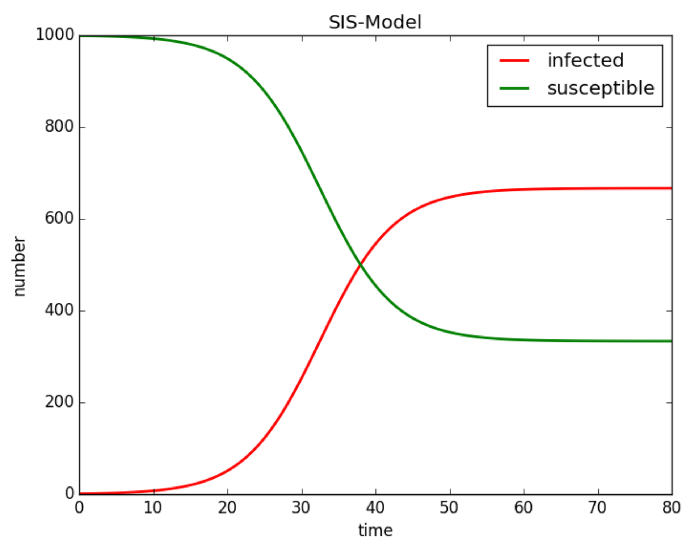


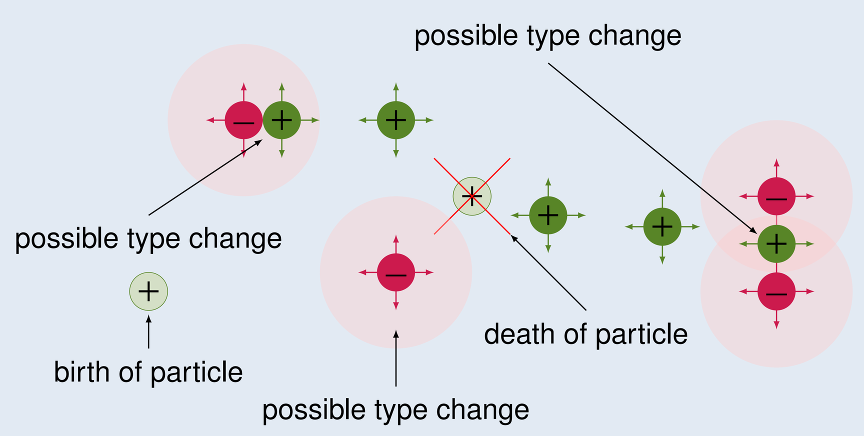
Figure 1: $N = 1000$, $S_0 = 999$, $I_0 = 1$, $r = 6$, $\beta = 0.03$, $\alpha = 0.8$

- no birth, no death, no migration of individuals

For an overview of compartmental models see e.g. [CH11].

1.2 Modeling via Interacting Particle Systems

- interacting particles of two types (susceptible (+) and infected (-)) in $\mathbb{R}^2$-space
- particles can move in space (mobility)
- particles interact via an interaction potential
- particles can change their types according to certain rates
- new particles can appear (birth process)
- particles can disappear (death process)



2 The mathematical background

The general mathematical background is provided in [FKO13].

2.1 The setting

- one-component configuration space Γ

$$\Gamma := \Gamma_{\mathbb{R}^2} := \left\{\gamma \subset \mathbb{R}^2 \,\middle|\, \#(\gamma \cap K) < \infty \text{ for all } K \subset \mathbb{R}^2 \text{ compact}\right\},$$

where $\#S$ denotes the cardinality of a set S
- one can identify each $\gamma \in \Gamma$ with a positive, integer-valued Radon measure
- let $\mathcal{B}(\Gamma)$ denote the Borel-σ-algebra corresponding to the vague topology on Γ
- we equip the measurable space $(\Gamma, \mathcal{B}(\Gamma))$ with a probability measure μ and obtain the probability space $(\Gamma, \mathcal{B}(\Gamma), \mu)$
- one-component configuration space of finite configurations

$$\Gamma_0 := \coprod_{n=0}^{\infty} \Gamma^{(n)}, \text{ where } \Gamma^{(n)} := \{\gamma \in \Gamma \,|\, \#\gamma = n\} \text{ for } n \in \mathbb{N} \text{ and } \Gamma^{(0)} := \{\varnothing\}$$

