博　士　論　文

L^p-independence of Growth Bounds of Generalized Feynman-Kac Semigroups

一般化された Feynman-Kac 半群の増大度の L^p 独立性

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Chapter 1

Introduction

The Gärtner-Ellis theorem is a useful theorem for the proof of the large deviation principle. To employ the Gärtner-Ellis theorem, we need to prove the existence and the differentiability of logarithmic moment generating functions. For the existence of the logarithmic moment generating function of an additive functional with bounded variation, it is enough to show the L^p-independence of growth bounds of the associated Feynman-Kac semigroup. We thus consider the L^p-independence of growth bounds of Feynman-Kac semigroups. Our main objective is to extend results in [39], [42] and [45] to more general Feynman-Kac semigroups by applying the Donsker-Varadhan type large deviation theorem.

Let X be a locally compact separable metric space and m a positive Radon measure on X with full support. Let $\mathbb{M} = (X_t, \mathbb{P}_x)$ be a conservative m-symmetric Hunt process on X and make some assumptions on $\mathbb{M}$ (Assumptions (I)–(IV) in Section 2.1). We denote by $(N(x, dy), H_t)$ the Lévy system of $\mathbb{M}$ (see (2.3) below). Let μ be a signed smooth Radon measure on X in the class $\mathcal{K}_\infty$ (Definition 2.1) and F a symmetric function on $X \times X$ in the class $\mathcal{J}_\infty$ (Definition 2.2). We define an additive functional $A_t(\mu + F)$ by

$$A_t(\mu + F) = A_t(\mu) + A_t(F) = A_t(\mu) + \sum_{0<s\leq t} F(X_{s-}, X_s).$$

Here, $A_t(\mu)$ is the continuous additive functional with the Revuz correspondence to μ (see (2.2) below). The second term $A_t(F)$ is the pure-jump additive functional which varies when the Hunt process X_t jumps on the support of F. The additive functional $A_t(\mu+F)$ is quite general. In fact, it is known from a result due to Motoo that if an additive functional is purely discontinuous and quasi-left-continuous, then it is of form $A_t(F)$ (Watanabe [47]). We formally define a Schrödinger type operator by

$$\mathcal{H}^{\mu+F}f = \mathcal{L}f + \mu f + \mu_H \mathbf{F} f, \quad \mu_H \mathbf{F} f = \left(\int_X \left(e^{F(x,y)} - 1 \right) f(y) N(x, dy) \right) \mu_H(dx),$$

where $\mathcal{L}$ is the generator of $\mathbb{M}$ and μ_H is the measure in the Revuz correspondence to the positive continuous additive functional H_t. We see that the semigroup $p_t^{\mu+F}$ generated by $\mathcal{H}^{\mu+F}$, $p_t^{\mu+F} = \exp(t\mathcal{H}^{\mu+F})$, is expressed by the generalized Feynman-Kac semigroup,

$$p_t^{\mu+F} f(x) = \mathbb{E}_x \left[\exp(A_t(\mu + F)) f(X_t) \right].$$

We define the L^p-growth bound of $\{p_t^{\mu+F}\}_{t>0}$ by

$$\lambda_p(\mu + F) = -\lim_{t\to\infty} \frac{1}{t} \log \|p_t^{\mu+F}\|_{p,p} \quad 1 \le p \le \infty,$$

where $\|\cdot\|_{p,p}$ is the operator norm from $L^p(X;m)$ to $L^p(X;m)$. The L^p-independence is defined by

$$\lambda_p(\mu + F) = \lambda_2(\mu + F), \quad 1 \le \forall p \le \infty.$$

Now we can state the main theorem in this thesis as follows:

Theorem 1.1. *Suppose that a Hunt process* $\mathbb{M}$ *satisfies Assumptions* (I)–(IV) *and that for a measure* μ *and a function* F *belong to the class* $\mathcal{K}_\infty$ *and the class* $\mathcal{J}_\infty$ *respectively. Then,* $\lambda_p(\mu + F)$ *is independent of* p *if and only if* $\lambda_2(\mu + F) \le 0$.

Theorem 1.1 says that the L^p-independence is completely determined by the L^2-growth bound.

Takeda [39] proved this theorem for Feynman-Kac semigroups with local potential $A_t(\mu)$. We in [42] extended it to non-local Schrödinger operators whose principal part is the fractional Laplacian, $(1/2)(-\Delta)^{\alpha/2}$, and we in [45] further extended it to more general operators whose principal parts are generators of symmetric Hunt processes. In those papers, the transience of $\mathbb{M}$ was assumed; however, in this thesis, we deal with recurrent Hunt processes as well as transient ones.

Simon [33] first proved the L^p-independence for classical Schrödinger operators $(1/2)\Delta - V$ on $\mathbb{R}^d$. Sturm [35, 36] extended it to Schrödinger operators on Riemannian manifolds. For the proof of the L^p-independence, they used the heat kernel estimates of Schrödinger operators. Our method in this thesis is completely different from their methods in [33], [35] and [36]. The approach in this thesis is similar to that in [39], [42] and [45]; we will use arguments in Donsker-Varadhan's large deviation theory. Donsker and Varadhan [16, 18] proved the large deviation principle for the occupation time. Kim [26] extended the lower bound estimate to symmetric Markov processes with Feynman-Kac functionals $\exp(A_t(\mu + F))$ (Theorem 3.10). However, he proved the upper bound estimate only for each compact set of the space of probability measures. Hence we apply his theorem to an extended Markov process. More precisely, we consider the Markov process on the one-point compactification X_∞ by making the adjoined point ∞ a trap, and use Kim's upper bound estimate for this extended Markov process. As a result, the rate function, say $\bar{I}_{\mu+F}$, is different from the original one. Indeed, $\bar{I}_{\mu+F}$ is a function on the space of probability measures on X_∞ not on X. For the proof of the main theorem, it is necessary to prove that the infimum of the extended I-function is equal to the infimum of the original one. To this end, we show that $\bar{I}_{\mu+F}(\delta_\infty) = 0$, that is, the contribution of adjoined point ∞ is null. For the proof of this fact, some properties of the Feynman-Kac semigroup $\{p_t^{\mu+F}\}_{t>0}$ are necessary. In particular, the invariance of $C_u(X)$, $p_t^{\mu+F}(C_u(X)) \subset C_u(X)$, is crucial, where $C_u(X)$ is the space of uniformly continuous bounded functions on X such that $\lim_{x\to\infty} f(x)$ exists. In addition, we use the fact that the Feynman-Kac semigroup $\{p_t^{\mu+F}\}_{t>0}$ possesses the *doubly Feller property*, that is, the strong Feller property, $p_t^{\mu+F}(\mathcal{B}_b(X)) \subset C_b(X)$, and the invariance of C_∞, $p_t^{\mu+F}(C_\infty(X)) \subset C_\infty(X)$. Here $C_\infty(X)$ is the space of continuous functions on X vanishing at infinity. Chung in [13] introduced the notion of the doubly Feller property and proved the stability of this property under transforms by multiplicative functionals. Applying his result, we show the doubly Feller property of $\{p_t^{\mu+F}\}_{t>0}$. For the proof of the invariance of $C_u(X)$, we find several properties equivalent to the invariance of $C_\infty(X)$ (Proposition 2.1). It should be emphasized that Proposition 2.1 is an extension of a relevant results due to Azencott [4], where he treated diffusion processes. Moreover, the method for showing the properties of Feynman-Kac semigroups is more general than that in [42]. Indeed, we in [42] used the heat kernel estimate for the symmetric Lévy processes on $\mathbb{R}^d$, due to Bass and Levin [6]; if the Lévy measure of symmetric Lévy process process is equivalent to that of the symmetric α-stable process, then the Lévy process has a continuous heat kernel equivalent to the heat kernel of the symmetric α-stable process. This method is not applicable for general Hunt processes treated in this thesis because we do not know the heat kernel estimates of them.

In order to illustrate the power of our main theorem, we will consider, in Chapter 5, some examples of symmetric Markov processes: one-dimensional diffusion processes, time changed diffusion processes, symmetric α-stable processes, Brownian motions on hyperbolic spaces and "α-stable processes" on hyperbolic spaces.

For one-dimensional diffusion processes on an interval of $\mathbb{R}$, Takeda [41] proved that if no boundaries of the interval are natural, then the L^p-independence of $\lambda_p(\mu)$ holds and that if one of the boundaries of the interval is natural, then $\lambda_p(\mu)$ is independent of p if and only if $\lambda_2(\mu) \le 0$. In addition, Ogura, Tomisaki and Kaneko [25] obtained a necessary and sufficient condition for $\lambda_2(0) > 0$. Combining these results, we have a necessary and sufficient condition for the L^p-independence in terms of speed measures and scale functions. We next consider the diffusion process with the generator $(1/2)|x|^\alpha\Delta$. This process is also probabilistically constructed by time change transform. Employing the result for one-dimensional diffusion processes above, we show that the L^p-independence of this process holds if and only if $\alpha \neq 2$. For the α-stable process on $\mathbb{R}^d$, we proved that for any $\mu \in \mathcal{K}_\infty$ and any $F \in \mathcal{J}_\infty$, the Feynman-Kac semigroup $\{p_t^{\mu+F}\}_{t>0}$ has L^p-independent growth bounds.

Let Δ be the Laplace-Beltrami operator on the hyperbolic space. It is well-known that the spectral bounds of Laplace-Beltrami operator on the hyperbolic space is equal to $(d-1)^2/8$ (e.g. Davies [14]). Thus the growth bounds of the Brownian semigroup is L^p-dependent if $d \ge 2$. However, applying Lemma C.4, we construct a positive measure $\mu \in \mathcal{K}_\infty$ such that the growth bounds of $(1/2)\Delta - \mu$ is L^p-independent. Owing to results in McGillivray [29] and Ôkura [30], we can prove that Assumptions (I)–(IV) are preserved under a certain subordination. In particular, the main theorem is applicable for the α-stable process generated by the fractional Laplace-Beltrami operator $(1/2)(-\Delta)^{\alpha/2}$, because it is constructed by the subordination of the Brownian motion. The L^2-spectral bound of the α-stable process is equal to $(d-1)^\alpha/2^{1+\alpha}$ by the spectral theorem, and consequently the L^p-independence does not hold. Nonetheless, we can construct a non-local potential

$F \in \mathcal{J}_\infty$ such that $\lambda_2(F) \leq 0$ (Lemma 5.7 and Lemma 5.8). We thus conclude that the growth bounds of $-(1/2)(-\Delta)^{\alpha/2} - \mu_V \mathbf{F}$ is L^p-independent, where μ_V is the Riemannian volume.

As mentioned above the L^p-independence implies the existence of logarithmic moment generating functions of additive functionals. For symmetric α-stable processes, the differentiability of moment generating function was proved in [44]. Combining this result with the main theorem, we can derive the large deviation principle of discontinuous additive functionals. In Appendix A, we make a comment on this topic. We will in Appendix B give an application of the identification between I-functions and Schrödinger forms; we prove the existence of ground state and establish the full large deviation principle for normalized symmetric Markov processes. Theorem B.6 concerns large deviations from not invariant measures but ground states of Schrödinger operators. In Appendix C, we make comments on time change transform which are used to make some examples in Section 5.4.

We close the introduction with some words on notation. For a topological space X, we use $\mathcal{B}(X)$ to denote the set of all Borel sets (or functions) on X. If $\mathcal{C} \subset \mathcal{B}(X)$, then $\mathcal{C}_b$ (resp. $\mathcal{C}_+$) denotes the set of bounded (resp. non-negative) functions in $\mathcal{C}$. For a set $A \subset X$, we denote by 1_A the indicator function of A, by A^c the complement of A and by A^o the interior of A. For functions f and g, notation "$f \sim g$" means that there exist constants $c_2 > c_1 > 0$ such that $c_1 g \leq f \leq c_2 g$. We use $c, C, ...$, etc as positive constants which may be different at different occurrences.

Chapter 2

Preliminaries and Notations

In this chapter, we first review the general theory of Dirichlet forms, symmetric Hunt processes and Feynman-Kac semigroups.

2.1 Dirichlet Forms and Symmetric Hunt Processes

Let X be a locally compact separable metric space and X_∞ the one-point compactification of X with adjoined point ∞. Let m be a positive Radon measure on X with full support. Let $\mathbb{M} = (\Omega, \mathcal{M}, \mathcal{M}_t, \theta_t, X_t, \mathbb{P}_x, \zeta)$ be an m-symmetric, Hunt process on X. Here, $\{\mathcal{M}_t\}$ is the minimal (augmented) admissible filtration, θ_t, $t \geq 0$ is the shift operator satisfying $X_s(\theta_t) = X_{s+t}$ identically for $s, t \geq 0$ and ζ is the lifetime of $\mathbb{M}$, $\zeta = \inf\{t > 0 : X_t = \infty\}$. Let us denote by $\{p_t\}_{t>0}$ the semigroup of $\mathbb{M}$, $p_t f(x) = \mathbb{E}_x[f(X_t)]$.

Assumption 1. We impose the following conditions on the semigroup $\{p_t\}_{t>0}$ of the Hunt process $\mathbb{M}$:

(I) (Irreducibility) If a Borel set A is p_t-invariant, that is, for any $f \in L^2(X; m) \cap \mathcal{B}_b(X)$ and any $t > 0$, $p_t(1_A f)(x) = 1_A(x) p_t f(x)$ m-a.e. x, then A satisfies either $m(A) = 0$ or $m(X \setminus A) = 0$.

(II) (Conservativeness) $p_t 1 = 1$.

(III) (Strong Feller Property) $p_t(\mathcal{B}_b(X)) \subset C_b(X)$.

(IV) (Invariance of $C_\infty(X)$) $p_t(C_\infty(X)) \subset C_\infty(X)$. Here, $C_\infty(X)$ is the space of continuous functions on X vanishing at the infinity.

Remark 2.1. We see from Assumption (III) that the semigroup $\{p_t\}_{t>0}$ admits an integral kernel $\{p(t,x,y)\}_{t>0}$ with respect to the measure m.

Let $\{G_\beta\}_{\beta>0}$ be the resolvent kernel defined by

$$G_\beta(x,y) = \int_0^\infty e^{-\beta t} p(t,x,y)dt, \quad \beta > 0.$$

If the Hunt process $\mathbb{M}$ is transient, then $G_0(x,y) < \infty \; x \neq y$. In this case, we simply write $G(x,y)$ for $G_0(x,y)$ and call it the *Green function*.

By the right continuity of sample paths of $\mathbb{M}$, $\{p_t\}_{t>0}$ can be extended to an $L^2(X;m)$-strongly continuous contraction semigroup, $\{T_t\}_{t>0}$ ([20, Lemma 1.4.3]). Then the *Dirichlet form* $(\mathcal{E},\mathcal{F})$ on $L^2(X;m)$ generated by $\mathbb{M}$ is defined by

$$\begin{cases} \mathcal{F} = \left\{ u \in L^2(X;m) : \lim_{t\to 0} \frac{1}{t}(u - T_t u, u)_m < \infty \right\}, \\ \mathcal{E}(u,v) = \lim_{t\to 0} \frac{1}{t}(u - T_t u, v)_m, \quad u,v \in \mathcal{F}, \end{cases} \tag{2.1}$$

where $(u,v)_m = \int_X u(x)v(x)m(dx)$ is the inner product on $L^2(X;m)$. The Dirichlet form $(\mathcal{E},\mathcal{F})$ is said to be *regular* if there exists a set $\mathcal{C} \subset \mathcal{F} \cap C_0(X)$ such that $\mathcal{C}$ is dense in $\mathcal{F}$ with respect to $\mathcal{E}_1$-norm and dense in $C_0(X)$ with respect to uniform norm. Here, $C_0(X)$ is the space of continuous functions on X with compact support and $\mathcal{E}_1(u,u) = \mathcal{E}(u,u) + (u,u)_m$.

We define the *(1-)capacity* Cap associated with the Dirichlet form $(\mathcal{E},\mathcal{F})$ as follows: for an open set $O \subset X$,

$$\mathrm{Cap}(O) = \inf\{\mathcal{E}_1(u,u) : u \in \mathcal{F}, u \geq 1,\ m\text{-a.e. on } O\}$$

and for a Borel set $A \subset X$,

$$\mathrm{Cap}(A) = \inf\{\mathrm{Cap}(O) : O \text{ is open, } O \supset A\}.$$

Let A be a subset of X. A statement depending on $x \in A$ is said to hold q.e. on A if there exists a set $N \subset A$ of zero capacity such that the statement is true for every $x \in A \setminus N$. "q.e." is an abbreviation of "quasi-everywhere". A real valued function u defined q.e. on X is said to be *quasi continuous* if for any $\epsilon > 0$ there exists an open set $G \subset X$ such that $\mathrm{Cap}(G) < \epsilon$ and $u|_{X \setminus G}$ is finite and continuous. Here, $u|_{X \setminus G}$ denotes the restriction of u to $X \setminus G$. It follows from Assumption (IV) that $(\mathcal{E}, \mathcal{F})$ is regular (Barlow, Bass, Kumagai and Teplyaev [5, Lemma 2.8]). Thus each function u in $\mathcal{F}$ admits a quasi-continuous version $\tilde{u}$, that is, $u = \tilde{u}$ m-a.e. In the sequel, we always assume that every function $u \in \mathcal{F}$ is represented by its quasi-continuous version.

