# Large deviations for intersection measures of some Markov processes

Takahiro Mori*

The aim of this talk is to analyze the intersection measures in a general setting.

Let $p$ be an integer with $p \geq 2$, $E$ be a locally compact, separable metric space and $m$ be a Radon measure on $E$ with supp$[m] = E$. Let $X^{(1)}, \cdots, X^{(p)}$ be $p$ independent irreducible Hunt processes on $E$, with life time $\zeta^{(1)}, \cdots, \zeta^{(p)}$ and start $x_1, \cdots, x_p \in E$, respectively. For simplicity, we also assume that $X^{(1)}, \cdots, X^{(p)}$ have the same law. For each $t > 0$, under the condition that all life times $\zeta^{(1)}, \cdots, \zeta^{(p)}$ are grater than $t$, *the intersection measre* $\ell_t^{\mathrm{IS}}$ is formally witten as

$$\ell_t^{\mathrm{IS}}(A) = \int_A \int_{[0,t]^p} \prod_{i=1}^{p} \delta_x(X^{(i)}(s_i)) ds_1 \cdots ds_p m(dx) \quad \text{for } A \subset E \text{ Borel}.$$

Intersection measure is firstly introduced in [LG92], and until now, only for the cases of Brownian motion and stable processes, its deep natures have been obtained. ([CR05], [KM13])

To deal with the intersection measure in a general setting, we now make five assumptions on $X^{(i)}$. Here $R_1$ is the 1-order resolvent, and $\{T_t\}$ is the linear operator determined by the transition probability $p_t$ of $X^{(i)}$:

**(A1)** (Tightness) $\forall \varepsilon > 0, \exists K$: compact, such that $\sup_{x \in E} R_1 1_{K^c}(x) \leq \varepsilon$.

**(A2)** (Transition density) $\forall t > 0$ and $\forall x \in E$, the transition probability $p_t(x, dy)$ is absolutely continuous with respect to $m$, and its density $p_t(\cdot, \cdot)$ is continuous and bounded on $E \times E$.

**(A3)** (Trace estimate) There exist $\rho > 0$, $t_0 > 0$ and $C > 0$ such that

$$C^{-1} t^{-\rho/2} \leq \int_E p_t(x, x) m(dx) \leq C t^{-\rho/2}, \quad \text{for all } t \in (0, t_0].$$

**(A4)** (Ultra-contractivity) There exist $\mu \in (2, \frac{2p}{p-1})$, $C > 0$ and $t_1 > 0$ such that

$$\|T_t\|_{1 \to \infty} \leq C t^{-\mu/2}, \quad \text{for all } t \in (0, t_1].$$

**(A5)** (Green function estimate)

$$\sup_{x \in E} \int_E R_1(x, y)^p m(dy) < \infty, \quad \lim_{\delta \downarrow 0} \sup_{x \in E} \int_E R_{1,\delta}(x, y)^p m(dy) = 0,$$

where

$$R_1(x, y) = \int_0^\infty e^{-t} p_t(x, y) dt, \quad R_{1,\delta}(x, y) = \int_0^\delta e^{-t} p_t(x, y) dt \quad \text{for } x, y \in E.$$

We can check these assumptions easily, if $X^{(i)}$ enjoys (sub-)Gaussian or jump-type heat kernel estimates.

*Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-8502, JAPAN. tmori@kurims.kyoto-u.ac.jp

The occupation measure $\ell_t^{(i)}$ and for each $\varepsilon > 0$ the approximated occupation measure $\ell_{\varepsilon,t}^{(i)}$ of $X^{(i)}$ until time $t > 0$ are defined by

$$\ell_t^{(i)}(A) := \int_0^t 1_A(X^{(i)}(s))ds, \qquad \ell_{\varepsilon,t}^{(i)}(A) := \int_A \left[\int_{[0,t]} p_\varepsilon^{(i)}(X^{(i)}(s), x)ds\right] m(dx)$$

for $A \subset E$, on the event $\{t < \zeta^{(i)}\}$.