$(\Gamma_0, \mathcal{B}(\Gamma_0), \lambda)$ denotes the Lebesgue–Poisson space

- two-component configuration space

Given two copies of the space Γ, denoted by Γ^+ and Γ^-, let

$$\Gamma^2 := \left\{(\gamma^+, \gamma^-) \in \Gamma^+ \times \Gamma^- \,\middle|\, \gamma^+ \cap \gamma^- = \varnothing\right\}.$$

Similarly, given two copies of the space Γ_0, denoted by Γ_0^+ and Γ_0^-, let

$$\Gamma_0^2 := \left\{(\eta^+, \eta^-) \in \Gamma_0^+ \times \Gamma_0^- \,\middle|\, \eta^+ \cap \eta^- = \varnothing\right\}.$$

Using the product structure we obtain

$$\left(\Gamma^2, \mathcal{B}(\Gamma^2), \mu^2\right) \quad \text{and} \quad \left(\Gamma_0^2, \mathcal{B}(\Gamma_0^2), \lambda^{\otimes 2}\right)$$

as state spaces, where μ^2 is a probability measure on $\left(\Gamma^2, \mathcal{B}(\Gamma^2)\right)$.

2.2 The strategy

Evolution of observables

- heuristically, the stochastic evolution of an infinite two-component particle system is described by a Markov process on Γ^2
- determined by its Markov generator L defined on a proper space of functions on Γ^2
- it provides a solution to the Kolmogorov backward equation

$$\frac{d}{dt}F_t = LF_t, \quad F_t\big|_{t=0} = F_0. \qquad \text{(EvO)}$$

Evolution of states

- stochastic evolution in terms of mean values
- for functions $F : \Gamma^2 \to \mathbb{R}$ integrable with respect to a probability measure μ^2 on $\mathcal{B}(\Gamma^2)$, i.e., a state of the system, the expected values are given by

$$\langle F, \mu \rangle := \int_{\Gamma^2} F(\gamma^+, \gamma^-)\, d\mu^2(\gamma^+, \gamma^-)$$

- time evolution problem on states

$$\frac{d}{dt}\langle F, \mu_t^2 \rangle = \langle LF, \mu_t^2 \rangle, \quad \mu_t^2\big|_{t=0} = \mu_0^2 \qquad \text{(EvS)}$$

Evolution of correlation functionals

- for F being of type $F = KG$, where $G : \Gamma_0^2 \to \mathbb{R}$ is bounded, measurable and of bounded support and K denotes the K-transform, (EvS) may be rewritten in terms of correlation functionals $k_t := k_{\mu_t^2}$ corresponding to the measures μ_t^2 provided, these functionals exist
- time evolution problem on correlation functionals in weak formulation:

$$\frac{d}{dt}\langle\langle G, k_t \rangle\rangle = \langle\langle \hat{L}G, k_t \rangle\rangle, \quad k_t\big|_{t=0} = k_0, \qquad \text{(wEvC)}$$

where $\hat{L} := K^{-1}LK$ and $\langle\langle \cdot, \cdot \rangle\rangle$ is the usual pairing

$$\langle\langle G, k \rangle\rangle := \int_{\Gamma_0^2} G(\eta^+, \eta^-)\, k(\eta^+, \eta^-)\, d\lambda^{\otimes 2}(\eta^+, \eta^-) \qquad \text{(P)}$$

in strong formulation:

$$\frac{d}{dt}k_t = \hat{L}^* k_t, \quad k_t\big|_{t=0} = k_0 \qquad \text{(sEvC)}$$

for $\hat{L}^*$ being the dual operator of $\hat{L}$ in the sense defined in (P)