We call a Borel measure μ on X *smooth* if it satisfies the following conditions:

(i) $\mathrm{Cap}(A) = 0$ implies $\mu(A) = 0$ for all $A \in \mathcal{B}(X)$.

(ii) there exists an increasing sequence $\{F_n\}$ of closed sets such that $\mu(F_n) < \infty$ for all n and $\lim_{n \to \infty} \mathrm{Cap}(K \setminus F_n) = 0$ for any compact set K.

Let us call a family of extended real valued function $\{A_t\}_{t \geq 0}$ on Ω an *additive functional* (AF in abbreviation) if the following conditions hold:

(i) $A_t(\cdot)$ is $\mathcal{M}_t$-measurable for all $t \geq 0$,

(ii) there exists a set $\Lambda \in \mathcal{M}_\infty = \sigma\left(\cup_{t \geq 0} \mathcal{M}_t\right)$ such that $\mathbb{P}_x(\Lambda) = 1$, for all $x \in X$, $\theta_t \Lambda \subset \Lambda$ for all $t > 0$, and for each $\omega \in \Lambda$, $A_\cdot(\omega)$ is a function satisfying: $A_0 = 0$, $A_t(\omega) < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_\zeta(\omega)$ for $t \geq 0$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$ for $s, t \geq 0$.

If an AF $\{A_t\}_{t \geq 0}$ is positive and continuous with respect to t for each $\omega \in \Lambda$, the AF is called a *positive continuous additive functional* (PCAF in abbreviation).

By [20, Theorem 5.1.4], there exists a one-to-one correspondence between smooth measures and PCAFs as follows: for each smooth measure μ, there exists a unique PCAF $\{A_t\}_{t \geq 0}$ such that for any $f \in \mathcal{B}_+(X)$ and γ-excessive function h ($\gamma \geq 0$), that is, $e^{-\gamma t} p_t h \leq h$,

$$\lim_{t \to 0} \frac{1}{t} \mathbb{E}_{h \cdot m}\left[\int_0^t f(X_s)\, dA_s\right] = \int_X f(x) h(x) \mu(dx). \tag{2.2}$$

Here, $\mathbb{E}_{h \cdot m}[f(X_t)] = \int_X \mathbb{E}_x[f(X_t)] h(x) m(dx)$. The equation (2.2) is called the *Revuz correspondence*. We denote by $A_t(\mu)$ the PCAF of the smooth measure μ. For a signed smooth measure $\mu = \mu^+ - \mu^-$, we define $A_t(\mu) = A_t(\mu^+) - A_t(\mu^-)$.

Let N be a kernel on $(X_\infty, \mathcal{B}(X_\infty))$ such that $N(x, \{x\}) = 0$ for any $x \in X$ and H_t a PCAF of $\mathbb{M}$. The pair (N, H_t) is said to be the *Lévy system* of $\mathbb{M}$ if for any non-negative $(X_\infty \times X_\infty)$-measurable function F vanishing on the diagonal set $\triangle = \{(x,x) : x \in X_\infty\}$, it holds that

$$\mathbb{E}_x \left[\sum_{0<s\le t} F(X_{s-}, X_s) \right] = \mathbb{E}_x \left[\int_0^t \int_{X_\infty} F(X_s, y) N(X_s, dy) dH_s \right], \tag{2.3}$$

where $X_{t-} = \lim_{s \uparrow t} X_s$. For the existence of the Lévy system, see Benveniste and Jacod [7]. We remark that

$$A_t(F) - \int_0^t \int_{X_\infty} F(X_s, y) N(X_s, dy) dH_s$$

is a martingale additive functional. Moreover, for any additive functional $A_t(\mu + F) = A_t(\mu) + A_t(F)$, there exists a unique continuous additive functional $A_t^p(\mu + F)$ such that $A_t(\mu+F) - A_t^p(\mu+F)$ is a martingale (see Rogers and Williams [31, Chapter VI]). We call $A_t^p(\mu + F)$ the *dual predictable projection* of $A_t(\mu + F)$. According to (2.3), we see that

$$A_t^p(\mu + F) = A_t(\mu) + \int_0^t \int_{X_\infty} F(X_s, y) N(X_s, y) dH_s.$$

The regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ is expressed by

$$\begin{aligned}\mathcal{E}(u,v) =& \mathcal{E}^{(c)}(u,v) \\ &+ \int_{X \times X \setminus \triangle} (u(x) - u(y))(v(x) - v(y)) J(dx, dy) + \int_X u(x) v(x) k(dx)\end{aligned}$$

(Beurling-Deny formula ([20, Theorem 3.2.1])). The first term $\mathcal{E}^{(c)}$ is called *local part* of $(\mathcal{E}, \mathcal{F})$. $\mathcal{E}^{(c)}$ is a symmetric Dirichlet form satisfying the *strong local property*, that is, $\mathcal{E}^{(c)}(u,v) = 0$ for $u, v \in \mathcal{F} \cap C_0(X)$ such that u is constant on $\mathrm{Supp}[v]$. In addition, there exists uniquely a positive Radon measure $\mu_{\langle u \rangle}$, $u \in \mathcal{F}$, satisfying

$$\mathcal{E}^{(c)}(u,u) = \frac{1}{2} \mu_{\langle u \rangle}(X).$$

If we introduce a bounded signed measure $\mu_{\langle u,v \rangle}$, $u, v \in \mathcal{F}$, by

$$\mu_{\langle u,v \rangle} = \frac{1}{2} \left(\mu_{\langle u+v \rangle} - \mu_{\langle u \rangle} - \mu_{\langle v \rangle} \right),$$

then,

$$\mathcal{E}^{(c)}(u,v) = \frac{1}{2}\mu_{\langle u,v\rangle}(X).$$

The second term is called the *jumping part* of $(\mathcal{E},\mathcal{F})$. J is a symmetric positive Radon measure on the product space $X \times X$ off the diagonal set $\triangle$ and called *jumping measure*. Using the Lévy system of $\mathbb{M}$, we have the following expression:

$$J(dx,dy) = \frac{1}{2}N(x,dy)\mu_H(dx),$$

where μ_H is the Revuz measure of the PCAF $\{H_t\}_{t\geq 0}$ of the Lévy system. The last term is called the *killing part*. k is a Radon measure on X and called *killing measure*. Moreover, it is expressed as

$$k(dx) = N(x,\infty)\mu_H(dx).$$

Remark 2.2. If the Hunt process $\mathbb{M}$ is conservative, then the Dirichlet form has no killing part. Thus we may replace X_∞ by X in the definition of Lévy system and the corresponding Dirichlet form has following expression:

$$\mathcal{E}(u,u) = \frac{1}{2}\mu_{\langle u\rangle}(X) + \frac{1}{2}\int_{X\times X}(u(x)-u(y))^2 N(x,dy)\mu_H(dx).$$

2.2 Generalized Feynman-Kac Semigroups

In this section, we introduce classes of local and non-local potentials (Definitions 2.1 and 2.2) which play a crucial role in this thesis. We also consider properties of Feynman-Kac semigroups associated with these potentials.

Definition 2.1 (Kato measure and Green tight measure)**.** Suppose that μ is a signed smooth measure associated with the positive continuous additive functional $A_t(\mu)$.

(i) A smooth measure μ is said to be the *Kato measure* (in notation, $\mu \in \mathcal{K}$) if

$$\lim_{t\to 0}\sup_{x\in X}\mathbb{E}_x[A_t(|\mu|)] = 0.$$

(ii) A measure $\mu \in \mathcal{K}$ is said to be the *β-Green tight measure* (in notation, $\mu \in \mathcal{K}_{\infty,\beta}$) if for any $\epsilon > 0$ there exist a compact subset K and a positive constant $\delta > 0$ such that

$$\sup_{x\in X}\int_{K^c} G_\beta(x,y)|\mu|(dy) \leq \epsilon,$$

and for any Borel set $B \subset K$ with $|\mu|(B) < \delta$,

$$\sup_{x \in X} \int_B G_\beta(x,y)|\mu|(dy) < \epsilon.$$

For a positive measure μ on X, denote

$$G_\beta \mu(x) = \int_X G_\beta(x,y)\mu(dy).$$

We note that for any $\beta > 0$, $\mathcal{K}_{\infty,\beta} = \mathcal{K}_{\infty,1}$. Indeed, for a positive measure μ on X, let $\mu_{K^c}(\cdot) = \mu(K^c \cap \cdot)$. Since by the resolvent equation

$$G_\beta \mu_{K^c} = G_\gamma \mu_{K^c} + (\gamma - \beta) G_\beta G_\gamma \mu_{K^c}, \quad 0 < \beta < \gamma,$$

we have

$$\|G_\beta \mu_{K^c}\|_\infty \leq \|G_\gamma \mu_{K^c}\|_\infty + \frac{\gamma - \beta}{\beta} \|G_\gamma \mu_{K^c}\|_\infty = \frac{\gamma}{\beta} \|G_\gamma \mu_{K^c}\|_\infty.$$

We simply write $\mathcal{K}_\infty$ for $\mathcal{K}_{\infty,1}$ and call a measure in $\mathcal{K}_\infty$ a *Green tight measure*. Moreover, if the Hunt process is transient, a measure $\mu \in \mathcal{K}_{\infty,0}$ is called a *Green tight measure in the strict sense*. We remark that $\mathcal{K}_{\infty,0} \subset \mathcal{K}_\infty \subset \mathcal{K}$.

Definition 2.2 (Class $\mathcal{J}$, $\mathcal{J}_\infty$ and $\mathcal{J}_{\infty,0}$)**.** Let F be a bounded measurable function on $X \times X$ vanishing on the diagonal set. We say that F belongs to the class $\mathcal{J}$ (resp. $\mathcal{J}_\infty$, $\mathcal{J}_{\infty,0}$) if

$$\mu_F(dx) = \left(\int_X F(x,y) N(x,dy) \right) \mu_H(dx) \in \mathcal{K} \text{ (resp. } \mathcal{K}_\infty, \mathcal{K}_{\infty,0}).$$

In the remainder of this thesis, we assume that F is symmetric, $F(x,y) = F(y,x)$. We write $\mu + F \in \mathcal{K}_\infty + \mathcal{J}_\infty$ if $\mu \in \mathcal{K}_\infty$ and $F \in \mathcal{J}_\infty$. For $\mu + F \in \mathcal{K}_\infty + \mathcal{J}_\infty$, we define the symmetric Dirichlet form $(\mathcal{E}_F, \mathcal{F})$ by

$$\begin{aligned}\mathcal{E}_F(u,u) =& \mathcal{E}^{(c)}(u,u) \\ &+ \frac{1}{2} \int_{X \times X} (u(x) - u(y))^2 e^{F(x,y)} N(x,dy) \mu_H(dx) + \int_X u(x)^2 k(dx).\end{aligned}$$

We set

$$F_1 = e^F - 1.$$

We easily see that the function F_1 also belongs to the class $\mathcal{J}_\infty$, and define another bilinear form $\mathcal{E}^{\mu+F}$ by

$$\begin{aligned}\mathcal{E}^{\mu+F}(u,u) &= \mathcal{E}_F(u,u) - \left(\int_X u^2 d\mu + \int_X u^2 d\mu_{F_1}\right) \\ &= \mathcal{E}(u,u) - \left(\int_X u^2 d\mu \right. \\ &\quad \left. + \int_{X\times X} u(x)u(y)F_1(x,y)N(x,dy)\mu_H(dx)\right), \quad u \in \mathcal{F}.\end{aligned}$$

We see by Albeverio and Ma [2, Theorem 4.1] and [3, Proposition 3.3] that $(\mathcal{E}^{\mu+F}, \mathcal{F})$ is a lower semi-bounded closed symmetric form. Denote by $\mathcal{L}^F$ the self-adjoint operator associated with $(\mathcal{E}_F, \mathcal{F})$ and $\mathcal{H}^{\mu+F}$ the self-adjoint operator associated with $(\mathcal{E}^{\mu+F}, \mathcal{F})$. Then $\mathcal{L}^F$ and $\mathcal{H}^{\mu+F}$ are formally written by

$$\mathcal{L}^F f = \mathcal{L}f + \left(\int_X (f(y) - f(x))F_1(x,y)N(x,dy)\right)\mu_H(dx)$$

and

$$\mathcal{H}^{\mu+F} f = \mathcal{L}f + \mu_H \mathbf{F} f + \mu f = \mathcal{L}^F f + \mu_H V^F f + \mu f,$$

where

$$\begin{aligned}\mu_H \mathbf{F} f &= \left(\int_X f(y)F_1(x,y)N(x,dy)\right)\mu_H(dx), \\ \mu_H V^F f &= \left(\int_X F_1(x,y)N(x,dy)\right) f(x)\mu_H(dx).\end{aligned}$$

Let $\{p_t^{\mu+F}\}_{t>0}$ be the L^2-semigroup generated by $\mathcal{H}^{\mu+F}$: $p_t^{\mu+F} = \exp(t\mathcal{H}^{\mu+F})$. Then the semigroup $\{p_t^{\mu+F}\}_{t>0}$ is expressed by

$$p_t^{\mu+F} f(x) = \mathbb{E}_x[\exp(A_t(\mu+F))f(X_t)],$$

where $A_t(\mu+F) = A_t(\mu) + \sum_{0<s\le t} F(X_{s-}, X_s)$. In fact, for $\mu + F \in \mathcal{K}_\infty + \mathcal{J}_\infty$, define a local martingale $M_t = A_t(\mu+F_1) - A_t^p(\mu+F_1)$ where $A_t^p(\mu+F_1)$ is the dual predictable projection of $A_t(\mu+F_1)$,

$$A_t^p(\mu+F_1) = A_t(\mu) + \int_0^t \left(\int_X F_1(X_s,y)N(X_s,dy)\right) dH_s.$$

Then the Doléans-Dade exponential M_t^F of M_t, a unique solution of the stochastic differential equation $Z_t = 1 + \int_0^t Z_{s-} dM_s$, is given by

$$M_t^F = \prod_{0<s\leq t} (1 + \Delta M_s) \exp(-\Delta M_s), \ \Delta M_s = M_s - M_{s-}$$

(cf. He, Wang and Yan [23, Theorem 9.39]). Noting that

$$\Delta M_{s-} = A_s(F_1) - A_{s-}(F_1) = F_1(X_{s-}, X_s),$$

we have

$$\begin{aligned} M_t^F &= \exp(A_t(F_1) - A_t^p(F) + A_t(F) - A_t(F_1)) \\ &= \exp(A_t(F) - A_t^p(F_1)). \end{aligned} \tag{2.4}$$

The semigroup

$$T_t^F f(x) = \mathbb{E}_x \left[M_t^F f(X_t) \right]$$

is identical to the one generated by $(\mathcal{E}_F, \mathcal{F})$ (Chen and Song [12, Theorem 4.8]). Let $(X_t, \mathbb{P}_x^M)$ be the transformed process of $\mathbb{M}$ by M_t^F: $\mathbb{P}_x^M(d\omega) = M_t^F \cdot \mathbb{P}_x(d\omega)$. We then see from (2.4) that the transformed semigroup by the non-local Feynman-Kac functional $\exp(A_t(\mu + F))$ is identical to the transformed semigroup of $\mathbb{P}_x^M$ by the local Feynman-Kac functional $\exp(A_t(\mu) + A_t^p(F_1))$:

$$\begin{aligned} p_t^{\mu+F} f(x) &= \mathbb{E}_x[\exp(A_t(\mu + F)) f(X_t)] \\ &= \mathbb{E}_x[\exp(A_t(F) - A_t^p(F_1) + A_t^p(F_1) + A_t(\mu)) f(X_t)] \\ &= \mathbb{E}_x^M[\exp(A_t(\mu) + A_t^p(F_1)) f(X_t)]. \end{aligned} \tag{2.5}$$

The next proposition is an extension of Proposition 3.1 in Azencott [4]. We state the proposition in a complete way, while we only use a part of the statement.

Proposition 2.1. *Let $\mathbb{M}$ be a Hunt process that possesses Assumptions* (I) *and* (III). *Then the following statements are equivalent:*

(A) $\mathbb{M}$ *possesses Assumption* (IV), *that is, for $t > 0$ and $f \in C_\infty(X)$,*

$$\lim_{x\to\infty} p_t f(x) = 0.$$

(B) *For $\beta > 0$ and $f \in C_\infty(X)$,*

$$\lim_{x\to\infty} G_\beta f(x) = 0.$$

(C) *For $t > 0$ and a compact set K,*

$$\lim_{x\to\infty} P_x(\sigma_K \le t) = 0.$$

(D) *For $\beta > 0$ and a compact set K,*

$$\lim_{x\to\infty} \mathbb{E}_x[e^{-\beta\sigma_K}] = 0.$$

Proof. (A) $\Rightarrow$ (B): Let f be a strictly positive function in $C_\infty(X)$. By Assumptions (I) and (IV), $G_\beta f$ is a strictly positive continuous function in $C_\infty(X)$.