For each $\varepsilon > 0$, the approximated (mutual) intersection measure $\ell_{\varepsilon,t}^{\mathrm{IS}}$ of $X^{(1)}, \cdots, X^{(p)}$ until time $t > 0$ is defined by

$$\ell_{\varepsilon,t}^{\mathrm{IS}}(A) := \int_A \left[\int_{[0,t]^p} \prod_{i=1}^p p_\varepsilon^{(i)}(X^{(i)}(s_i), x)ds_1 \cdots ds_p\right] m(dx)$$

for $A \subset E$, on the event $\{t < \zeta^{(1)} \wedge \cdots \wedge \zeta^{(p)}\}$.

We define the function $\mathbf{J} : \mathcal{M}_f(E) \times (\mathcal{M}_1(E))^p \to [0, \infty]$ by

$$\mathbf{J}(\mu; \mu_1, \cdots, \mu_p) := \begin{cases} \displaystyle\sum_{i=1}^p \{\mathcal{E}(\psi_i, \psi_i) - \lambda_1\} & ; \text{ if } \psi_i = \sqrt{\dfrac{d\mu_i}{dm}} \in \mathcal{F} \text{ and } \displaystyle\prod_{i=1}^p \dfrac{d\mu_i}{dm} = \dfrac{d\mu}{dm}, \\ \infty & ; \text{otherwise} \end{cases}$$

for $(\mu; \mu_1, \cdots, \mu_p) \in \mathcal{M}_f(E) \times (\mathcal{M}_1(E))^p$, where $(\mathcal{E}, \mathcal{F})$ is the associated regular Dirichlet form of $X^{(i)}$ and $\lambda_1 := \inf\{\mathcal{E}(\psi, \psi); \psi \in \mathcal{F}, \|\psi\|_2 = 1\}$ is the bottom of spectrum.

The first result enables us to deal with $\ell_t^{\mathrm{IS}}$ in our setting:

**Proposition 0.1.** *Suppose* **(A5)** *and let $t > 0$. Then, there exists the random measure $\ell_t^{IS} \in \mathcal{M}_f(E)$ such that, in the vague topology of $\mathcal{M}_f(E)$,*

$$\ell_{t,\varepsilon}^{IS} \to \ell_t^{IS} \quad \text{in distribution, as } \varepsilon \to 0,$$

*with respect to the probability measure $\widetilde{\mathbb{P}}_t := \mathbb{P}(\,\cdot \mid \{t < \zeta^{(1)} \wedge \cdots \wedge \zeta^{(p)}\})$.*

□

The following is *one of* our main results. This extends [KM13], in which the large deviation principle is established for the case of $d$-dimensional Brownian motions before exiting some bounded open set $B \subset \mathbb{R}^d$:

**Theorem 0.2** (Large deviation principle)**.** *Suppose* **(A1) - (A5)**. *Then the tuple*

$$\left(\frac{1}{t^p}\ell_t^{IS}; \frac{1}{t}\ell_t^{(1)}, \cdots, \frac{1}{t}\ell_t^{(p)}\right) \in \mathcal{M}_f(E) \times (\mathcal{M}_1(E))^p$$

*satisfises the large deviation principle as $t \to \infty$, with probability $\widetilde{\mathbb{P}}_t$, scale $t$ and the good rate function* $\mathbf{J}$.

□

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# Behavior of fundamental solutions for critical Schrödinger operators

福島大学人間発達文化学類　　和田正樹

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## 1　準備及び考えたい問題

$\{X_t\}_{t\geq 0}$ を生成作用素 $\mathcal{L} = -(-\Delta)^{\alpha/2}$ $(0 < \alpha \leq 2)$ とする $\mathbb{R}^d$ 上の過渡的な対称 $\alpha$-安定過程とする。このとき、対応するディリクレ形式が $\mathcal{E}(u,u) = (-\mathcal{L}u, u)_m$ として、推移確率密度関数 $p(t,x,y)$ が方程式 $\partial u/\partial t = \mathcal{L}u$ の基本解として与えられる。ここで、$m$ は $\mathbb{R}^d$ 上のルベーグ測度、$(\cdot,\cdot)_m$ は $L^2(\mathbb{R}^d)$ における内積を表す。更に、グリーン核 $G(x,y) = \displaystyle\int_0^\infty p(t,x,y)dt$ により、以下の条件が成立している非負測度 $\mu$ を考える。

$$\lim_{a\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|\leq a} G(x,y)\mu(dy) = 0, \quad \lim_{R\to\infty}\sup_{x\in\mathbb{R}^d}\int_{|y|>R} G(x,y)\mu(dy) = 0,$$
$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} G(x,y)\mu(dy)\mu(dx) < \infty.$$