3 Application

3.1 Modeling infection and recovery: the flip generator

Infection of particles

- Markov pre-generator

$$(L_{\text{flip}}^{\text{inf}} F)(\gamma^+, \gamma^-) := \sum_{x \in \gamma^+} c^{+-}(x, \gamma^-) \times \left[F\left(\gamma^+ \setminus \{x\}, \gamma^- \cup \{x\}\right) - F\left(\gamma^+, \gamma^-\right)\right], \quad F \in \mathcal{D},$$

where $c^{+-}(x, \gamma^-) \geq 0$ is the rate at which a +-particle at $x \in \gamma^+$ flips to a −-particle in dependence of the surrounding −-particles and $\mathcal{D}$ is a suitable domain of functions $F : \Gamma^2 \to \mathbb{R}$.

- specification of the flip rate for $L_{\text{flip}}^{\text{inf}}$

$$c^{+-}(x, \gamma^-) := \sum_{y \in \gamma^-} \phi\left(|x - y|\right), \quad x \in \mathbb{R}^2, \quad \gamma^- \in \Gamma^-$$

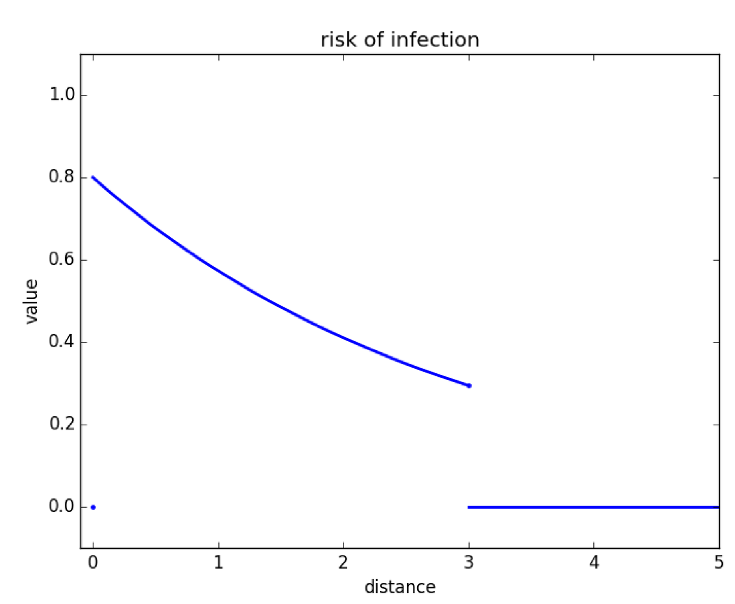


Figure 2: rate of infection ϕ for a single particle

- The corresponding operator $\hat{L}_{\text{flip}}^{\text{inf}*}$ reads

$$\begin{aligned}\left(\hat{L}_{\text{flip}}^{\text{inf}*} k\right)(\eta^+, \eta^-) = &\sum_{\{x,y\} \subset \eta^-} \phi\left(|x - y|\right) k(\eta^+ \cup \{x\}, \eta^- \setminus \{x\}) \\ &- \sum_{x \in \eta^+} \sum_{y \in \eta^-} \phi\left(|x - y|\right) k(\eta^+, \eta^-) \\ &+ \sum_{y \in \eta^-} \int_{\mathbb{R}^2} \phi\left(|x - y|\right) k\left(\eta^+ \cup \{y\}, \eta^- \setminus \{y\} \cup \{x\}\right) dx \\ &- \sum_{x \in \eta^+} \int_{\mathbb{R}^2} \phi\left(|x - y|\right) k\left(\eta^+, \eta^- \cup \{y\}\right) dy\end{aligned}$$

acting on correlation functionals
- one-particle-correlation functionals

$$\mathbb{R}^2 \ni x \mapsto k^+(x) := k^{(1,0)}(x) = k\left(\{x\}, \varnothing\right) \quad \text{and} \quad \mathbb{R}^2 \ni x \mapsto k^-(x) := k^{(0,1)}(x) := k\left(\varnothing, \{x\}\right)$$