(B) $\Rightarrow$ (C): Put $c = \inf_{x\in K} G_\beta f(x) > 0$. Since for $\beta > 0$,

$$\mathbb{P}_x[\sigma_K \le t] \le e^{\beta t}\mathbb{E}_x\left[e^{-\beta\sigma_k}\right] \le \frac{e^{\beta t}}{c}\mathbb{E}_x\left[e^{-\beta\sigma_K}G_\beta f(X_{\sigma_K})\right]$$

and

$$\begin{aligned}\mathbb{E}_x\left[e^{-\beta\sigma_K}G_\beta f(X_{\sigma_K})\right] &= \mathbb{E}_x\left[e^{-\beta\sigma_K}\mathbb{E}_{X_{\sigma_K}}\left[\int_0^\infty e^{-\beta t}f(X_t)dt\right]\right]\\ &\le \mathbb{E}_x\left[\int_{\sigma_K}^\infty e^{-\beta t}f(X_t)dt\right] \le G_\beta f(x),\end{aligned}$$

we have the implication.

(C)$\Rightarrow$(A): Let f be a non-negative function in $C_\infty(X)$. By Assumption (III), we have only to show that $\lim_{x\to\infty} p_t f(x) = 0$. For any $\epsilon > 0$, there exists a compact set K such that $f(x) < \epsilon$ for all $x \notin K$. Then $f(X_t) < \|f\|_\infty 1_{\{\sigma_K\le t\}} + \epsilon 1_{\{\sigma_K>t\}} \le \|f\|_\infty 1_{\{\sigma_K\le t\}} + \epsilon$. Thus,

$$p_t f(x) = \mathbb{E}_x[f(X_t)] < \|f\|_\infty \mathbb{P}_x(\sigma_K \le t) + \epsilon.$$

(C) $\Rightarrow$ (D): By the property (C), for any $\beta > 0$ and compact set K,

$$\begin{aligned}\lim_{x\to\infty}\mathbb{E}_x[e^{-\beta\sigma_K}] &= \lim_{x\to\infty}\mathbb{E}_x[e^{-\beta\sigma_K};\sigma_K \le t] + \lim_{x\to\infty}\mathbb{E}_x[e^{-\beta\sigma_K};\sigma_K > t]\\ &\le \lim_{x\to\infty}\mathbb{P}_x(\sigma_K \le t) + e^{-\beta t}\lim_{x\to\infty}\mathbb{P}_x(\sigma_K > t)\\ &\le e^{-\beta t}.\end{aligned}$$

By letting $t \to \infty$, we have the desired claim.

(D) $\Rightarrow$ (C): This implication from that

$$\mathbb{E}_x[e^{-\beta\sigma_K}] \ge \mathbb{E}_x[e^{-\beta\sigma_K}1_{\{\sigma_K\le t\}}] \ge e^{-\beta t}\mathbb{P}_x(\sigma_K \le t).$$

$\square$

We will show some properties of the generalized Feynman-Kac semigroup $\{p_t^{\mu+F}\}_{t>0}$, $\mu + F \in \mathcal{K}_\infty + \mathcal{J}_\infty$. Let A be a Borel set and σ_A the *first hitting time* of A, $\sigma_A = \inf\{t > 0 : X_t \in A\}$. To know properties of the generalized Feynman-Kac semigroup $\{p_t^{\mu+F}\}_{t>0}$, we need next two important theorems.

Theorem 2.2 (Chung [13, Theorem 2])**.** *Suppose that the Markov semigroup $\{p_t\}_{t>0}$ possesses doubly Feller property (Assumptions* (III) *and* (IV)*). Assume that $A_t(\mu + F)$ satisfies the following conditions:*

(a) *For some $t > 0$,*

$$\sup_{x\in X} \sup_{0\le s\le t} \mathbb{E}_x[\exp(A_t(\mu + F))] < \infty;$$

(b) *for each $t > 0$, there exists a number $\alpha > 1$ such that*

$$\sup_{x\in X} \mathbb{E}_x[\exp(\alpha A_t(\mu + F))] < \infty;$$

(c) *for each compact subset $K \subset X$, we have*

$$\lim_{t\to 0} \sup_{x\in K} \mathbb{E}_x[| \exp(A_t(\mu + F)) - 1|] = 0.$$

Then, $p_t^{\mu+F}(C_\infty(X)) \subset C_\infty(X)$ and $p_t^{\mu+F}(\mathcal{B}_b(X)) \subset C_b(X)$.

Theorem 2.3 (Generalized Khas'minskii's Lemma [48, Lemma 2.1 (a)])**.** *Let $\mu+F \in \mathcal{K}_\infty + \mathcal{J}_\infty$. If $\sup_{x\in X} \mathbb{E}_x[A_t(\mu + F)] = \lambda < 1$, then for all $x \in X$,*

$$\mathbb{E}_x\left[e^{A_t(\mu)} \prod_{0<s\le t} (1 + \Delta A_s(F))\right] \le \frac{1}{1-\lambda}, \quad \Delta A_s(F) = A_s(F) - A_{s-}(F).$$

The following theorem is crucial to study the rate function defend in Chapter 3.

Theorem 2.4. *Let $\mu + F \in \mathcal{K}_\infty + \mathcal{J}_\infty$. Then the following assertions hold.*

(i) *There exist constants c and $\kappa(\mu + F)$ such that*

$$\|p_t^{\mu+F}\|_{p,p} \le ce^{\kappa(\mu+F)t}, \quad 1 \le \forall p \le \infty,\ t > 0.$$

Here, $\|\cdot\|_{p,p}$ means the operator norm from $L^p(X; m)$ to $L^p(X; m)$;

(ii) *$\{p_t^{\mu+F}\}_{t>0}$ is a strongly continuous symmetric semigroup on $L^2(X; m)$ and the closed form corresponding to $\{p_t^{\mu+F}\}_{t>0}$ is identical to $(\mathcal{E}^{\mu+F}, \mathcal{F})$;*

(iii) $p_t^{\mu+F}(\mathcal{B}_b(X)) \subset C_b(X)$;

(iv) $p_t^{\mu+F}(C_\infty(X)) \subset C_\infty(X)$;

(v) $p_t^{\mu+F}(C_u(X)) \subset C_u(X)$ *and* $\lim_{x\to\infty} p_t^{\mu+F} f(x) = \lim_{x\to\infty} f(x)$, *where* $C_u(X)$ *is the space of uniformly continuous bounded functions on* X *such that* $\lim_{x\to\infty} f(x)$ *exists.*

Proof. The statements (i) and (ii) follow from results in Albeverio, Blanchard and Ma [1, Theorem 4.1]. The statements (iii) and (iv), that is, the strong Feller property and the invariance of $C_\infty(X)$ of $p_t^{\mu+F}$ follow from Theorem 2.2. In fact, since

$$\mathbb{E}_x\left[A_t(\mu+F)\right] = \mathbb{E}_x\left[A_t(\mu) + \int_0^t \left(\int_X F(X_s,y)N(X_s,dy)\right) dH_s\right],$$

$\lim_{t\to 0}\sup_{x\in X}\mathbb{E}_x\left[A_t(|\mu|+|F|)\right] = 0$ by the definitions of $\mathcal{K}_\infty$, $\mathcal{J}_\infty$. We have

$$\begin{aligned}\mathbb{E}_x\left[\exp(A_t(\mu+F))\right] &= \mathbb{E}_x\left[\exp\left(A_t(\mu) + \sum_{0<s\le t} F(X_{s-},X_s)\right)\right] \\ &= \mathbb{E}_x\left[\exp(A_t(\mu)) \prod_{0<s\le t}\exp(1+F_1(X_{s-},X_s))\right].\end{aligned} \tag{2.6}$$

Furthermore, the Stieltjes exponential of $A_t(\mu+F_1)$ is equal to

$$\exp(A_t(\mu)) \prod_{0<s\le t}\exp(1+F_1(X_{s-},X_s))$$

(cf. Sharpe [32, Section 71] and Ying [48]). Theorem 2.3 says that the right hand side of (2.6) is less than or equal to $(1-\sup_{x\in X}\mathbb{E}_x[A_t(\mu+F_1)])^{-1}$. Thus, the functional $\exp(A_t(\mu+F))$ satisfies conditions (a)-(c) in Theorem 2.2. Hence we show (iii) and (iv).

(v): Since $f(x)-f(\infty) \in C_\infty(X)$ and $p_t^{\mu+F}f(x) = p_t^{\mu+F}(f(x)-f(\infty)) + f(\infty)p_t^{\mu+F}1(x)$, it is enough to prove that

$$\lim_{x\to\infty} p_t^{\mu+F}1(x) = \lim_{x\to\infty}\mathbb{E}_x\left[\exp(A_t(\mu+F))\right] = 1.$$

For a non-negative $\mu \in \mathcal{K}_\infty$ a non-negative $F \in \mathcal{J}_\infty$ and $B \in \mathcal{B}(X)$, let $\mu_B(dx) = 1_B(x)\mu(dx)$, $F_B(x,y) = 1_B(x)F(x,y)$, and $A_t((\mu+F)_B) = A_t(\mu_B) + A_t(F_B)$. We then have for a compact set $K \subset X$,

$$\begin{aligned}\mathbb{E}_x\left[\exp(A_t((\mu+F)_K))\right] &= \mathbb{E}_x\left[\exp(A_t((\mu+F)_K)); \sigma'_K > t\right] \\ &\quad + \mathbb{E}_x\left[\exp(A_t((\mu+F)_K)); \sigma'_K \le t\right] \\ &= \mathbb{P}_x(\sigma'_K > t) + \mathbb{E}_x\left[\exp(A_t((\mu+F)_K)); \sigma'_K \le t\right].\end{aligned}$$

Here, $\sigma'_K = \inf\{t > 0 : X_{t-} \in K\}$. By Theorem A.2.3 in [20] and Proposition 2.1, $\lim_{x\to\infty} \mathbb{P}_x(\sigma'_K > t) \ge \lim_{x\to\infty} \mathbb{P}_x(\sigma_K > t) = 1$. Since

$$\mathbb{E}_x[\exp(A_t((\mu+F)_K)); \sigma'_K \le t] \le \mathbb{E}_x[\exp(A_t(2(\mu+F)_K))]^{1/2}\mathbb{P}_x(\sigma'_K \le t)^{1/2},$$

we see

$$\lim_{x\to\infty} \mathbb{E}_x\left[\exp(A_t((\mu+F)_K))\right] = 1.$$

Moreover, using Theorem 2.3 again, we have

$$\begin{aligned}\sup_{x\in X} \mathbb{E}_x[\exp(A_t((\mu+F)_{K^c}))] &= \sup_{x\in X} \mathbb{E}_x\left[\exp A_t(\mu_{K^c}) \prod_{0<s\le t} (1 + A_s(F_{1,K^c}))\right] \\ &\le \frac{1}{1 - \sup_{x\in X} \mathbb{E}_x[A_t((\mu+F_1)_{K^c})]}.\end{aligned}$$

By the definition of $\mathcal{K}_\infty$ and $\mathcal{J}_\infty$, for any $\epsilon > 0$ there exists a compact set K such that

$$\begin{aligned}\sup_{x\in X} \mathbb{E}_x\left[A_t((\mu+F_1)_{K^c})\right] &\le e^t \sup_{x\in X} \int_{K^c} G_1(x,y)(\mu + \mu_{F_1})(dy) \\ &\le \epsilon.\end{aligned}$$

We then have

$$\begin{aligned}&\lim_{x\to\infty} \sup_{x\in X} \mathbb{E}_x[\exp(A_t(\mu+F))] \\ &\qquad \le \lim_{x\to\infty} \left(\mathbb{E}_x[\exp(2A_t((\mu+F)_K))]^{1/2} \cdot \mathbb{E}_x[\exp(2A_t((\mu+F)_{K^c}))]^{1/2}\right) \\ &\qquad \le \lim_{x\to\infty} \left(\mathbb{E}_x[\exp(A_t((2\mu+2F)_K))]^{1/2} \cdot \mathbb{E}_x[\exp(A_t((2\mu+2F)_{K^c}))]^{1/2}\right) \\ &\qquad \le 1.\end{aligned}$$

On the other hand,

$$\begin{aligned}\liminf_{x\to\infty} \mathbb{E}_x[\exp(A_t(\mu+F))] &\ge \liminf_{x\to\infty} \mathbb{E}_x[\exp(-A_t((\mu+F)^-)] \\ &\ge \left\{\limsup_{x\to\infty} \mathbb{E}_x[\exp(A_t((\mu+F)^-)]\right\}^{-1} \\ &\ge 1.\end{aligned}$$

Therefore, we can conclude that for any $\mu + F \in \mathcal{K}_\infty + \mathcal{J}_\infty$,

$$\lim_{x \to \infty} \mathbb{E}_x \left[\exp(A_t(\mu + F)) \right] = 1.$$

□

Chapter 3

Donsker-Varadhan Type Large Deviation Principle

In this chapter, we consider the asymptotic properties for generalized Feynman-Kac semigroups. Although, we only use the upper bound estimate (Theorem 3.11) to prove the main theorem, we give a proof of the lower bound estimate (Theorem 3.10) for completeness.

Let $\{R_\alpha^{\mu+F}\}_{\alpha>\kappa(\mu+F)}$ be the resolvent of the generalized Schrödinger operator $\mathcal{H}^{\mu+F}$, that is, for $f \in \mathcal{B}_b(X)$,

$$\begin{aligned} R_\alpha^{\mu+F} f(x) &= \int_0^\infty e^{-\alpha t} p_t^{\mu+F} f(x) dt \\ &= \mathbb{E}_x \left[\int_0^\infty \exp(-\alpha t + A_t(\mu+F)) f(X_t) dt \right]. \end{aligned}$$

Here, $\kappa(\mu+F)$ is the constant in Theorem 2.4 (i). Set

$$\mathcal{D}_+^2(\mathcal{H}^{\mu+F}) = \{\phi = R_\alpha^{\mu+F} g : \alpha > \kappa(\mu+F), g \in L^2(X;m) \cap C_b(X), \text{ with } g \geq 0\}.$$

For $\phi = R_\alpha^{\mu+F} g \in \mathcal{D}_+^2(\mathcal{H}^{\mu+F})$, let

$$\mathcal{H}^{\mu+F}\phi = \alpha\phi - g.$$

Let $\mathcal{P}(X)$ be the set of probability measures on X equipped with the weak topology. We define the *I-function* $I_{\mu+F}$ on $\mathcal{P}(X)$ by

$$I_{\mu+F}(\nu) = -\inf_{\phi \in \mathcal{D}_+^2(\mathcal{H}^{\mu+F})} \int_X \frac{\mathcal{H}^{\mu+F}\phi}{\phi} d\nu.$$

It is known in Takeda [37, Proposition 4.3] that

$$I_{\mathcal{E}^{\mu+F}}(\nu) = I_{\mu+F}(\nu) \quad \text{for } \nu \in \mathcal{P}(X)$$

(see also Proposition B.3).

For $\phi \in \mathcal{D}^2_+(\mathcal{H}^{\mu+F})$, let

$$H_t^\phi = \frac{\phi(X_t)}{\phi(X_0)} \exp\left(-\int_0^t \frac{\mathcal{H}^{\mu+F}\phi}{\phi}(X_s)ds\right)$$

and put

$$\begin{aligned} M_t^\phi = &\exp(A_t(\mu) + A_t(F))\phi(X_t) - \phi(X_0) \\ &- \int_0^t \exp(A_s(\mu) + A_s(F))\mathcal{H}^{\mu+F}\phi(X_s)ds. \end{aligned}$$

Then M_t^ϕ is a martingale with respect to $\mathbb{P}_x$. Indeed, $\mathbb{E}_x[M_s^\phi] = 0$ and by the Markov property,

$$\begin{aligned} \mathbb{E}_x[M_{s+t}^\phi \mid \mathcal{M}_s] &= M_s^\phi + \exp(A_t(\mu) + A_t(F))\mathbb{E}_{X_s}[M_t^\phi] \\ &= M_s^\phi. \end{aligned}$$

Consequently, $\phi(X_t) - \phi(X_0)$ is a semi-martingale:

$$\phi(X_t) - \phi(X_0) = M_t^{[\phi]} + N_t^{[\phi]},$$

where $M_t^{[\phi]}$ and $N_t^{[\phi]}$ are the martingale part and the bounded variation part of $\phi(X_t) - \phi(X_0)$ respectively.