このとき、シュレディンガー形式を $\mathcal{E}^\mu(u,u) = \mathcal{E}(u,u) - \displaystyle\int_{\mathbb{R}^d} u^2(x)\mu(dx)$ で定め、対応する作用素を $\mathcal{L}^\mu$ とする。方程式 $\partial u/\partial t = \mathcal{L}^\mu u$ にも基本解 $p^\mu(t,x,y)$ が存在することが知られているが ([1])、ここではその挙動を元の基本解 $p(t,x,y)$ の挙動と比較する。

## 2　先行結果と主結果

直感的には $\mu$ が「十分小さい」ときには、$p^\mu(t,x,y)$ は元の $p(t,x,y)$ と似たような挙動をすること（基本解の安定性という）が予想される。そこで、まずは測度 $\mu$ の大きさを表すための指標を

$$\lambda(\mu) = \inf\left\{\mathcal{E}(u,u) \mid \int_{\mathbb{R}^d} u^2(x)\mu(dx) = 1\right\}$$

で定める。[4] や [5] では、基本解の安定性が成立するための必要十分条件が、$\lambda(\mu) > 1$ を満たすこと、すなわち $\mu$ が劣臨界的であることが示された。

$\lambda(\mu) = 1$ のとき、$\mu$ は臨界的であるといい、$p^\mu(t,x,y)$ は元の $p(t,x,y)$ とは異なる挙動をすることがわかる。しかし、具体的な振る舞いが与えられているのは、元のマルコフ過程が 3 次元ブラウン運動、$(d,\alpha) = (3,2)$ の場合に限られており ([2, 4])、次の通りである。

$$p^\mu(t,x,y) \asymp c_1\left(1+\frac{\sqrt{t}}{1+|x|}\right)\left(1+\frac{\sqrt{t}}{1+|y|}\right)t^{-\frac{3}{2}}\exp\left(-c_2\frac{|x-y|^2}{t}\right) \tag{2.1}$$

一般の安定過程の枠組みでは、$p^\mu(t,x,y)$ そのものの具体的な振る舞いは未だ確立されていないが、今回の主結果では時間無限大での漸近振る舞いが、次の通り得られた。

**定理 2.1.** $\mu$ は臨界的で $h(x)$ はシュレディンガー形式における基底状態とする。$K_{d,\alpha}$ を適切な正数とするとき、$p^\mu(t,x,y)$ は以下の漸近挙動をもつ。

$$\begin{aligned} \lim_{t\to\infty} t^{2-\frac{d}{\alpha}} p^\mu(t,x,y) &= K_{d,\alpha} h(x)h(y) \quad (1 < d/\alpha < 2) \\ \lim_{t\to\infty} (\log t) p^\mu(t,x,y) &= K_{d,\alpha} h(x)h(y) \quad (d/\alpha = 2) \\ \lim_{t\to\infty} p^\mu(t,x,y) &= K_{d,\alpha} h(x)h(y) \quad (d/\alpha > 2) \end{aligned}$$

ここで、基底状態については $h(x) \asymp 1 \wedge |x|^{\alpha-d}$ の評価が成り立っており、いることに注意すると、定理 2.1 は、$(d,\alpha)=(3,2)$ における先行結果 (2.1) を、$t\to\infty$ のときの振る舞いに限って拡張したものである。また、[3] では零臨界的なときの基本解が時間無限大で消えることを示しているが、本結果により、そのオーダーが厳密に確定できている。