- time evolution on one-particle-correlation functionals, $x \in \mathbb{R}^2$ and $t \geq 0$, conform to (sEvC),

$$\begin{aligned}\frac{d}{dt}k_t^+(x) &= -\int_{\mathbb{R}^2} \phi\left(|x - y|\right) k_t\left(\{x\}, \{y\}\right) dy \\ \frac{d}{dt}k_t^-(x) &= \int_{\mathbb{R}^2} \phi\left(|x - y|\right) k_t\left(\{x\}, \{y\}\right) dy\end{aligned} \qquad \text{(sEvC1)}$$

- note that the two-particles-correlation functionals

$$k_t^{(1,1)}(x, y) := \begin{cases} k_t(\eta) & \text{if } \eta := \left(\{x\}, \{y\}\right) \in \Gamma_0^2 \\ 0 & \text{else} \end{cases}, \quad t \geq 0,$$

are involved

Vlasov Scaling

- in order to tackle equations (sEvC1) we apply a mean field-type scaling, the so called Vlasov scaling, to obtain

$$\begin{aligned}\frac{d}{dt}k_t^+(x) &= -\left(\phi * k_t^-\right)(x)\, k_t^+(x) \\ \frac{d}{dt}k_t^-(x) &= \left(\phi * k_t^-\right)(x)\, k_t^+(x),\end{aligned} \qquad \text{(ssEvC1)}$$

a closed system of equations with $x \in \mathbb{R}^2$

Recovery of particles

- Markov pre-generator

$$(L_{\text{flip}}^{\text{rec}} F)(\gamma^+, \gamma^-) := \alpha \sum_{y \in \gamma^-} \left[F\left(\gamma^+ \cup \{y\}, \gamma^- \setminus \{y\}\right) - F\left(\gamma^+, \gamma^-\right)\right], \quad F \in \mathcal{D},$$

where $\alpha \in [0,1]$ is the constant rate at which a −-particle at $x \in \gamma^-$ flips to a +-particle and $\mathcal{D}$ is a suitable domain of functions $F : \Gamma^2 \to \mathbb{R}$

Resulting equations for infection and recovery of particles

- applying the above procedure yields

$$\begin{aligned}\frac{d}{dt}k_t^+(x) &= -\left(\phi * k_t^-\right)(x)\, k_t^+(x) + \alpha\, k_t^-(x) \\ \frac{d}{dt}k_t^-(x) &= \left(\phi * k_t^-\right)(x)\, k_t^+(x) - \alpha\, k_t^-(x), \quad x \in \mathbb{R}^2\end{aligned} \qquad \text{(ssEvC2)}$$

4 Outlook

Mobility of particles

- Markov pre-generator for hopping +-particles

$$\begin{aligned}(L_{\text{mov}} F)(\gamma^+, \gamma^-) := &\sum_{x \in \gamma^+} \int_{\mathbb{R}^2} c^+\left(x, x', \gamma^+ \setminus \{x\}, \gamma^-\right) \\ &\times \left[F\left(\gamma^+ \setminus \{x\} \cup \{x'\}, \gamma^-\right) - F\left(\gamma^+, \gamma^-\right)\right] dx' \\ &+ \sum_{y \in \gamma^-} \int_{\mathbb{R}^2} c^-\left(y, y', \gamma^+, \gamma^- \setminus \{y\}\right) \\ &\times \left[F\left(\gamma^+, \gamma^- \setminus \{y\} \cup \{y'\}\right) - F\left(\gamma^+, \gamma^-\right)\right] dy', \quad F \in \mathcal{D},\end{aligned}$$

where $c^+(x, x', \gamma^+ \setminus \{x\}, \gamma^-) \geq 0$ indicates the rate at which a +-particle located at $x \in \gamma^+$ hops to a free site $x' \in \mathbb{R}^2$. $\mathcal{D}$ is a suitable domain of functions $F : \Gamma^2 \to \mathbb{R}$, see [FKO13].

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