Lemma 3.1 ([26, Lemma 3.1])**.** *M_t^ϕ can be written as*

$$M_t^\phi = \int_0^t e^{A_{s-}(\mu+F)}dM_s^{[\phi]} + \int_0^t e^{A_{s-}(\mu+F)}dZ_s(\phi, F_1),$$

where

$$Z_t(\phi, F_1) = \sum_{0<s\le t} \phi(X_s)F_1(X_{s-}, X_s) - \int_0^t \int_X \phi(y)F_1(X_s, y)N(X_s, dy)dH_s.$$

Proof. We apply Itô's formula ([23, Theorem 9.35]) to the semi-martingale $\phi(X_t)$, $e^{A_t(\mu+F)}$ and the function $G(x, y) = xy$. Since

$$de^{A_t(\mu+F)} = e^{A_t(\mu+F)}dA_t(\mu) + e^{A_{t-}(\mu+F)}dA_t(F_1),$$

we have

$$
\begin{aligned}
e^{A_t(\mu+F)}\phi(X_t)-\phi(X_0) &= G(e^{A_t(\mu+F)},\phi(X_t))-G(e^{A_0(\mu+F)},\phi(X_0)) \\
&= \int_0^t e^{A_{s-}(\mu+F)}d\phi(X_s)+\int_0^t \phi(X_{s-})de^{A_t(\mu+F)} \\
&= \int_0^t e^{A_{s-}(\mu+F)}dM_s^{[\phi]}+\int_0^t e^{A_{s-}(\mu+F)}dN_s^{[\phi]} \\
&\qquad +\int_0^t \phi(X_s)e^{A_s(\mu+F)}dA_s(\mu) \\
&\qquad\quad +\int_0^t \phi(X_s)e^{A_{s-}(\mu+F)}dA_s(F_1).
\end{aligned} \tag{3.1}
$$

Put $A_t(\phi,F_1)=\sum_{0<s\le t}\phi(X_s)F_1(X_{s-},X_s)$. Then the dual predictable projection $A_t^p(\phi,F_1)$ of $A_t(\phi,F_1)$ is

$$
A_t^p(\phi,F_1)=\int_0^t\int_X N(X_s,dy)\phi(y)F_1(X_s,y)dH_s,
$$

and $Z_t(\phi,F_1)=A_t(\phi,F_1)-A_t^p(\phi,F_1)$ is a martingale with respect to $\mathbb{P}_x$, $x\in X$. Then the last term of the right hand side of (3.1) equals

$$
\begin{aligned}
\int_0^t \phi(X_s)e^{A_{s-}(\mu+F)}dA_s(F_1) &= \int_0^t e^{A_{s-}(\mu+F)}dA_s(\phi,F_1) \\
&= \int_0^t e^{A_{s-}(\mu+F)}dZ_s(\phi,F_1)+\int_0^t e^{A_{s-}(\mu+F)}\int_X N(X_s,dy)\phi(y)F_1(X_s,y)dH_s.
\end{aligned}
$$

□

Define the local martingale N_t^ϕ by

$$
N_t^\phi=\int_0^t \frac{1}{\phi(X_{s-})}dM_s^{[\phi]},
$$

and the multiplicative functional L_t^ϕ by

$$
L_t^\phi=e^{A_t(\mu+F)}\frac{\phi(X_t)}{\phi(X_0)}\exp\left(-\int_0^t \frac{\mathcal{H}^{\mu+F}\phi}{\phi}(X_s)ds\right).
$$

Theorem 3.2. *L_t^ϕ satisfies*

$$
L_t^\phi=1+\int_0^t L_{s-}^\phi dN_s^\phi,
$$

that is, L_t^ϕ is identical to the Doléans-Dade exponential of N_t^ϕ.

Proof. See [20, Section 6.3]. □

Lemma 3.3. *It holds that*

$$\mathbb{E}_x\left[e^{A_t(\mu+F)}\frac{\phi(X_t)}{\phi(X_0)}\exp\left(-\int_0^t\frac{\mathcal{H}^{\mu+F}\phi}{\phi}(X_s)ds\right)\right]\leq 1.$$

Proof. L_t^ϕ is a supermartingale multiplicative functional. Indeed, let $K_n = \{x \in X : \phi(x) \geq 1/n\}$. We have

$$\begin{aligned}
&d\left(e^{A_t(\mu+F)}\phi(X_s)\exp\left(-\int_0^t\frac{\mathcal{H}^{\mu+F}\phi}{\phi}(X_u)du\right)\right)\\
&\quad=\exp\left(-\int_0^t\frac{\mathcal{H}^{\mu+F}\phi}{\phi}(X_u)du\right)\left(d(e^{A_t(\mu+F)}\phi(X_s))-e^{A_t(\mu+F)}\mathcal{H}^{\mu+F}\phi(X_s)ds\right).
\end{aligned}$$

For $n\in\mathbb{N}$,

$$L^\phi_{t\wedge\tau_n}-1=\int_0^{t\wedge\tau_n}\frac{1}{\phi(X_0)}\exp\left(-\int_0^s\frac{\mathcal{H}^{\mu+F}\phi}{\phi}(X_u)du\right)dM_s^\phi,\quad \mathbb{P}_x\text{-a.e., } x\in X,$$

where $\tau_n = \inf\{t > 0 : X_t \notin K_n^o\}$. Using Fatou's lemma and the fact that $L^\phi_{t\wedge\tau_n}$ is a martingale,

$$\mathbb{E}_x[L_t^\phi]\leq\liminf_{n\to\infty}\mathbb{E}_x[L^\phi_{t\wedge\tau_n}]=1.$$

□

We denote by $\mathbb{M}^\phi = (\Omega, X_t, \mathbb{P}^\phi_x)$ the transformed process of $\mathbb{M}$ by L_t^ϕ:

$$\mathbb{P}_x^\phi(d\omega)=L_t^\phi(\omega)\cdot\mathbb{P}_x(d\omega)$$

and by $\{p_t^\phi\}_{t>0}$ the semigroup of $\mathbb{M}^\phi$.

Lemma 3.4. $\mathbb{M}^\phi$ *is a* $\phi^2 m$*-symmetric right process on* X*.*

Proof. Since $\mathbb{M}^\phi$ is a right process (see [32, Theorem 62.19]), we have only to show the symmetry of $\mathbb{M}^\phi$. Let r_t be a reversal operator on Ω, that is,

$$r_t(\omega)(s)=\begin{cases}\omega((t-s)-) & \text{if } 0\leq s\leq t,\\ \omega(0) & \text{if } s>t.\end{cases}$$

Note that $\mathbb{M}^\phi$ is reversible under $\mathbb{P}^\phi_m$-a.e. ω. Indeed, for any $\mathcal{M}_t$-measurable function f,

$$\mathbb{E}_m[f(r_t(\cdot))]=\mathbb{E}_m[f(\cdot)].$$

We have $A_t(\mu+F)(r_t\omega) = A_t(\mu+F)(\omega)$, $\mathbb{P}_m$-a.e. because of the fact that F is symmetric on $X \times X$ and Theorem 5.1.1 in [20]. Thus we have for $f, g \in \mathcal{B}(X)$,

$$\begin{aligned}
&(p_t^{\phi} f, g)_{\phi^2 m} \\
&= \left(\mathbb{E}_{\cdot} \left[e^{(A_t(\mu+F))} \frac{\phi(X_t)}{\phi(X_0)} \exp\left(- \int_0^t \frac{\mathcal{H}^{\mu+F}\phi}{\phi}(X_s)ds \right) f(X_t) \right], g \right)_{\phi^2 m} \\
&= \mathbb{E}_m \left[e^{A_t(\mu+F)} \phi(X_t)\phi(X_0) f(X_0) g(X_t) \exp\left(- \int_0^t \frac{\mathcal{H}^{\mu+F}\phi}{\phi}(X_s)ds \right) \right] \\
&= (f, p_t^{\phi} g)_{\phi^2 m}.
\end{aligned}$$

Hence the proof is complete. □

Denote by $(\mathcal{E}^{\phi}, \mathcal{F}^{\phi})$ the Dirichlet form on $L^2(X; \phi^2 m)$ associated with $\mathbb{M}^{\phi}$.

Proposition 3.5 ([11, Theorem 2.6])**.** *Suppose that $\mu + F \in \mathcal{K}_\infty + \mathcal{J}_\infty$, then $\mathcal{F} \subset \mathcal{F}^{\phi}$ and for $u \in \mathcal{F}$,*

$$\begin{aligned}
\mathcal{E}^{\phi}(u,u) = &\frac{1}{2} \int_X \phi^2 d\mu_{\langle u \rangle} \\
&+ \frac{1}{2} \int_{X \times X \setminus \triangle} (u(x) - u(y))^2 \phi(x)\phi(y) N(x, dy) \mu_H(dx).
\end{aligned}$$

Proposition 3.6 ([26, Proposition 3.2])**.** *Let $\phi \in \mathcal{D}_+^2(\mathcal{H}^{\mu+F})$. Then, $1 \in \mathcal{F}^{\phi}$ and $\mathcal{E}^{\phi}(1,1) = 0$.*

Proposition 3.7. *Suppose that $m(X) < \infty$. Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form associated with an m-symmetric right process $\mathbb{M}$ on X. If $1 \in \mathcal{F}$ and $\mathcal{E}(1,1) = 0$, then the following assertions are equivalent:*

(i) *The semigroup $\{p_t\}_{t>0}$ is irreducible (see Assumption* (I)*).*

(ii) *If $f \in \mathcal{F}$ and $\mathcal{E}(f,f) = 0$, then f is constant m-a.e.*

(iii) *If $f \in L^2(X; m)$ and $p_t f = f$ for all $t > 0$, then f is constant m-a.e.*

Proof. (i) $\Rightarrow$ (ii): We first note that if $f \in \mathcal{F}$ and $\mathcal{E}(f,f) = 0$, then f is p_t-invariant. Indeed, by the Cauchy-Schwarz inequality for the non-negative definite symmetric form $\mathcal{E}$, we have $\mathcal{E}(f,g) = 0$ and hence $\mathcal{E}_\alpha(f,g) = \alpha(f,g)_m$ for any $g \in \mathcal{F}$ and $\alpha > 0$. Since $\mathcal{E}_\alpha(G_\alpha f, g) = (f,g)_m$, we have $\alpha G_\alpha f = f$. This implies $p_t f = f$ for all $t > 0$. By the hypothesis, for any $\lambda \in \mathbb{R}$, $f - \lambda \in \mathcal{F}$ and $\mathcal{E}(f - \lambda, f - \lambda) = 0$.

Let $\varphi^+(t) = t \vee 0$, $t \in \mathbb{R}$, then φ^+ is a normal contraction function. Thus $f_\lambda = \varphi^+ \circ (f - \lambda) \in \mathcal{F}$ and $\mathcal{E}(f_\lambda, f_\lambda) \leq \mathcal{E}(f, f) = 0$. We have that f_λ is p_t-invariant. We put $B_\lambda = \{x \in E : f_\lambda(x) = 0\}$, then $p_t(1_{B_\lambda^c} f_\lambda) = p_t(f_\lambda) = 0$ m-a.e. on B_λ. Then B_λ is a p_t-invariant set. Indeed, by the Markov property of p_t, $p_t(1_{B_\lambda^c} 1_{\{f_\lambda \geq 1/n\}}) = 0$ m-a.e. on B_λ. Letting $n \to \infty$, we have $1_{B_\lambda} p_t(1_{B_\lambda^c}) = 0$, i.e. $1_{B_\lambda} p_t(1_{B_\lambda^c} f) = 0$ for all $f \in L^2(X; m)$. Assumption (I) says that $m(B_\lambda) = 0$ or $m(B_\lambda^c) = 0$. We define $\lambda_0 = \sup\{\lambda : m(B_\lambda) = 0\}$, then we have that $m(B_\lambda) \neq 0$ for any $\lambda > \lambda_0$. This implies $m(B_\lambda^c) = 0$. That is, $m(\{f > \lambda_0\}) = 0$. On the other hand, for any $\lambda < \lambda_0$, $m(B_\lambda = 0) = 0$ i.e. $m(\{f < \lambda_0\}) = 0$. Hence we have $f = \lambda_0$ m-a.e.

(ii) $\Rightarrow$ (iii): Let f be a p_t-invariant function in $L^2(X; m)$. By the definition of the Dirichlet form (see (2.1)), we have $f \in \mathcal{F}$ and $\mathcal{E}(f, f) = 0$.

(iii) $\Rightarrow$ (i): Note that $1 \in \mathcal{F} \subset L^2(X; m)$. Since $\mathcal{E}(1, 1) = 0$,

$$0 = \mathcal{E}(u, u) = \sup_{t>0} \frac{1}{t}(1 - p_t 1, 1)_m.$$

We then have $p_t 1 = 1$. Hence for any p_t-invariant set $A \in \mathcal{B}(X)$, $p_t 1_A = 1_A p_t 1 = 1_A$. Thus $m(A) = 0$ or $m(A^c) = 0$. $\square$

Theorem 3.8 ([26, Lemma 3.1]). *Assume the hypotheses in Theorem 3.7. Then, one of the assertions in Theorem 3.7 is true if and only if $\mathbb{M}$ is ergodic, that is, if $\Lambda \in \mathcal{M}^0$ is θ_t-invariant, $(\theta_t)^{-1}(\Lambda) = \Lambda$, then $\mathbb{P}_x(\Lambda) = 0$ for all $x \in X$ or $\mathbb{P}_x(\Omega \setminus \Lambda) = 0$ for all $x \in X$, where $\mathcal{M}^0 = \sigma\{X_t : 0 \leq t < \infty\}$ for all $x \in X$.*

Theorem 3.9. *The transformed process $\mathbb{M}^\phi$ is ergodic.*

Proof. Let Λ be a θ_t-invariant set. On account of positivity of L_t^ϕ, $(\mathcal{E}^\phi, \mathcal{F}^\phi)$ is irreducible. Therefore, it follows from Propositions 3.6 and 3.7 that $\mathbb{P}^\phi_{\phi^2 m}(\Lambda) = 0$ or $\mathbb{P}^\phi_{\phi^2 m}(\Omega \setminus \Lambda) = 0$. Remark 2.1 implies that $\mathbb{M}^\phi$ also admits a transition density. Hence we have $\mathbb{P}^\phi_x(\Lambda) = 0$ or $\mathbb{P}^\phi_x(\Omega \setminus \Lambda) = 0$. $\square$

Let L_t be the *occupation distribution*, that is,

$$L_t(A) = \frac{1}{t} \int_0^t 1_A(X_s) ds, \quad t > 0, A \in \mathcal{B}(X). \tag{3.2}$$

Then $L_t \in \mathcal{P}(X)$.

Define the function $I_{\mathcal{E}^{\mu+F}}$ on $\mathcal{P}(X)$by

$$I_{\mathcal{E}^{\mu+F}}(\nu) = \begin{cases} \mathcal{E}^{\mu+F}(\sqrt{f}, \sqrt{f}) & \text{if } \nu = f \cdot dm, \\ \infty & \text{otherwise.} \end{cases}$$

We now prove the lower bound estimate.

Theorem 3.10 ([26, Theorem 4.1])**.** *For each open set $G \subset \mathcal{P}(X)$,*

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp(A_t(\mu + F)); L_t \in G\right] \geq -\inf_{\nu\in G} I_{\mu+F}(\nu). \tag{3.3}$$

Proof. Let G be an open subset of $\mathcal{P}(X)$ and $\phi = R_\alpha^{\mu+F} f$ a positive function in $L^2(X; m) \cap C_b(X)$ with $\phi^2 dm \in G$. Then it holds that

$$\begin{aligned}
\mathbb{E}_x &\left[\exp(A_t(\mu + F))\ ; L_t \in G\right] \\
&= \mathbb{E}_x \left[e^{A_t(\mu+F)} H_t^\phi \left(H_t^\phi\right)^{-1}; L_t \in G\right] \\
&= \mathbb{E}_x \left[L_t^\phi \left(H_t^\phi\right)^{-1}; L_t \in G\right] = \mathbb{E}_x^\phi \left[\left(H_t^\phi\right)^{-1}; L_t \in G\right] \\
&\geq \exp\left(t\left(\int_X \phi \mathcal{H}^{\mu+F} \phi dm - \epsilon\right)\right) \mathbb{E}_x^\phi \left[\frac{\phi(X_0)}{\phi(X_t)}; S(t, \epsilon)\right] \\
&\geq \exp\left(t\left(\int_X \phi \mathcal{H}^{\mu+F} \phi dm - \epsilon\right)\right) \frac{\phi(x)}{\|\phi\|_\infty} \mathbb{P}_x^\phi(S(t, \epsilon)),
\end{aligned}$$

where

$$\begin{aligned}
&S(t,\epsilon) \\
&= \left\{\omega \in \Omega : \left|\int_X \frac{\mathcal{H}^{\mu+F}\phi}{\phi}(x) L_t(\omega, dx) - \int_X \phi \mathcal{H}^{\mu+F} \phi dm\right| < \epsilon, L_t(\omega) \in G\right\}.
\end{aligned}$$

Let

$$\begin{aligned}
\Omega_1 &= \left\{\omega \in \Omega : \lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{\mathcal{H}^{\mu+F}\phi}{\phi}(X_s(\omega)) ds = \int_X \phi \mathcal{H}^{\mu+F} \phi dm\right\}, \\
\Omega_2 &= \left\{\omega \in \Omega : L_t(\omega) \text{ weakly converges to } \phi^2 dm \text{ as } t \to \infty\right\}.
\end{aligned}$$

We then have $\mathbb{P}_x^\phi(\Omega_i) = 1$, $\phi^2 m$-a.e. from the invariance of Ω_i $(i = 1, 2)$ and Theorem 3.9. Therefore $\lim_{t\to\infty} \mathbb{P}_x^\phi(S(t, \epsilon)) = 1$ for all $x \in X$. Consequently, we get