証明の大まかな流れは次の通りである。

(1) [6] にて扱われている、基本解 $p^\mu(t,x,y)$ におけるエルゴード型定理。すなわち、$k(t)$ を $d/\alpha$ に応じて適切に選ぶことで、適切な正数 $C_{d,\alpha}$ により $\displaystyle\lim_{t\to\infty}\frac{1}{k(t)}\int_0^t p^\mu(\epsilon+s,x,y)ds = C_{d,\alpha}h(x)h(y)$ がいえる。特に $d/\alpha>2$ のときは $k(t)=t$ であり、エルゴード定理そのものである。

(2) $L^2(h^2\cdot m)$ 上のマルコフ半群 $P_t^{\mu,h}f(x) = \mathbb{E}_x\left[\exp(A_t^\mu)\dfrac{h(X_t)}{h(X_0)}f(X_t)\right]$ において、$(P_t^{\mu,h}f,f)_{h^2\cdot m}$ は $t$ に関して単調減少である。ここで、$A_t^\mu$ は $\mu$ とルヴューズ対応するような正値連続な加法汎関数である。

(3) 任意の非負の実数 $c_1, c_2$ と $\mathbb{R}^d$ の 2 点 $x_0, y_0$ に対して、関数列 $f_n(x)$ を適切に選ぶことで、$(P_t^{\mu,h}f_n,f_n)_{h^2\cdot m}$ は $c_1^2p^\mu(t,x_0,x_0)+2c_1c_2p^\mu(t,x_0,y_0)+c_2^2p^\mu(t,y_0,y_0)$ に収束させることができる。特に、この収束値も $t$ に関して単調減少である。

(4) (1) と (3) より、$\dfrac{p^\mu(t,x,y)}{k'(t)}$ は $t\to\infty$ のとき $C_{d,\alpha}h(x)h(y)$ に収束する。

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# IRREDUCIBLE DECOMPOSITION FOR MARKOV PROCESSES

Kazuhiro Kuwae

(Fukuoka University)

## 1. Main Theorem

Let $E$ be a separable metric space and $\mathfrak{m}$ a $\sigma$-finite Borel measure on $E$. We consider a quasi-regular semi-Dirichlet form $(\mathcal{E},\mathcal{F})$ with a lower bound $-\gamma$ on $L^2(E;\mathfrak{m})$ ($\gamma \geq 0$). Under the quasi-regularity of $(\mathcal{E},\mathcal{F})$, we may (and do) assume that $E$ is a Lusin topological space, i.e., $E$ is homeomorphic to a Borel subset of a compact metric space. An $\mathfrak{m}$-measurable subset $B$ of $E$ is said to be *weakly* $(T_t)_{t\geq 0}$*-invariant* if $\mathbf{1}_{B^c}T_t\mathbf{1}_B u = 0$ for any $t > 0$ and $u \in L^2(E;\mathfrak{m})$, equivalently $B^c$ is weakly $(\hat{T}_t)_{t\geq 0}$-invariant. An $\mathfrak{m}$-measurable subset $B$ of $E$ is said to be *(strongly)* $(T_t)_{t\geq 0}$*-invariant* if $T_t\mathbf{1}_B u = \mathbf{1}_B T_t u$ for any $t > 0$ and $u \in L^2(E;\mathfrak{m})$. Clearly, the strong $(T_t)_{t\geq 0}$-invariance implies the weak one. Any semi-Dirichlet form $(\mathcal{E},\mathcal{F})$ with a lower bound $-\gamma$ on $L^2(E;\mathfrak{m})$ is said to be *strictly irreducible* (resp. *irreducible*) if for any weakly (resp. strongly) $(T_t)_{t\geq 0}$-invariant set $B$ relative to the $C_0$-semigroup $(T_t)_{t\geq 0}$ of (resp. *irreducible*) $(\mathcal{E},\mathcal{F})$, $\mathfrak{m}(B) = 0$ or $\mathfrak{m}(B^c) = 0$. The process $\mathbf{X}$ is called $\mathfrak{m}$*-irreducible* if the corresponding semi-Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(E;\mathfrak{m})$ is irreducible (see [4]). A set $B(\subset E_\partial)$ is called *nearly Borel* if there exist Borel subsets $B_1, B_2$ of $E_\partial$ such that $B_1 \subset B \subset B_2$ and $\mathbf{P}_\mu(X_t \in B_2 \setminus B_1, \exists t \in [0,\infty[) = 0$ for all $\mu \in \mathcal{P}(E_\partial)$. Denote by $\mathcal{B}^n(E_\partial)$ (resp. $\mathcal{B}^n(E)$) the family of nearly Borel subsets of $E_\partial$ (resp. $E$). A set $A$ is called *finely open* if for each $x \in A$, there exists a $B \in \mathcal{B}^n(E)$ such that $E \setminus A \subset B$ and $\mathbf{P}_x(\sigma_B > 0) = 1$. The family of finely open sets defines a topology on $E$ which is called the *fine topology* of $\mathbf{X}$. $\mathbf{X}$ is said to be *finely irreducible* if for any finely open nearly Borel set $D$ with $D \neq \emptyset$, $\mathbf{P}_x(\sigma_D < \infty) > 0$ for all $x \in E$ (see [4]). A set $B \subset E$ is said to be $\mathbf{X}$*-invariant* if $B \in \mathcal{B}^n(E)$ and