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp(A_t(\mu + F)); L_t \in G\right] \geq \int_X \phi \mathcal{H}^{\mu+F} \phi dm - \epsilon.$$

Since $\{\phi = R_\alpha^{\mu+F} g : g \in L^2(X; m) \cap C_b(X), \alpha > \kappa(\mu + F)\}$ is a dense subset of $\mathcal{F}$ with respect to the norm $\mathcal{E}_1^{\mu+F}$, the proof is complete. $\square$

We now prove the upper bound estimate by the same argument as in [17] and [37]. Let

$$\mathcal{D}_{++}(\mathcal{H}^{\mu+F}) = \{\phi = R_\alpha^{\mu+F} g : \alpha > \kappa(\mu+F), g \in C_u(X) \text{ with } g \geq \exists\epsilon > 0\}.$$

Theorem 3.11 (Kim [26, Remark 4.1])**.** *For each compact set $K \subset \mathcal{P}(X)$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp(A_t(\mu+F)); L_t \in K\right] \leq - \inf_{\nu\in K} I_{\mathcal{E}^{\mu+F}}(\nu).$$

Proof. Take $\phi \in \mathcal{D}_{++}(\mathcal{H}^{\mu+F})$. Then, by Lemma 3.3,

$$\mathbb{E}_x \left[e^{A_t(\mu+F)} \left(\frac{\phi(X_t)}{\phi(X_0)} \right) \exp\left(- \int_0^t \frac{\mathcal{H}^{\mu+F}\phi}{\phi}(X_s) ds \right) \right] \leq 1,$$

we have

$$\mathbb{E}_x \left[e^{A_t(\mu+F)} \exp\left(- \int_0^t \frac{\mathcal{H}^{\mu+F}\phi}{\phi}(X_s) ds \right) \right] \leq \frac{\phi(x)}{\inf_{x\in X}\phi(x)}.$$

Hence for any Borel set $C \subset \mathcal{P}(X)$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp(A_t(\mu+F)); L_t \in C\right] \leq \inf_{\phi\in\mathcal{D}_{++}(\mathcal{H}^{\mu+F})} \sup_{\nu\in C} \int_X \frac{\mathcal{H}^{\mu+F}\phi}{\phi} d\nu. \quad (3.4)$$

Let K be a compact set of $\mathcal{P}(X)$ and put

$$l = \sup_{\nu\in K} \inf_{\phi\in\mathcal{D}_{++}(\mathcal{H}^{\mu+F})} \int_X \frac{\mathcal{H}^{\mu+F}\phi}{\phi} d\nu.$$

Then for any $\nu \in K$ and any $\delta > 0$ there exists a function $\phi_\nu \in \mathcal{D}_{++}(\mathcal{H}^{\mu+F})$ such that

$$\int_X \frac{\mathcal{H}^{\mu+F}\phi_\nu}{\phi_\nu} d\nu \leq l + \delta.$$

Since the function $\mathcal{H}^{\mu+F}\phi_\nu/\phi_\nu$ is bounded and continuous on X, there exists a neighborhood $N(\nu)$ of ν such that for $\lambda \in N(\nu)$,

$$\int_X \frac{\mathcal{H}^{\mu+F}\phi_\nu}{\phi_\nu} d\lambda \leq l + 2\delta.$$

The set $\{N(\nu) : \nu \in K\}$ is an open covering of the compact set K. Hence there exists $\nu_i \in K$, $i = 1, \dots, k$ with $K \subset \cup_{i=1}^k N(\nu_i)$, and for any $1 \leq i \leq k$,

$$\sup_{\nu\in N(\nu_i)} \int_X \frac{\mathcal{H}^{\mu+F}\phi_{\nu_i}}{\phi_{\nu_i}} d\nu \leq l + 2\delta.$$

Moreover,

$$\max_{1\le i\le k} \inf_{\phi\in\mathcal{D}_{++}(\mathcal{H}^{\mu+F})} \sup_{\nu\in N(\nu_i)} \int_X \frac{\mathcal{H}^{\mu+F}\phi_{\nu_i}}{\phi_{\nu_i}} d\nu \le l + 2\delta.$$

Using (3.4), we have

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp(A_t(\mu+F)); L_t \in K\right] \\
&\le \max_{1\le i\le k} \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x[\exp(A_t(\mu+F)); L_t \in N_i] \\
&\le \max_{1\le i\le k} \inf_{\phi\in\mathcal{D}_{++}(\mathcal{H}^{\mu+F})} \sup_{\nu\in N_i} \int_X \frac{\mathcal{H}^{\mu+F}\phi}{\phi} d\nu \\
&\le l + 2\delta.
\end{aligned}$$

□

Chapter 4

L^p-independence of Growth Bounds

In this chapter, we prove the main theorem (Theorem 4.5). Let $\mathbb{M}$ be a Hunt process satisfying Assumptions (I)–(IV). First, we extended of the Hunt process $\mathbb{M}$ and the I-function.

We define the transition density $\bar{p}_t(x, dy)$ on $(X_\infty, \mathcal{B}(X_\infty))$: for $E \in \mathcal{B}(X_\infty)$,

$$\bar{p}_t(x, E) = \begin{cases} p_t(x, E \setminus \{\infty\}), & x \in X, \\ \delta_\infty(E), & x = \infty. \end{cases}$$

Let $\bar{\mathbb{M}}$ be the Markov process on X_∞ with transition probability $\bar{p}_t(x, dy)$, that is, an extension of $\mathbb{M}$ with ∞ being a trap. Furthermore, for $\mu + F \in \mathcal{K}_\infty + \mathcal{J}_\infty$, we denote the semigroup $\{\bar{p}_t^{\mu+F}\}_{t>0}$ and the resolvent $\{\bar{R}_\alpha^{\mu+F}\}_{\alpha > \kappa(\mu+F)}$ of $\bar{\mathbb{M}}$ by

$$\bar{p}_t^F f(x) = \bar{\mathbb{E}}_x[\exp(A_t(\mu + F))f(X_t)],$$
$$\bar{R}_\alpha^{\mu+F} f(x) = \int_0^\infty e^{-\alpha t} \bar{p}_t^{\mu+F} f(x) dt, \quad f \in \mathcal{B}_b(X_\infty).$$

Here, $\kappa(\mu + F)$ is the constant in Theorem 2.4 (i). Then $\bar{R}_\alpha^{\mu+F} f(x) = R_\alpha^{\mu+F} f(x)$ for $x \in X$ and $\bar{R}_\alpha^{\mu+F} f(\infty) = f(\infty)/\alpha$. Set

$$\mathcal{D}_{++}(\bar{\mathcal{H}}^{\mu+F}) = \{\phi = \bar{R}_\alpha^{\mu+F} g : \alpha > \kappa(\mu + F), g \in C(X_\infty) \text{ with } g > 0\}.$$

We see that for $\phi = \bar{R}_\alpha^{\mu+F} g \in \mathcal{D}_{++}(\bar{\mathcal{H}}^{\mu+F})$, $\lim_{x \to \infty} \phi(x) = g(\infty)/\alpha$ by Theorem 2.4 (v).

Let us define the function $\bar{I}_{\mu+F}$ on $\mathcal{P}(X_\infty)$, the set of probability measures on X_∞, by

$$\bar{I}_{\mu+F}(\nu) = - \inf_{\phi \in \mathcal{D}_{++}(\bar{\mathcal{H}}^{\mu+F})} \int_{X_\infty} \frac{\bar{\mathcal{H}}^{\mu+F} \phi}{\phi} d\nu, \quad \nu \in \mathcal{P}(X_\infty),$$

where $\bar{\mathcal{H}}^{\mu+F}\phi = \alpha \bar{R}_\alpha^{\mu+F} g - g$ for $\phi = \bar{R}_\alpha^{\mu+F} g \in \mathcal{D}_{++}(\bar{\mathcal{H}}^{\mu+F})$. We then have

$$\bar{I}_{\mu+F}(\delta_\infty) = 0, \tag{4.1}$$

because $\bar{\mathcal{H}}^{\mu+F}\phi(\infty) = \alpha\phi(\infty) - g(\infty) = g(\infty) - g(\infty) = 0$ for any $\phi \in \mathcal{D}_{++}(\bar{\mathcal{H}}^{\mu+F})$.

Note that $\mathcal{P}(X_\infty) \setminus \{\delta_\infty\}$ and $(0,1] \times \mathcal{P}(X)$ are in one-to-one correspondence through the map:

$$\nu \in \mathcal{P}(X_\infty) \setminus \{\delta_\infty\} \mapsto (\nu(X), \hat{\nu}(\cdot) = \nu(\cdot)/\nu(X)) \in (0,1] \times \mathcal{P}(X). \tag{4.2}$$

Lemma 4.1. *For* $\nu \in \mathcal{P}(X_\infty) \setminus \{\delta_\infty\}$,

$$\bar{I}_{\mu+F}(\nu) = I_{\mu+F}(\nu) = \nu(X) \cdot I_{\mathcal{E}^{\mu+F}}(\hat{\nu}).$$

Proof. For $\phi = \bar{R}_\alpha^{\mu+F} g \in \mathcal{D}_{++}(\bar{\mathcal{H}}^{\mu+F})$, $\bar{\mathcal{H}}^{\mu+F}\phi(\infty) = 0$ and $\bar{\mathcal{H}}^{\mu+F}\phi(x) = \mathcal{H}^{\mu+F}\phi(x)$ for $x \in X$. Hence for $\nu \in \mathcal{P}(X_\infty)$,

$$\begin{aligned}
\bar{I}_{\mu+F}(\nu) &= -\inf_{\phi \in \mathcal{D}_{++}(\bar{\mathcal{H}}^{\mu+F})} \int_{X_\infty} \frac{\bar{\mathcal{H}}^{\mu+F}\phi}{\phi} d\nu \\
&= -\inf_{\phi \in \mathcal{D}_{++}(\mathcal{H}^{\mu+F})} \int_X \frac{\mathcal{H}^{\mu+F}\phi}{\phi} d\nu \\
&= -\inf_{\phi \in \mathcal{D}_{++}(\mathcal{H}^{\mu+F})} \nu(X) \int_X \frac{\mathcal{H}^{\mu+F}\phi}{\phi} d\hat{\nu} \\
&= \nu(X) \cdot I_{\mathcal{E}^{\mu+F}}(\hat{\nu}).
\end{aligned}$$

□

We have the next equality through the one-to-one map (4.2).

$$\inf_{\nu \in \mathcal{P}(X_\infty)\setminus\{\delta_\infty\}} \bar{I}_{\mu+F}(\nu) = \inf_{0<\theta\le 1, \nu \in \mathcal{P}(X)} (\theta I_{\mathcal{E}^{\mu+F}}(\nu))$$

Moreover, $\bar{I}_{\mu+F}(\delta_\infty) = 0$ by Lemma 4.1. Hence we have the next corollary.

Corollary 4.2.

$$\inf_{\nu \in \mathcal{P}(X_\infty)} \bar{I}_{\mu+F}(\nu) = \inf_{0\le\theta\le 1, \nu \in \mathcal{P}(X)} (\theta I_{\mathcal{E}^{\mu+F}}(\nu)) = \inf_{0\le\theta\le 1} (\theta \inf_{\nu \in \mathcal{P}(X)} I_{\mathcal{E}^{\mu+F}}(\nu)). \tag{4.3}$$

Let us denote by $\|p_t^{\mu+F}\|_{p,p}$ the operator norm of $p_t^{\mu+F}$ from $L^p(X;m)$ to $L^p(X;m)$, and define

$$\lambda_p(\mu+F) = -\lim_{t\to\infty} \frac{1}{t} \log \|p_t^{\mu+F}\|_{p,p}, \quad 1 \le p \le \infty.$$

We then have:

Corollary 4.3. *For $\mu + F \in \mathcal{K}_\infty + \mathcal{J}_\infty$,*

$$\lambda_\infty(\mu + F) \geq \inf_{0\leq\theta\leq 1}\left(\theta \inf_{\nu\in\mathcal{P}(X)} I_{\mathcal{E}^{\mu+F}}(\nu)\right) = \inf_{0\leq\theta\leq 1}(\theta\lambda_2(\mu + F)). \tag{4.4}$$

Proof. By the positivity of $p_t^{\mu+F}$,

$$\sup_{x\in X} \mathbb{E}_x\left[\exp(A_t(\mu + F))\right] = \sup_{x\in X} p_t^{\mu+F} 1(x) = \|p_t^{\mu+F}\|_{\infty,\infty}.$$

We thus see that

$$\lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in X} \mathbb{E}_x\left[\exp(A_t(\mu + F))\right] = -\lambda_\infty(\mu + F).$$

Hence we have the first inequality in (4.4) by Theorem 3.11 and the equation (4.3).

By the spectral theorem, $\lambda_2(\mu + F)$ is identical to the bottom of the spectrum of $-\mathcal{H}^{\mu+F}$ and by the variational formula for the bottom of spectrum

$$\lambda_2(\mu + F) = \inf_{\nu\in\mathcal{P}(X)} I_{\mathcal{E}^{\mu+F}}(\nu).$$

Therefore we have the second equality in (4.4). □

If $\lambda_2(\mu + F) \leq 0$, then $\inf_{0\leq\theta\leq 1}(\theta\lambda_2(\mu + F)) = \lambda_2(\mu + F)$. Hence we have:

Corollary 4.4. *If $\lambda_2(\mu + F) \leq 0$, then*

$$\lambda_\infty(\mu + F) \geq \lambda_2(\mu + F).$$

The inequality, $\lambda_2(\mu + F) \geq \lambda_\infty(\mu + F)$, generally holds. Indeed, by Schwarz's inequality,

$$\begin{aligned} p_t^{\mu+F} f(x) &= \mathbb{E}_x[\exp(A_t(\mu + F))f(X_t)] \\ &\leq (\mathbb{E}_x[\exp(A_t(\mu + F))f^2(X_t)])^{1/2} \cdot (\mathbb{E}_x[\exp(A_t(\mu + F))])^{1/2}. \end{aligned}$$

Hence we see

$$\begin{aligned} \|p_t^{\mu+F} f\|_2^2 &\leq \|p_t^{\mu+F}(f^2)\|_1 \sup_{x\in X} \mathbb{E}_x[\exp(A_t(\mu + F))] \\ &\leq \|f\|_2^2 \sup_{x\in X} \|p_t^{\mu+F}\|_{\infty,\infty}^2. \end{aligned}$$

The last inequality follows from the fact that, by the symmetry and the positivity of $p_t^{\mu+F}$,

$$\|p_t^{\mu+F}(f^2)\|_1 = \int_X f(x)^2 (p_t^{\mu+F} 1(x)) m(dx) \leq \|f\|_2^2 \cdot \|p_t^{\mu+F}\|_{\infty,\infty}^2.$$

We then have $\|p_t^{\mu+F}\|_{2,2} \le \|p_t^{\mu+F}\|_{\infty,\infty}$. Thus,

$$\|p_t^{\mu+F}\|_{2,2} \le \|p_t^{\mu+F}\|_{p,p} \le \|p_t^{\mu+F}\|_{\infty,\infty}, \quad 1 \le \forall p \le \infty,$$

by the Riesz-Thorin interpolation theorem. Therefore, we can conclude that

$$\lambda_2(\mu + F) \le 0 \Longrightarrow \lambda_p(\mu + F) = \lambda_2(\mu + F), \quad 1 \le \forall p \le \infty.$$

We would like to remark that the inequality $\lambda_2(\mu+F) \ge \lambda_\infty(\mu+F)$ follows from Theorem 3.10.

We now state the main theorem.