$$\mathbf{P}_x(X_t \in B \text{ for all } t \in [0,\zeta[,\ X_{t-} \in B \text{ for all } t \in ]0,\zeta[) = 1, \qquad x \in B.$$

A set $N \subset E$ is called *properly exceptional* if $N$ is a nearly Borel $\mathfrak{m}$-negligible set and $E \setminus N$ is $\mathbf{X}$-invariant. If $\mathbf{X}$ has a decomposition $E = B_1 + B_2 + N$ such that each $B_i$ ($i = 1,2$) is $\mathbf{X}$-invariant and $N$ is properly exceptional, then each $B_i$ ($i = 1,2$) is strongly $(T_t)_{t\geq 0}$-invariant.

We consider the following conditions:

**(A)** $\mathbf{X}$ is a diffusion process or $\mathfrak{m}$-symmetric.

**(AC)** $P_t(x, \mathrm{d}y) \ll \mathfrak{m}(\mathrm{d}y)$ for each $x \in E$ and $t > 0$.

**(AC)**$'$ For some fixed $\alpha > 0$, $R_\alpha(x, \mathrm{d}y) \ll \mathfrak{m}(\mathrm{d}y)$ for each $x \in E$.

**(RSF)** $R_\alpha(\mathcal{B}_b(E)) \subset C_b(E)$ for each $\alpha > 0$.

**Remark 1.1.** (1) In fact, (**AC**) is equivalent to $(\mathbf{AC})'$. This is proved in [2, Theorem 4.2.2] under the $\mathfrak{m}$-symmetry of $\mathbf{X}$, whose proof remains valid provided $\mathbf{X}$ is properly associated with $(\mathcal{E},\mathcal{F})$.

(2) $(\mathbf{AC})'$ is equivalent to Meyer's hypothesis (**L**) (see [1, pp. 112]).

(3) In view of resolvent equation, $(\mathbf{AC})'$ is equivalent to that $R_\alpha(x,\mathrm{d}y)\ll\mathfrak{m}(\mathrm{d}y)$ for each $x\in E$ and any $\alpha>0$.

(4) It is known that $(\mathbf{AC})'$ holds if and only if every exceptional is polar (see [2, Theorem 4.2.4]).

(5) Clearly, (**RSF**) implies $(\mathbf{AC})'$.

Our main theorems are the following:

**Theorem 1.1.** *Assume* (**A**)*. For each $x\in E$, there exists an $\mathcal{E}$-quasi-open and $\mathcal{E}$-quasi -closed nearly Borel set $E_x$ satisfying the following:*

(1) *$x\in E_x$.*

(2) *There exists a properly exceptional set $N$ such that $E_x\setminus N$ and $E_x^{\rm c}\setminus N$ are $\mathbf{X}$-invariant.*

(3) *The part process $\mathbf{X}_{E_x}$ is $\mathfrak{m}$-irreducible, i.e., the part space $(\mathcal{E}_{E_x},\mathcal{F}_{E_x})$ on $L^2(E_x;\mathfrak{m})$ is irreducible.*