Theorem 4.5. *Let $\mu+F \in \mathcal{K}_\infty+\mathcal{J}_\infty$. Then $\lambda_2(\mu+F) = \lambda_p(\mu+F)$ for all $1 \le p \le \infty$ if and only if $\lambda_2(\mu + F) \le 0$. In particular, if $\lambda_2(\mu + F) > 0$, then $\lambda_\infty(\mu + F) = 0$.*

Proof. We have already proved the "if" part. To prove that "only if" part, suppose that $\lambda_2(\mu + F) > 0$. Then

$$\lambda_\infty(\mu + F) \ge \inf_{0\le\theta\le1} \theta \inf_{\nu\in\mathcal{P}(X)} I_{\mathcal{E}^{\mu+F}}(\nu) = \inf_{0\le\theta\le1} \theta(\lambda_2(\mu + F)) = 0$$

by Corollary 4.3. Then it is enough to prove $\lambda_\infty(\mu + F) \le 0$. By Theorem 2.4 (v), $\lim_{x\to\infty} p_t^{\mu+F}1(x) = 1$, which implies that $\|p_t^{\mu+F}\|_{\infty,\infty} \ge 1$. Hence,

$$-\lambda_\infty(\mu + F) = \lim_{t\to\infty} \frac{1}{t} \log \|p_t^{\mu+F}\|_{\infty,\infty} \ge 0.$$

□

Corollary 4.6. *Suppose that the Hunt process $\mathbb{M}$ is transient. If $\lambda_2(0) = 0$, then the growth bound of the Feynman-Kac semigroup $\{p_t^{\mu+F}\}_{t>0}$ is L^p-independent for any $\mu + F \in \mathcal{K}_{\infty,0} + \mathcal{J}_{\infty,0}$.*

Proof. The boundedness of F implies that there exists a constant C' such that $\mathcal{E}_F(u, u) \le C'\mathcal{E}(u, u)$ for all $u \in \mathcal{F}$. Consequently, we have

$$\begin{aligned}\mathcal{E}^{\mu+F}(u, u) &= \mathcal{E}_F(u, u) - \left(\int_X u^2 d\mu_{F_1} + \int_X u^2 d\mu\right)\\ &\le C'\mathcal{E}(u, u) - \left(\int_X u^2 d\mu_{F_1} + \int_X u^2 d\mu\right).\end{aligned}$$

Hence to show that $\lambda_2(\mu + F) \le 0$, it is enough to prove that $\lambda_2(\mu) \le 0$ for any $\mu \in \mathcal{K}_{\infty,0}$. To this end, we have only to prove that for any positive $\mu \in \mathcal{K}_{\infty,0}$,

$$\lambda_2(\mu) = \inf\left\{\mathcal{E}(u,u) + \int_X u^2 d\mu : u \in \mathcal{F}, \|u\|_2 = 1\right\} = 0.$$

We see from [34, Theorem 3.1], for any $u \in \mathcal{F}$ such that $\|u\|_2 = 1$,

$$\int_X u^2 d\mu \le \|G\mu\|_\infty \mathcal{E}(u,u),$$

and thus

$$\lambda_2(\mu) \le \left(\mathcal{E}(u,u) + \int_X u^2 d\mu\right) \le (1 + \|G\mu\|_\infty)\,\mathcal{E}(u,u).$$

Take a minimizing sequence $\{u_n\}$ of $\mathcal{F}$, i.e., $\|u_n\|_2 = 1$ and $\lim_{n\to\infty} \mathcal{E}(u_n, u_n) = \lambda_2(0) = 0$, we have the desired claim. □

For a compact set $K \subset X$, define the subspace $\mathcal{F}_{K^c}$ of $\mathcal{F}$ by

$$\mathcal{F}_{K^c} = \{u \in \mathcal{F} : u = 0 \text{ q.e. on } K\}.$$

Then, identifying the space $L^2_{K^c}(X;m) = \{u \in L^2(X;m) : u = 0 \ m\text{-a.e. on } K\}$ with $L^2(K^c;m)$, we see that $(\mathcal{E}, \mathcal{F}_{K^c})$ is regarded as a regular Dirichlet form on $L^2(K^c;m)$. The Dirichlet form $(\mathcal{E}, \mathcal{F}_{K^c})$ is said to be *the part of the Dirichlet form* $(\mathcal{E}, \mathcal{F})$ on the open set K^c. Denote by $\mathcal{L}_K$ the self-adjoint operator associated with $(\mathcal{E}, \mathcal{F}_{K^c})$.

Remark 4.1. Let $\sigma(\mathcal{L}_K)$ be the spectrum of $\mathcal{L}_K$. Assume that for any compact set K,

$$\inf \sigma(\mathcal{L}_K) = 0. \tag{4.5}$$

Then, Corollary 4.6 holds without the transience condition. Indeed, we have for $\mu \in \mathcal{K}_\infty$,

$$\begin{aligned}\lambda_2(\mu) &= \inf\left\{\mathcal{E}(u,u) + \int_X u^2 d\mu : u \in \mathcal{F}, \|u\|_2 = 1\right\} \\ &\le \inf\left\{\mathcal{E}(u,u) + \int_X u^2 d\mu : u \in \mathcal{F}_{K^c}, \|u\|_2 = 1\right\},\end{aligned}$$

we have from the assumption, the right hand side equals

$$\inf\left\{\int_X u^2 d\mu_{K^c} : u \in \mathcal{F}_{K^c}, \|u\|_2 = 1\right\}$$

($\mu_{K^c}(\cdot) = \mu(K^c \cap \cdot)$). Since $\int_X u^2 d\mu_{K^c} \le \|G_1\mu_{K^c}\|_\infty \cdot \mathcal{E}_1(u,u)$, the right hand side tends to zero as $K \uparrow X$. We see that the assumption (4.5) is fulfilled for spatially homogeneous symmetric Lévy processes.

Let us consider a spatially homogeneous symmetric Lévy process with Lévy measure J. The Lévy measure J is said to be *exponentially localized* if there exists a positive constant δ such that

$$\int_{|x|>1} e^{\delta|x|} J(dx) < \infty. \tag{4.6}$$

For example, the Lévy measure of the *relativistic Schrödinger process*, the symmetric Lévy process generated by $\sqrt{-\Delta + m^2} - m$, $m > 0$, satisfies (4.6) (Carmona, Master and Simon [8]). Assuming that J is exponentially localized we can prove in the same way as in [37] that if $\mu + F$ belongs to the class $\mathcal{K} + \mathcal{J}$, then $\lambda_p(\mu + F)$ is independent of p. The Lévy measure of the symmetric α-stable process on $\mathbb{R}^d$ is $\left(K(d,\alpha)/|x|^{d+\alpha}\right) dx$, and is not exponentially localized. This is the reason why we need to restrict the class of potentials to $\mathcal{K}_\infty + \mathcal{J}_\infty$.

Chapter 5

Examples

In this chapter, to illustrate the power of our main theorem, we apply Theorem 4.5 to some examples of symmetric Hunt processes; for one-dimensional diffusion processes we can obtain a necessary and sufficient condition for the L^p-independence. It is known that the bottom of L^2-spectrum of the Brownian motion (or "the α-stable process") on hyperbolic space is strictly positive, and thus the growth bound depend on p; however, by adding a suitable potential to the Laplace-Beltrami operator, we can construct a Feynman-Kac semigroup satisfying the L^p-independence.

5.1 One-Dimensional Diffusion Processes

Let $s(x)$ be a strictly increasing continuous function on (r_0, r_1) and $m(x)$ a strictly increasing right-continuous function $m(x)$ on (r_0, r_1). We define

$$D_s^+ f(x) = \lim_{h\downarrow 0} \frac{f(x+h) - f(x)}{s(x+h) - s(x)}, \quad D_m f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{m(x+h) - m(x)},$$

provided the limits exist. We consider the L^p-independence of one-dimensional diffusion processes. Let $I = (r_0, r_1)$, $-\infty \le r_0 < c < r_1 \le \infty$, be an interval in $\mathbb{R}$ generated by $D_m D_s^+$.

Definition 5.1 (Feller's boundary classification (Itô and McKean [24, p.108] or Mandl [28, pp.24-25])). Let

$$\rho(x) = \int_c^x \left(\int_c^y dm(z) \right) ds(y), \quad \sigma(x) = \int_c^x \left(\int_c^y ds(z) \right) dm(y).$$

We call

(i) r_i a *regular boundary* if $\rho < \infty$, $\sigma < \infty$,

(ii) r_i an *exit boundary* if $\rho < \infty$, $\sigma = \infty$,

(iii) r_i an *entrance boundary* if $\rho = \infty$, $\sigma < \infty$,

(iv) r_i a *natural boundary* if $\rho = \infty$, $\sigma = \infty$.

As a typical example, let us consider a one-dimensional differential operator

$$A = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$$

on I. Here, $a(x)$ and $b(x)$ are strictly positive continuous functions on I and $a(x)$ is strictly positive on I. We set

$$B(x) = \int_c^x b(y)a(y)^{-1}dy,$$
$$dm(x) = a(x)^{-1}e^{B(x)}dx, \quad ds(x) = e^{-B(x)}dx,$$
$$\frac{d}{dm(x)} = D_m, \quad \frac{d}{ds(x)} = D_s.$$

Then the operator A is expressed by

$$A = a(x)e^{-B(x)}\frac{d}{dx}\left(e^{B(x)}\frac{d}{dx}\right) = D_m D_s.$$

The Dirichlet form on $L^2(I;m)$ generated by $\mathbb{M}$ is written as

$$\mathcal{E}(u,v) = -\int_{r_0}^{r_1} D_m D_s^+ u \cdot v dm = \int_{r_0}^{r_1} D_s^+ u(x) \cdot D_s^+ v(x) ds(x).$$

We denote by $\mathbb{M} = (\mathbb{P}_x, X_t)$ be the minimal diffusion process generated by $D_m D_s^+$.

Theorem 5.1 (Takeda [41, Theorem 5.1])**.** *Let $\mu \in \mathcal{K}_\infty$. If no boundaries are natural, then $\lambda_p(\mu)$ is independent of p. If one of the boundaries is natural, the diffusion process satisfies Assumptions* (I)–(IV)*, that is, $\lambda_p(\mu)$ is independent of p if and only if $\lambda_2(\mu) \leq 0$.*

We consider the diffusion process on I generated by $D_m D_s^+$. We define an increasing right-continuous function $\tilde{m}(x)$ by $\tilde{m}(x) = m(s^{-1}(x))$ and hereafter write m for $\tilde{m}$ and $s(x)$ for x.

For a fixed $c \in [r_0, r_1]$, define

$$A_0(m;c) = \sup_{x \in (r_0, c)} (x - r_0) m((x, c]),$$
$$A_1(m;c) = \sup_{x \in (c, r_1)} (r_1 - x) m((c, x]).$$

Theorem 5.2 (Kaneko, Ogura and Tomisaki [25, Theorem 2])**.** $\lambda_2(0) > 0$ *if and only if* $A_0(m;c) < \infty$ *and* $A_1(m;c) < \infty$.

On account of Theorem 4.5, Theorem 5.1 and Theorem 5.2, we have the following theorem:

Theorem 5.3. *Let* $\mathbb{M}$ *be a diffusion process on an interval* $I = (r_0, r_1)$. *Then the growth bounds of the Markov semigroup is* L^p*-independent if and only if one of the following conditions is fulfilled:*

(i) *no boundaries are natural,*

(ii) *if* r_i *is natural, then* $A_i(m;c) = \infty$.

5.2 Time Changed Diffusion Processes

Applying the results in the previous section, we prove the L^p-independence for multi-dimensional diffusion processes.

For $\alpha \geq 0$, we define the function

$$\rho_\alpha(x) = \begin{cases} 1, & |x| < 1, \\ \dfrac{1}{|x|^\alpha}, & |x| \geq 1. \end{cases}$$

Let $\mathbb{M}$ be a diffusion process on $\mathbb{R}^d$ $(d \geq 3)$ with the corresponding Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d; \rho_\alpha dx)$ defined by

$$\begin{cases} \mathcal{E}(u,v) = \dfrac{1}{2} \displaystyle\int_{\mathbb{R}^d} (\nabla u \cdot \nabla v) dx, \\ \mathcal{F} = \overline{C_0^\infty(\mathbb{R}^d)}^{\left(\mathcal{E}(\cdot,\cdot) + \|\cdot\|_{L^2(\mathbb{R}^d; \rho_\alpha dx)}\right)^{1/2}}. \end{cases}$$

Let $(B_t, \mathbb{P}_x)$ be the d-dimensional Brownian motion and define

$$A_t^\alpha = \int_0^t \rho_\alpha(B_s) ds.$$

Then the diffusion process $\mathbb{M}$ is the time changed process of the Brownian motion by the PCAF A_t^α.

Theorem 5.4. *The L^p-independence of growth bounds of the Markov semigroup of $\mathbb{M}$ holds if and only if $\alpha \neq 2$.*

This theorem tells us that for Markov semigroups, the L^p-independence holds quite generally. Making use of the rotation invariance of ρ_α, we may consider the one-dimensional diffusion process $\mathbb{M}_1$ on $[0, \infty)$ generated by

$$dm(r) = \begin{cases} 1, & (0 \leq r < 1), \\ r^{d-\alpha-1}dr, & (1 \leq r), \end{cases} \qquad ds(r) = \begin{cases} 1, & (0 \leq r < 1), \\ r^{1-d}dr, & (1 \leq r). \end{cases} \tag{5.1}$$

Thus, the corresponding Dirichlet form on $L^2([0, \infty); m)$ is

$$\mathcal{E}(u, v) = \int_0^\infty \frac{du}{ds}\frac{dv}{ds} ds \tag{5.2}$$

Theorem 5.5. *Let $\mathbb{M}_1$ be the diffusion process on $[0, \infty)$ generated by $D_m D_s^+$ defined as above. Then we have:*

(i) *0 is a regular boundary.*

(ii) *If $d < \alpha$, then ∞ is a regular boundary.*

(iii) *If $2 < \alpha \leq d$, then ∞ is an exit boundary.*

(iv) *If $\alpha \leq 2$, then ∞ is a natural boundary.*

Proof. (i): $\rho(0) < \infty$ and $\sigma(0) < \infty$ follow from the definitions of m and s.

(ii)–(iv): Because of

$$\rho(x) = \int_1^x \left(\int_1^y dm(z) \right) ds(y) \sim \int_1^x (y^{1-\alpha} - y^{1-d}) dy, \quad \text{as } x \to \infty,$$

hence we have $\rho(\infty) < \infty$ if and only if $\alpha > 2$ and $d > 2$. On the other hand,

$$\sigma(x) = \int_1^x \left(\int_1^y ds(z \right) dm(y) \sim \int_1^x (y^{1-\alpha} - y^{d-\alpha-1}) dy, \quad \text{as } x \to \infty,$$

we thus have $\sigma(\infty) < \infty$ if and only if $\alpha > 2$ and $\alpha > d$. Consequently, we have the desired claim. □

Theorem 5.5 says that no boundaries are natural if $\alpha > 2$. Moreover, Theorem 5.1 says that if $\alpha > 2$, then the L^p-independence holds and that if $\alpha \leq 2$, then $\mathbb{M}$ satisfies Assumptions (I)–(IV). By the same way as in the proof of Example 5.6 in [37], L^2-spectral bounds of the Markov semigroup of $\mathbb{M}$ is equal to zero if and only if $\alpha < 2$. Thus we have the assertion of Theorem 5.4.

5.3 α-Stable Processes on Euclidean Spaces

Let $\mathbb{M} = (X_t, \mathbb{P}_x)$ be the symmetric α-stable process on $\mathbb{R}^d$ $(0 < \alpha < 2)$, the pure jump process generated by $\frac{1}{2}(-\Delta)^{\alpha/2}$. Let $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ be the symmetric Dirichlet form generated by $\mathbb{M} = (X_t, \mathbb{P}_x)$:

$$\begin{cases} \mathcal{E}^{(\alpha)}(u,v) = \dfrac{K(d,\alpha)}{2} \displaystyle\iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \triangle} \dfrac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}} dxdy, \\ \mathcal{F}^{(\alpha)} = \left\{ u \in L^2(\mathbb{R}^d) : \displaystyle\iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \triangle} \dfrac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} dxdy < \infty \right\}, \end{cases}$$

where

$$K(d,\alpha) = \frac{\alpha \Gamma(\frac{d+\alpha}{2})}{2^{1-\alpha}\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}.$$

Theorem 5.6. *Let $\mu + F \in \mathcal{K}_\infty + \mathcal{J}_\infty$. Then*

$$\lambda_p(\mu + F) = \lambda_2(\mu + F), \quad 1 \le \forall p \le \infty.$$

Proof. Noting that the α-stable process satisfies Assumptions (I)–(IV). Thus, on account of Remark 4.1, we have this theorem. □

5.4 Brownian Motions and "α-Stable Processes" on Hyperbolic Spaces

In this section, we consider Brownian motions and "α-stable processes" on hyperbolic spaces generated by subordination procedure (Section C.2).

Let $\mathbb{H}^d$ be the hyperbolic space of dimension d $(d \ge 2)$ with volume element v, that is,

$$\begin{cases} \mathbb{H}^d = \{x = (y,z) : y \in \mathbb{R}^{d-1} \text{ and } 0 < z < \infty\}, \\ v(dx) = z^{-d-2} dydz. \end{cases}$$

The Laplace-Beltrami operator Δ is given by

$$\Delta = z^2 \left(\Delta_y + \frac{\partial^2}{\partial z^2} \right) - (d-2) z \frac{\partial}{\partial z},$$

where Δ_y denotes the Euclidean Laplacian on $\mathbb{R}^{d-1}$. Let $\mathbb{M}$ be the Brownian motion on $\mathbb{H}^d$ with the Dirichlet form $(\mathcal{E}, \mathcal{F})$:

$$\mathcal{E}(u,u) = \frac{1}{2} \int_{\mathbb{H}^d} (\nabla u, \nabla u) dv = \int_0^\infty \lambda d(E_\lambda u, u), \quad u \in \mathcal{F}, \tag{5.3}$$

where $\mathcal{F}$ is the closure of $C_0^\infty(\mathbb{H}^d)$ with respect to the norm, $\mathcal{E}_1(\cdot,\cdot)^{1/2} = (\mathcal{E}(\cdot,\cdot) + (\cdot,\cdot))^{1/2}$. Then, we can apply Theorem 4.5 to the Brownian motion. Indeed, we see from Example 3.3 in Grigor'yan [21], it is transient and satisfies Assumption (II). Assumptions (I), (III) and (IV) are also fulfilled (see [14]). In fact, an explicit expression of the corresponding transition density $p(t,x,y)$ is known (Grigor'yan and Noguchi [22, Theorem 1.1]): if $d = 2m+1$, then

$$p(t,x,y) = \frac{(-1)^m}{2^m\pi^m}\frac{1}{(4\pi t)^{1/2}}\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m e^{-m^2t-\rho^2/4t},$$

and if $d = 2m+2$, then

$$\begin{aligned} p(t,x,y) = &\frac{(-1)^{m+5/2}}{2^{m+3/2}\pi^m}t^{-3/2}e^{-(2m+1)^2t/4}\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m \\ &\times\int_r^\infty \frac{se^{-s^2/4t}}{(\cosh s - \cosh r)^{1/2}}ds, \end{aligned}$$

where $r = |x-y|$. We can check Assumptions (I), (III) and (IV) directly using this expression.

We further construct examples of Feynman-Kac semigroups with the L^p-independence.

Example 5.1. Let $\mathbb{M}$ be the Brownian motion on $\mathbb{H}^d$ whose Dirichlet form is defined in (5.3). It is known in [14, p.177] that

$$\lambda_2(0) = \inf\left\{\mathcal{E}(u,u) : u \in \mathcal{F}, \int u^2 dv = 1\right\} = \frac{1}{2}\left(\frac{(d-1)}{2}\right)^2 > 0 \qquad (5.4)$$

On the other hand, $\lambda_\infty(0) = 0$ because of the conservativeness of the Brownian motion. Hence the L^p-independence does not hold. Let μ be in $\mathcal{K}_\infty$ such that $\mu \geq 0$ and $\mu \not\equiv 0$. Lemma C.4 yields that

$$\inf\left\{\mathcal{E}^{\theta\mu}(u,u) : u \in \mathcal{F}, \int_{\mathbb{H}^d} u^2 dv = 1\right\} < 0$$

for sufficiently large θ. We can conclude that the L^p-independence holds for large θ.

Next two lemmas are used to show that it is possible to make $\lambda_2(F)$ less than or equal to zero by adding a non-local potential F. It is not trivial. Indeed, $\lambda_2(\theta F)$ does not always become small even if we take a large θ, because θ appear in two terms of the corresponding Schrödinger form:

$$\mathcal{E}^{\theta F}(u,u) = \frac{1}{2}\int_{\mathbb{H}^d\times\mathbb{H}^d}(u(x)-u(y))^2 e^{\theta F(x,y)}N(x,dy)\mu_H(dx) - \int_{\mathbb{H}^d} u(x)^2 d\mu_{\theta F_1}.$$

If θ becomes larger, then the first term becomes larger.

Lemma 5.7. *If*

$$\inf\left\{\mathcal{E}(u,u) : u \in \mathcal{F}, \int_{X\times X} u(x)u(y)F_1(x,y)N(x,dy)\mu_H(dx) = 1\right\} < 1,$$

then

$$\inf\left\{\mathcal{E}^F(u,u) : u \in \mathcal{F}, \|u\|_2 = 1\right\} < 0.$$

Proof. Take a function ϕ in $\mathcal{F}$ satisfying $\mathcal{E}(\phi,\phi) < 1$ and

$$\int_{X\times X} \phi(x)\phi(y)F_1(x,y)N(x,dy)\mu_H(dx) = 1.$$

Let $\psi = \phi/\|\phi\|_2$. Then we have

$$\begin{aligned}\mathcal{E}^F(\psi,\psi) &= \mathcal{E}(\psi,\psi) - \int_{X\times X} \psi(x)\psi(y)F_1(x,y)N(x,dy)\mu_H(dx) \\ &= \frac{1}{\|\phi\|_2^2}\left(\mathcal{E}(\phi,\phi) - \int_{X\times X} \phi(x)\phi(y)F_1(x,y)N(x,dy)\mu_H(dx)\right) < 0.\end{aligned}$$

□

Lemma 5.8. *Let $F \in \mathcal{J}_\infty$ with $F \geq 0$ and $F \not\equiv 0$, and define $F_1^\theta = e^{\theta F} - 1$. Then there exists $u \in \mathcal{F}$ such that*

$$\mathcal{E}(u,u) < 1 \text{ and } \int_{X\times X} u(x)u(y)F_1^\theta(x,y)N(x,dy)\mu_H(dx) = 1 \tag{5.5}$$

for a sufficiently large θ.

Proof. Take a non-negative function v in $\mathcal{F}$ such that

$$\int v(x)v(y)F_1(x,y)N(x,dy)\mu_H(dx) = 1.$$

Let

$$\begin{aligned}k(\theta) &= \frac{\int v(x)v(y)F_1(x,y)N(x,dy)\mu_H(dx)}{\int v(x)v(y)F_1^\theta(x,y)N(x,dy)\mu_H(dx)} \\ &= \frac{1}{\int v(x)v(y)F_1^\theta(x,y)N(x,dy)\mu_H(dx)}.\end{aligned}$$

Obviously, $k(\theta) \to 0$ as $\theta \to \infty$. Thus the function u defined by $u = \sqrt{k(\theta)}v$ satisfies (5.5) for a sufficiently large θ. □

Example 5.2. Let $\mathbb{M}^{(\alpha)}$ be a Hunt process defined by the arguments in Section C.2. Proposition C.5 enable us to we can apply Theorem 4.5 to $\mathbb{M}^{(\alpha)}$. We see from (5.4) and (C.1),

$$\inf\left\{\mathcal{E}^{(\alpha)}(u,u) : u \in \mathcal{F}^{(\alpha)}, \int u^2 dv = 1\right\} = \frac{1}{2}\left(\frac{(d-1)}{2}\right)^{\alpha}.$$

Hence the L^p-independence does not hold. Let F be in $\mathcal{J}_\infty$ such that $F \geq 0$ and $F \not\equiv 0$. Lemmas 5.7 and 5.8 yield that

$$\inf\left\{\mathcal{E}^{(\alpha),\theta F}(u,u) : u \in \mathcal{F}^{(\alpha)}, \int_{\mathbb{H}^d} u^2 dv = 1\right\} < 0$$

for sufficiently large θ. We can conclude that $\lambda_p(\theta F)$ is independent of p for large θ.

Appendix A

Large Deviation Principles for Discontinuous Additive Functionals

The symmetric α-stable process is a typical example of pure-jump processes. In this chapter, we study further some properties of the process . In the first section, we give an alternative proof of Theorem 2.4 by using these properties. As stated in Introduction, large deviations for additive functionals motivate us to show the L^p-independence of growth bounds of Feynman-Kac semigroups. In the second section, we study large deviations for purely discontinuous additive functionals of symmetric α-stable processes.

A.1 An Alternative Proof of Theorem 2.4

We give an alternative proof of Theorem 2.4 for symmetric α-stable processes. We will use the heat kernel estimates of α-stable processes due to Bass and Levin [6] and Komatsu [27]. This method enables us to reduce general non-local potentials to local potentials.

Let $\mathbb{M} = (X_t, \mathbb{P}_x)$ $(0 < \alpha < 2)$ be the symmetric α-stable process on $\mathbb{R}^d$ with the Dirichlet form $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$. We recall that the Doléans-Dade exponential M_t^F of $M_t = A_t(F_1) - A_t^p(F_1)$ is expressed by

$$\begin{aligned} M_t^F &= \exp(A_t(F_1) - A_t^p(F_1) + A_t(F) - A_t(F_1)) \\ &= \exp(A_t(F) - A_t^p(F_1)). \end{aligned}$$

We define the semigroup $T_t^F f(x) = \mathbb{E}_x[M_t^F f(X_t)]$. The corresponding symmetric Dirichlet form $\mathcal{E}_F^{(\alpha)}$ is defined as follows:

$$\mathcal{E}_F^{(\alpha)}(u,u) = \frac{K(d,\alpha)}{2} \int_{X \times X \setminus \triangle} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} e^{F(x,y)} dxdy$$

Note that the jumping measure $e^{F(x,y)}N(x,y)$ of $(\mathcal{E}_F^{(\alpha)}, \mathcal{F}^{(\alpha)})$ is equivalent with that of $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ because of the boundedness of F. We then see from [6] and [27] that T_t^F has a continuous integral kernel $T^F(t,x,y) \in C([0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d)$ with

$$C_1 \left(\frac{1}{t^{d/\alpha}} \wedge \frac{t}{|x-y|} \right) \le T^F(t,x,y) \le C_2 \left(\frac{1}{t^{d/\alpha}} \wedge \frac{t}{|x-y|} \right). \tag{A.1}$$

Theorem A.1. *Let $F \in \mathcal{J}_\infty$. Then the semigroup $\{T_t^F\}_{t>0}$ satisfies Assumptions* (I)–(IV)*.*

Proof. The estimate (A.1) imply the invariance of $C_\infty(\mathbb{R}^d)$ and the irreducibility. We show the conservativeness of $\{T_t^F\}_{t>0}$. Let $\{\delta_m\}_{m=1}^\infty$ be a sequence of non-negative functions such that $\delta_m \in C_0^\infty(\mathbb{R}^d)$, $\int \delta_m dx = 1$ and $\mathrm{Supp}[\delta_m] \subset (-m^{-1}, m^{-1})^d$ for all $m \ge 1$. We set

$$k_m(x,y) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{F(\xi,\eta)} \delta_m(x-\xi)\delta_m(y-\eta) d\xi d\eta$$

and define

$$\mathcal{E}_m(u,u) = \frac{K(d,\alpha)}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \triangle} \frac{(u(x)-u(y))^2 k_m(x,y)}{|x-y|^{d+\alpha}} dxdy, \quad u \in \mathcal{F}.$$

Let us denote by $T_m(t,x,y)$ the continuous integral kernel of the Dirichlet form $(\mathcal{E}_m, \mathcal{F})$. Then, noting that

$$\inf_{x,y} e^{F(x,y)} \le k_m(x,y) \le \sup_{x,y} e^{F(x,y)} \quad \text{for all } m,$$

we have from (A.1) that

$$C_1 \left(\frac{1}{t^{d/\alpha}} \wedge \frac{t}{|x-y|} \right) \le T_m(t,x,y) \le C_2 \left(\frac{1}{t^{d/\alpha}} \wedge \frac{t}{|x-y|} \right).$$

Moreover, we see from [27] that $\{T_m(t,x,y)\}_m$ is equi-continuous on any compact subset of $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$. By the Ascoli-Arzelà theorem, choosing a subsequence if necessary, we may suppose that $\{T_m(t,x,y)\}_m$ converges to a function $T^F(t,x,y)$

locally uniformly on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. Note that $\int_{\mathbb{R}^d} T_m(t,x,y)dy = 1$ for any $t>0$, $x\in\mathbb{R}^d$ and $m\geq 1$. Then we have $1=\lim_{m\to\infty}\int T_m(t,x,y)dy = \int T^F(t,x,y)dy$ because for any $\epsilon>0$, $t>0$ and $x\in\mathbb{R}^d$ there exists $R>0$ such that

$$\sup_m \int_{\{|y|>R\}} T_m(t,x,y)dy < \epsilon$$

by Theorem 3 in [27]. We can show the strong Feller property of $\{T_t^F\}_{t>0}$ by exactly the same way as that in Davies [14, Corollary 5.2.7]. In fact, for each $f\in\mathcal{B}_b(\mathbb{R}^d)$ and any $x_0\in\mathbb{R}^d$,

$$\liminf_{x\to x_0}\int_{\mathbb{R}^d} T^F(t,x,y)(\|f\|_\infty \pm f(y))dy \geq \|f\|_\infty \pm \int_{\mathbb{R}^d} T^F(t,x_0,y)f(y)dy$$

by Fatou's lemma and the conservativeness of $\{T_t^F\}_{t>0}$. Hence $T_t^F f$ is bounded lower and upper semicontinuous. □

The following theorem is the main part of Theorem 2.4.

Theorem A.2 ([39, Theorem 2.1])**.** *Suppose that a Hunt process satisfies Assumptions* (I)–(IV) *and a signed measure μ belongs to $\mathcal{K}_\infty$. Then the Feynman-Kac semigroup $\{p_t^\mu\}_{t>0}$ satisfies all properties in Theorem 2.4.*

We now turn to the alternative proof of Theorem 2.4.

An Alternative Proof of Theorem 2.4. Theorem A.1 says that the semigroup $\{T_t^F\}_{t>0}$ satisfies Assumptions (I)–(IV). Using the inequality (A.1), the resolvent of the Dirichlet form $(\mathcal{E}_F^{(\alpha)},\mathcal{F}^{(\alpha)})$ is also equivalent to one of the original Dirichlet form $(\mathcal{E}^{(\alpha)},\mathcal{F}^{(\alpha)})$. Hence the class $\mathcal{K}_\infty$ (resp. $\mathcal{K}_{\infty,0}$) of $(\mathcal{E}_F^{(\alpha)},\mathcal{F}^{(\alpha)})$ is the same of the original Dirichlet form. Hence, $\mu+\mu_{F_1}$ belongs to $\mathcal{K}_\infty$ of $(\mathcal{E}_F^{(\alpha)},\mathcal{F}^{(\alpha)})$ and Theorem A.2 is applicable for $\mu+\mu_{F_1}$ and $(\mathcal{E}_F,\mathcal{F})$. □

A.2 A Large Deviation Principle for Discontinuous Additive Functionals

The L^p-independence implies the existence of the logarithmic moment generating function of $A_t(\mu+F)$. Hence for the application of the Gärtner-Ellis theorem, it is necessary to show the differentiability of the logarithmic moment generating function. For symmetric α-stable processes, we know in [44] that the logarithmic moment generating function is differentiable. As a result, we can establish the large deviation principle of $A_t(F)$. In this section, we explain this topic.

We define a new class $\mathcal{A}_\infty$ of non-local potentials F.

Definition A.1. A function $F \in \mathcal{J}_\infty$ is said to be in the class $\mathcal{A}_\infty$, if for any $\epsilon > 0$ there exist a Borel set K of finite $\mu_{|F|}$-measure and a constant $\delta > 0$ such that for any measurable set $B \subset K$ with $\mu_{|F|}(B) < \delta$,

$$\sup_{(x,w)\in\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} \iint_{((K\setminus B)\times(K\setminus B))^c} \frac{G(x,y)G(z,w)}{G(x,w)} |F(y,z)| N(x,y) dz dy \le \epsilon.$$

Let F be a positive symmetric bounded function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \triangle$ in $\mathcal{A}_\infty$. We recall the symmetric closed form $(\mathcal{E}^{\theta F}, \mathcal{F}^{(\alpha)})$ by

$$\mathcal{E}^{\theta F}(u,v) = \mathcal{E}^{(\alpha)}(u,v) - \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} u(x)v(y)\left(e^{\theta F(x,y)} - 1\right) N(x,y) dx dy, \quad \theta \in \mathbb{R},$$

where

$$N(x,y) = \frac{K(d,\alpha)}{|x-y|^{d+\alpha}}.$$

Then its associated self-adjoint operator $\mathcal{H}^{\theta F}$ is formally written by

$$\mathcal{H}^{\theta F} f(x) = \frac{1}{2}(-\Delta)^{\alpha/2} f(x) - \int_{\mathbb{R}^d} f(y)\left(e^{\theta F(x,y)} - 1\right) N(x,y) dy.$$

We define the function $C(\theta)$ by

$$-C(\theta) = \lambda_2(\theta F) = \inf\left\{\mathcal{E}^{\theta F}(u,u) :\ u \in \mathcal{F}^{(\alpha)},\ \int_{\mathbb{R}^d} u^2 dx = 1\right\},$$

and let $I(\lambda)$ be the Fenchel-Legendre transform of $C(\theta)$:

$$I(\lambda) = \sup_{\theta\in\mathbb{R}}\{\lambda\theta - C(\theta)\}, \quad \lambda \in \mathbb{R}.$$

Then, the function $C(\theta)$ is a convex, non-negative and differentiable function on $\mathbb{R}$. This implies that its Fenchel-Legendre transform I is convex and *good* (e.g. [15, Lemma 2.3.9]); for every $l > 0$, the level set $\{\lambda \in \mathbb{R} :\ I(\lambda) \le l\}$ is compact. Now the main theorem in this chapter is as follows:

Theorem A.3. *Assume that $d \le 2\alpha$. Then for a positive function $F \in \mathcal{A}_\infty^+$, $A_t(F)/t$ obeys the large deviation principle with rate function $I(\lambda)$:*

(i) *For each closed set* $K \in \mathbb{R}$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x \left(\frac{A_t(F)}{t} \in K \right) \leq - \inf_{\lambda \in K} I(\lambda).$$

(ii) *For each open set* $G \subset \mathbb{R}$,

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x \left(\frac{A_t(F)}{t} \in G \right) \geq - \inf_{\lambda \in G} I(\lambda).$$

To apply the Gärtner-Ellis theorem, we will show first that the limit

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp(\theta A_t(F)) \right], \quad \theta \in \mathbb{R} \tag{A.2}$$

exists ([15, Assumption 2.3.2]). We call the limit the *logarithmic moment generating function* of $A_t(F)$. For the proof of the existence of the limit (A.2), we use Theorem 5.6; if the function F belongs to the class $\mathcal{J}_\infty$, it follows from Theorem 3.10 that

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp(\theta A_t(F)) \right] \geq -\lambda_2(\theta F).$$

Moreover, we see from the L^p-independence that

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp(\theta A_t(F)) \right] &\leq \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in\mathbb{R}^d} \mathbb{E}_x \left[\exp(\theta A_t(F)) \right] \\ &= -\lambda_\infty(\theta F) = -\lambda_2(\theta F) = C(\theta). \end{aligned}$$

We thus see that $-\lambda_2(\theta F)$ is the logarithmic moment generating function of Feynman-Kac functional $A_t(F)$. In [40], Takeda used an ergodic theorem due to Fukushima [19] to prove the existence of the limit (A.2).