*For each $x\in E$, $E_x$ satisfying* (1)–(3) *is unique in the sense that if $\widetilde{E}_x$ satisfies conditions* (1)–(3) *by replacing $E_x$ with $\widetilde{E}_x$, then $E_x=\widetilde{E}_x$ q.e. provided $E_x\cap\widetilde{E}_x$ is not $\mathfrak{m}$-polar. Moreover, for each $x,y\in E$, if $E_x\cap E_y$ is not $\mathfrak{m}$-polar, then $E_x=E_y$ q.e.*

**Theorem 1.2.** *Assume* (**A**) *and* (**AC**)*. For each $x\in E$, there exists a finely open and finely closed Borel set $E_x$ satisfying the following:*

(1) *$x\in E_x$.*

(2) *$E_x$ and $E_x^{\rm c}$ are $\mathbf{X}$-invariant.*

(3) *The part process $\mathbf{X}_{E_x}$ is finely irreducible.*

*If* (**A**) *and* (**RSF**) *are satisfied, then $E_x$ can be taken to be open and closed. For each $x\in E$, $E_x$ satisfying* (1)–(3) *is unique in the sense that if $\widetilde{E}_x$ satisfies all conditions* (1)–(3) *by replacing $E_x$ with $\widetilde{E}_x$, then $E_x=\widetilde{E}_x$. Moreover, for $x,y\in E$, if $E_x\cap E_y\neq\emptyset$, then $E_x=E_y$. More strongly, there exists a countable sets $\{x_i\}$ such that $E=\bigcup_{i=1}^\infty E_{x_i}$ is a disjoint union.*

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# $L^p$-INDEPENDENCE OF SPECTRAL RADIUS FOR GENERALIZED FEYNMAN-KAC SEMIGROUPS

Zhen-Qing Chen, Daehong Kim and Kazuhiro Kuwae

(University of Washington, Kumamoto University and Fukuoka University)

## 1. Preliminary

Let $E$ be a Lusin metric space and $\mathfrak{m}$ a $\sigma$-finite Borel measure on $E$ with full topological support. We add $\partial \notin E$ as an isolated point to $E$. Let $\mathbf{X} = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \mathbf{P}_x, x \in E_\partial)$ be an $\mathfrak{m}$-symmetric special standard process on $E$ with lifetime $\zeta := \inf\{t > 0 \mid X_t = \partial\}$. Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form on $L^2(E; \mathfrak{m})$ associated with $\mathbf{X}$. Then $(\mathcal{E}, \mathcal{F})$ is automatically quasi-regular. We further assume that $\mathbf{X}$ satisfies **(AC)**. Suppose $\mu$ is a signed smooth measure. Let $\mu^+$ and $\mu^-$ denote the positive and negative variation measure of $\mu$ in its the Jordan decomposition, which are smooth measures. We define $A^\mu := A^{\mu^+} - A^{\mu^-}$. Let $\dot{\mathcal{F}}_{\rm loc}$ be the family of all functions locally in $\mathcal{F}$ in the broad sense, i.e., $u \in \dot{\mathcal{F}}_{\rm loc}$ if and only if there exist an increasing sequence $\{O_n\}$ of finely open nearly Borel sets satisfying $\bigcup_{n=1}^\infty O_n = E$ and $\{u_n\} \subset \mathcal{F}$ such that $u = u_n$ $\mathfrak{m}$-a.e. on $O_n$. Since $(\mathcal{E}, \mathcal{F})$ is quasi-regular, every $u \in \dot{\mathcal{F}}_{\rm loc}$ admits an $\mathcal{E}$-quasi-continuous $\mathfrak{m}$-version $\tilde{u}$, and we omit tilde from $\tilde{u}$, i.e., we always assume $u \in \dot{\mathcal{F}}_{\rm loc}$ is represented by its $\mathcal{E}$-quasi-continuous version.

Let $N^u$ be the continuous additive functional of zero quadratic variation appeared in a Fukushima decomposition of $u(X_t) - u(X_0)$ up to the lifetime. Note that $N^u$ is not necessarily of bounded variation in general. Let $F$ be a bounded symmetric function on $E \times E$ which is extended to a function defined on $E_\partial \times E_\partial$ so that $F(x, \partial) = F(\partial, x) = F(x, x)$ for $x \in E_\partial$ (actually there is no need to define the value $F(\partial, y)$ for $y \in E$). Then $A_t^F := \sum_{0<s\le t} F(X_{s-}, X_s)$ (whenever it is summable) is an additive functional of $\mathbf{X}$. It is natural to consider the following generalized non-local Feynman-Kac transforms by the additive functionals $A := N^u + A^\mu + A^F$ of the form

$$e_A(t) := \exp(A_t), \quad t \in [0, \zeta[. \tag{1}$$

We define $Q_t f(x) := \mathbf{E}_x[e_A(t) f(X_t)]$ for any Borel function $f$. Let $\mathcal{Q}$ be the quadratic form defined by

$$\mathcal{Q}(f, g) := \mathcal{E}(f, g) + \mathcal{E}(u, fg) - \mathcal{H}(f, g), \tag{2}$$

where

$$\mathcal{E}(u, fg) := \frac{1}{2}\int_E f \, d\mu_{\langle u,g\rangle} + \frac{1}{2}\int_E g \, d\mu_{\langle u,f\rangle},$$

$$\mathcal{H}(f, g) := \int_E f(x)g(x)\mu(dx) + \int_E \int_E f(x)g(y)(e^{F(x,y)} - 1)N(x, dy)\mu_H(dx).$$

Here $(N,H)$ be a Lévy system of $\mathbf{X}$. Then $(\mathcal{Q},\mathcal{F})$ is lower bounded on $L^2(E;\mathfrak{m})$ and associated to $(Q_t)_{t>0}$ under the condition $(\mathbf{A})$:

$(\mathbf{A})$ $\mu^+ + N(e^{F^+}-1)\mu_H \in S^1_{EK}(\mathbf{X})$, $\quad \mu_{\langle u\rangle}\in S^1_K(\mathbf{X})$ and $\mu^- + N(F^-)\mu_H \in S^1_D(\mathbf{X})$.

One can define the $L^p$-spectral radius $\lambda_p(\mathbf{X},u,\mu,F)\in[-\infty,+\infty]$ by

$$\lambda_p(\mathbf{X},u,\mu,F) = -\downarrow\lim_{t\to\infty}\frac{1}{t}\log\|Q_t\|_{p,p} = -\inf_{t>0}\frac{1}{t}\log\|Q_t\|_{p,p}. \tag{3}$$

Using the symmetry of $\{Q_t; t\geq 0\}$ and interpolation, it is easy to deduce (cf. [1, (4.2)]) that

$$\|Q_t\|_{2,2}\leq\|Q_t\|_{p,p}\leq\|Q_t\|_{\infty,\infty}\quad\text{for all } p\in[1,+\infty]$$

and therefore

$$\lambda_2(\mathbf{X},u,\mu,F)\geq\lambda_p(\mathbf{X},u,\mu,F)\geq\lambda_\infty(\mathbf{X},u,\mu,F)\quad\text{for all } p\in[1,+\infty]. \tag{4}$$

Thus to establish the $L^p$-independence of spectral radius, it suffices to show $\lambda_2(\mathbf{X},u,\mu,F)\leq\lambda_\infty(\mathbf{X},u,\mu,F)$. For $\alpha>0$, denote by $\mathbf{X}^{(\alpha)}$ the $\alpha$-subprocess of $\mathbf{X}$. Let $S^1_{EK}(\mathbf{X})$ (resp. $S^1_K(\mathbf{X})$, $S^1_{LK}(\mathbf{X})$) denote the class of smooth measures in the strict sense of extended Kato class (resp. Kato class, local Kato class) with respect to $\mathbf{X}$. Let $S^1_{NK_\infty}(\mathbf{X})$ (resp. $S^1_{NK_1}(\mathbf{X})$) be the family of natural Green-tight measures of Kato class (resp. natural semi-Green-tight measures of extended Kato class) with respect to $\mathbf{X}$ and $S^1_{D_0}(\mathbf{X})$ the family of Green-bounded smooth measures with respect to $\mathbf{X}$.

## 2. Main Theorems

Our main results are the following:

**Theorem 2.1.** *Suppose that* $\mathfrak{m}(E)<\infty$ *and*

(5) *there is a* $t_0>0$ *so that* $P_{t_0}$ *is a bounded operator from* $L^2(E;\mathfrak{m})$ *to* $L^\infty(E;\mathfrak{m})$.

*If* $\mu^+ + N(e^{2F^+}-1)\mu_H\in S^1_{EK}(\mathbf{X})$, *then* $\lambda_p(\mathbf{X},u,\mu,F)$ *is independent of* $p\in[1,\infty]$.

**Theorem 2.2.** *Suppose that* $\mu^+ + N(e^{F^+}-1)\mu_H\in\cap_{\alpha>0}S^1_{NK_1}(\mathbf{X}^{(\alpha)})$ *and* $\mu_{\langle u\rangle}\in S^1_{NK_\infty}(\mathbf{X}^{(1)})$. *Then the following holds.*

1. $\lambda_\infty(\mathbf{X},u,\mu,F)\geq\min\{\lambda_2(\mathbf{X},u,\mu,F),0\}$. *Consequently,* $\lambda_p(\mathbf{X},u,\mu,F)$ *is independent of* $p\in[1,\infty]$ *provided* $\lambda_2(\mathbf{X},u,\mu,F)\leq 0$.
2. *Assume that* $\mathbf{X}$ *is conservative. Suppose one of the following holds:*
   (i) $\mathbf{X}$ *is transient and* $\mu^- + N(F^-)\mu_H\in S^1_{D_0}(\mathbf{X})$. *Assume one of the following:*
      (a) $u^- := \max\{-u,0\}\in L^p(E;\mathfrak{m})$ *for some* $p\in[1,+\infty]$.
      (b) $\mu_{\langle u\rangle}\in S^1_{D_0}(\mathbf{X})$ *and* $\mathfrak{m}(E)<\infty$.
      (c) $\mu_{\langle u\rangle}(E)<\infty$.
   (ii) $u\in\dot{\mathcal{F}}_{\rm loc}$ *is a bounded function and* $\mu^- + N(F^-)\mu_H\in S^1_{NK_\infty}(\mathbf{X}^{(1)})$.
   *Then* $\lambda_\infty(\mathbf{X},u,\mu,F)=0$ *if* $\lambda_2(\mathbf{X},u,\mu,F)>0$. *Hence* $\lambda_p(\mathbf{X},u,\mu,F)$ *is independent of* $p\in[1,\infty]$ *if and only if* $\lambda_2(\mathbf{X},u,\mu,F)\leq 0$.

**Theorem 2.3.** *Assume* $\mathfrak{m}\in S^1_{NK_\infty}(\mathbf{X}^{(1)})$. *Then* $\lambda_p(\mathbf{X},u,\mu,F)$ *is independent of* $p\in[1,\infty]$.

These three theorems extend the previous known results by the first author [1, 2], the second and third authors [4, 5], Takeda [7, 8, 9, 10], and Tawara [11].

One of the significant progress is to remove the irreducibility condition for $(\mathcal{E}, \mathcal{F})$. The irreducibility condition of the given Dirichlet form $(\mathcal{E}, \mathcal{F})$ has been assumed to apply the Donsker-Varadhan type large deviation principle, or to apply the analytic characterization for the gaugeability of (generalized) Feynman-Kac functionals for the $L^p$-independence of the spectral radius of (generalized) Feynman-Kac semigroup. Based on the irreducible decomposition for Markov processes (in the previous talk), we can remove the irreducibility condition by utilizing the Terkelsen's Minimax Principle (see [12]).

Secondly, we remove the boundedness condition for the function $u$ appeared in the generalized Feynman-Kac semigroup $(Q_t)_{t>0}$. The boundedness of $u$ has been needed to apply the Chen-Zhang type Girsanov transform in order to reduce the case for $u = 0$. Applying Terkelsen's Minimax Principle again, we can prove Theorems 2.1 and 2.2 by making an exhaustion of increasing finely open nearly Borel sets on which the function $u$ can be regarded to be bounded.

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