The existence of the logarithmic moment generating function leads us to the upper bound (i) in Theorem A.3. To prove the lower bound (ii) in Theorem A.3 by using the Gärtner-Ellis theorem, we need to show that the function $C(\theta)$ is *essentially smooth* in the sense of [15, Definition 2.3.5]. With regard to this property, the following theorem holds.

Theorem A.4 ([44, Theorem 7.2])**.** *Let* F *be in* $\mathcal{A}_\infty^+$. *Then if* $d \leq 2\alpha$, *the spectral function* $C(\theta)$ *is differentiable on* $\mathbb{R}$.

For the proof of the differentiability, we follow the arguments in [43]; we need the criticality theory for Schrödinger operators with non-local potential. The condition, $d \leq 2\alpha$, comes from the null criticality of the Schrödinger operator $\mathcal{H}^{\theta_0 F}$, where

$$\theta_0 = \inf\{\theta > 0 : C(\theta) > 0\}.$$

More precisely, we will prove that the operator $\mathcal{H}^{\theta_0 F}$ is *critical*, that is, $\mathcal{H}^{\theta_0 F}$ does not admit the minimal positive Green function but admits a positive continuous $\mathcal{H}^{\theta_0 F}$-harmonic function (this function is called a *ground state* and uniquely determined up to constant multiplication). We note that if $d \leq \alpha$, then the symmetric α-stable process is recurrent. On account of the recurrence, we can show that θ_0 equals 0 and the ground state is the positive constant function. In addition, we will prove that if $d \leq 2\alpha$, then $\mathcal{H}^{\theta_0 F}$ is *null critical*, that is, the ground state does not belong to $L^2(\mathbb{R}^d)$. In fact, denoting by h the ground state, for $\alpha < d$ there exist positive constants c, C such that

$$\frac{c}{|x|^{d-\alpha}} \leq h(x) \leq \frac{C}{|x|^{d-\alpha}}, \quad |x| > 1.$$

The criticality of Schrödinger type operators is studied by many people (M. Murata, Y. Pinchover, R. Pinsky,...). In particular, Z.-Q. Chen [10] considered the subcriticality of Schrödinger operators with non-local potential and obtained a necessary and sufficient condition for the subcriticality: For a positive F in $\mathcal{A}_\infty$, the operator $\mathcal{H}^F$ is subcritical if and only if

$$\begin{aligned} \inf\Bigg\{\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} &(u(x)-u(y))^2 e^{F(x,y)} N(x,y)dxdy : \\ &\int_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} u^2(x)\left(e^{F(x,y)}-1\right)N(x,y)dxdy = 1\Bigg\} > 1. \end{aligned} \tag{A.3}$$

In [44], an another necessary and sufficient condition for the subcriticality as follows was established:

$$\inf\left\{\mathcal{E}^{(\alpha)}(u,u) : \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} u(x)u(y)\left(e^{F(x,y)}-1\right)N(x,y)dxdy = 1\right\} > 1. \tag{A.4}$$

We can check the condition (A.4) more easily than Chen's one, because the minimizing form does not depend on the function F. However we can study the properties of the minimizing function in (A.3) more easily than that in (A.4). In fact, we know that the minimizing function in (A.3) is characterized as the ground state of an irreducible time changed Markov generator, and thus it is strictly positive and continuous. On the other hand, we have not known the meaning of the minimizing function in (A.4). We thus use these two conditions for proof of the null criticality of $\mathcal{H}^{\theta_0 F}$.

As remarked in [15, Section 2.3], the Gärtner-Ellis theorem is restrictive for the proof of the large deviation principle. In fact, the Gärtner-Ellis theorem is not applicable to the case $d > 2\alpha$ because the function $C(\theta)$ is not differentiable at $\theta = \theta_0$ (Remark A.1), while we believe that the large deviation itself holds even for $d > 2\alpha$.

Example A.1. Let K be a subset of $\mathbb{R}^d$ with finite Lebesgue measure. Then it is known in [10, Example 2.1] that the function F belongs to $\mathcal{A}_\infty$. In particular, for compact sets K_1, K_2 with $K_1 \cap K_2 = \emptyset$, the function $F(x,y) = 1_{K_1}(x)1_{K_2}(y) + 1_{K_2}(x)1_{K_1}(y)$ is in $\mathcal{A}_\infty$.

The corresponding additive functional is

$$A_t(F) = \sum_{0<s\le t} F(X_{s-}, X_s) = \sharp\{s : 0 < s \le t, X_{s-} \in K_i, X_s \in K_j, i \neq j\},$$

that is, the additive functional denotes the number that the α-stable process jumps between K_1 and K_2 up to t.

Remark A.1. For classical Schrödinger operators, the non-differentiability of spectral function was considered in [33]. The argument in [33, Theorem 2.1] can be adapted to prove that if $d > 2\alpha$, then $C(\theta)$ is not differentiable. Indeed, the ground state h belongs to $L^2(dx)$, that is, 0 is an eigenvalue of $\mathcal{H}^{\theta_0 F}$. We normalize the function h as $\|h\|_2 = 1$. Let

$$\mathcal{B}_F(u,u) = \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} u(x)u(y)F(x,y)N(x,y)dxdy.$$

Since for $\theta > \theta_0$,

$$-C(\theta) \le \mathcal{E}^{(\alpha)}(h,h) - \mathcal{B}_{(\theta F)_1}(h,h)$$

and

$$\mathcal{E}^{(\alpha)}(h,h) = \mathcal{B}_{(\theta_0 F)_1}(h,h),$$

we have

$$C(\theta) \ge \mathcal{B}_{(\theta F)_1}(h,h) - \mathcal{B}_{(\theta_0 F)_1}(h,h) \ge (\theta - \theta_0)\mathcal{B}_F(h,h).$$

Hence we see that

$$\liminf_{\theta\downarrow\theta_0} \frac{C(\theta)}{\theta - \theta_0} = \mathcal{B}_F(h,h) > 0.$$

Appendix B

A Large Deviation Principle for Normalized Markov Processes

Varadhan [46] gave an abstract formulation for the large deviation principle. Theorem 3.10 and Theorem 3.11 are slightly different form the lower estimate and the upper estimate in his formulation; at least, the rate function $I_{\mathcal{E}^{\mu+F}}$ is not positive. Moreover, the Hunt process is not supposed to be conservative, consequently has no invariant measure. Hence these theorem does not say the large deviation from invariant measure. In this chapter, we show that under Assumptions (I), (III) and (V), we can fit these theorem to Varadhan's formulation by considering normalized Markov processes. According to this modification, our theorem says the large deviation from the ground state of the generalized Schrödinger operator.

Let $\mathbb{M} = (\Omega, X_t, \mathbb{P}_x, \zeta)$ be an m-symmetric Markov process on a locally compact separable metric space X. Note that $\mathbb{M}$ is allowed to be explosive.

Assumption 2. (V) For any $\epsilon > 0$, there exists a compact set K such that

$$\sup_{x \in X} R_1 1_{K^c}(x) \leq \epsilon.$$

We make Assumptions (I), (III) and (V). Let $\mu \in \mathcal{K}_\infty$ and $F \in \mathcal{J}_\infty$. Let $A_t(\mu+F)$ be the additive functional defined in Chapter 2. We define the function $I_{\mathcal{E}^{\mu+F}}$ on $\mathcal{P}(X)$ in the same way as in Chapter 3:

$$I_{\mathcal{E}^{\mu+F}}(\nu) = \begin{cases} \mathcal{E}^{\mu+F}(\sqrt{f}, \sqrt{f}) & \text{if } \nu = f \cdot m,\ \sqrt{f} \in \mathcal{F}, \\ \infty & \text{otherwise.} \end{cases}$$

Let us define $\kappa(\mu + F)$ by

$$\kappa(\mu + F) = \lim_{t \to \infty} \frac{1}{t} \log \|p_t^{\mu+F}\|_{\infty,\infty}.$$

We see from Theorem 2.4 that $\kappa(\mu+F)$ is finite. If $\alpha > \kappa(\mu+F)$ and $f \in \mathcal{B}_b(X)$, we define the resolvent $R_\alpha^{\mu+F}$ by

$$R_\alpha^{\mu+F} f(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\alpha t + A_t(\mu+F)} f(X_t)dt; t < \zeta\right].$$

We set

$$\mathcal{D}_+(\mathcal{H}^{\mu+F}) = \left\{R_\alpha^{\mu+F} f : \alpha > \kappa(\mu+F), f \in L^2(X;m) \cap C_b(X), f \geq 0 \text{ and } f \not\equiv 0\right\}.$$

Each function $\phi = R_\alpha^{\mu+F} f \in \mathcal{D}_+(\mathcal{H}^{\mu+F})$ is strictly positive because $\mathbb{P}_x(\sigma_O < \zeta) > 0$ for any $x \in X$ by Assumption (I). Here O is a non-empty open set $\{x \in X : f(x) > 0\}$. We define the generator $\mathcal{H}^{\mu+F}$ by

$$\mathcal{H}^{\mu+F} u = \alpha u - f, \quad u = R_\alpha^{\mu+F} f \in \mathcal{D}_+(\mathcal{H}^\mu),$$

and the function I on $\mathcal{P}(X)$ by

$$I(\nu) = -\inf_{\substack{\phi \in \mathcal{D}_+(\mathcal{H}^{\mu+F}) \\ \epsilon > 0}} \int_X \frac{\mathcal{H}^{\mu+F}\phi}{\phi + \epsilon h} d\nu. \tag{B.1}$$

Here, $h(x)$ is the *gauge function*, that is, $h(x) = \mathbb{E}_x[e^{A_\zeta(\mu+F)}]$.

The gauge function $h(x)$ satisfies $0 < c \leq h(x) \leq C < \infty$. Indeed, for $\mu \in \mathcal{K}_\infty$ and $F \in \mathcal{J}_\infty$, by Proposition 2.2 in [9] and the definition of $\mathcal{J}_\infty$, $\sup_{x\in X} \mathbb{E}_x(A_\zeta(|\mu| + |F|)) < \infty$. By Jensen's inequality,

$$\inf_{x \in X} \mathbb{E}_x(\exp(A_\zeta(\mu+F))) > 0.$$

On the other hand, By the Gauge Theorem (see [9, Theorem 2.13]), the gauge function $h(x)$ is either bounded on X or identically ∞ on X. However, if $h(x) \equiv \infty$, then we may replace the Markov process $\mathbb{M}$ by the 1-subprocess $\mathbb{M}^{(1)} = (\Omega, X_t, \mathbb{P}_x^{(1)})$ of $\mathbb{M}$, that is, $\mathbb{M}^{(1)}$ be the Hunt process transformed by e^{-t}: $\mathbb{P}_x^{(1)}(d\omega) = e^{-t}\mathbb{P}_x(d\omega)$.

We need add a positive constant ϵ because the Markov process is not supposed to be conservative.

Denote by $\mathcal{B}_b^+(X)$ the set of non-negative bounded Borel functions on X. Let us define the function I_α on $\mathcal{P}(X)$ by

$$I_\alpha(\nu) = -\inf_{\substack{u \in \mathcal{B}_b^+(X) \\ \epsilon > 0}} \int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} u + \epsilon h}{u + \epsilon h}\right) d\nu.$$

Lemma B.1. *It holds that*

$$I_\alpha(\nu) \le \frac{I(\nu)}{\alpha}, \quad \nu \in \mathcal{P}(X).$$

Proof. For $u = R_\alpha^{\mu+F} f \in \mathcal{D}_+(\mathcal{H}^{\mu+F})$ and $\epsilon > 0$, set

$$\phi(\alpha) = -\int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} u + \epsilon h}{u + \epsilon h}\right) d\nu.$$

Then, noting that $\frac{d}{d\alpha}\left(R_\alpha^{\mu+F} u\right) = -\left(R_\alpha^{\mu+F}\right)^2 u$, we have

$$\frac{d\phi}{d\alpha}(\alpha) = -\int_X \frac{R_\alpha^{\mu+F} u - \alpha\left(R_\alpha^{\mu+F}\right)^2 u}{\alpha R_\alpha^{\mu+F} u + \epsilon h} d\nu = \int_X \frac{\mathcal{H}^{\mu+F}\left(R_\alpha^{\mu+F}\right)^2 u}{\alpha R_\alpha^{\mu+F} u + \epsilon h} d\nu.$$

Since

$$\left(\alpha\left(R_\alpha^{\mu+F}\right)^2 u - R_\alpha^{\mu+F} u\right)\left(\alpha^2\left(R_\alpha^{\mu+F}\right)^2 u + \epsilon h\right) \\ - \left(\alpha\left(R_\alpha^{\mu+F}\right)^2 u - R_\alpha^{\mu+F} u\right)\left(\alpha R_\alpha^{\mu+F} u + \epsilon h\right)$$

equals $\alpha\left(\alpha\left(R_\alpha^{\mu+F}\right)^2 u - R_\alpha^{\mu+F} u\right)^2 \ge 0$, we have

$$\frac{\alpha\left(R_\alpha^{\mu+F}\right)^2 u - R_\alpha^{\mu+F} u}{\alpha R_\alpha^{\mu+F} u + \epsilon h} \ge \frac{\alpha\left(R_\alpha^{\mu+F}\right)^2 u - R_\alpha^{\mu+F} u}{\alpha^2\left(R_\alpha^{\mu+F}\right)^2 u + \epsilon h},$$

and thus

$$\begin{aligned} \int_X \frac{\mathcal{H}^{\mu+F}\left(R_\alpha^{\mu+F}\right)^2 u}{\alpha R_\alpha^{\mu+F} u + \epsilon h} d\nu &\ge \int_X \frac{\mathcal{H}^{\mu+F}\left(R_\alpha^{\mu+F}\right)^2 u}{\alpha^2\left(R_\alpha^{\mu+F}\right)^2 u + \epsilon h} d\nu \\ &= -\frac{1}{\alpha^2}\left(-\int_X \frac{\mathcal{H}^{\mu+F}\left(R_\alpha^{\mu+F}\right)^2 u}{\left(R_\alpha^{\mu+F}\right)^2 u + \frac{\epsilon h}{\alpha^2}} d\nu\right) \\ &\ge -\frac{1}{\alpha^2} I(\nu). \end{aligned}$$

Therefore

$$\phi(\infty) - \phi(\alpha) = \int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} u + \epsilon h}{u + \epsilon h}\right) d\nu \ge -\frac{I(\nu)}{\alpha},$$

which implies

$$-\inf_{\substack{u \in \mathcal{D}_+(\mathcal{H}^{\mu+F}) \\ \epsilon > 0}} \int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} u + \epsilon h}{u + \epsilon h}\right) d\nu \le \frac{I(\nu)}{\alpha}.$$

Since $\|\beta R_\beta^{\mu+F} f\|_\infty \leq C\|f\|_\infty$, $\beta > 0$, and $\beta R_\beta^{\mu+F} f(x) \to f(x)$ as $\beta \to \infty$,

$$\int_X \log\left(\frac{\alpha R_\alpha^{\mu+F}(\beta R_\beta^{\mu+F} f) + \epsilon h}{\beta R_\beta^{\mu+F} f + \epsilon h}\right) d\nu \overset{\beta\to\infty}{\longrightarrow} \int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} f + \epsilon h}{f + \epsilon h}\right) d\nu. \tag{B.2}$$

Define the measure ν_α by

$$\nu_\alpha(A) = \int_X \alpha R_\alpha^{\mu+F}(x, A) d\nu(x), \quad A \in \mathcal{B}(X).$$

Given $v \in \mathcal{B}_b^+(X)$, take a sequence $\{g_n\}_{n=1}^\infty \subset C_b^+(X) \cap L^2(X; m)$ such that

$$\int_X |v - g_n| d(\nu_\alpha + \nu) \to 0 \text{ as } n \to \infty.$$

We then have

$$\int_X |\alpha R_\alpha^{\mu+F} v - \alpha R_\alpha^{\mu+F} g_n| d\nu \leq \int_X \alpha R_\alpha^{\mu+F}(|v - g_n|) d\nu = \int_X |v - g_n| d\nu_\alpha \to 0$$

as $n \to \infty$, and so

$$\int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} g_n + \epsilon h}{g_n + \epsilon h}\right) d\nu \overset{n\to\infty}{\longrightarrow} \int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} v + \epsilon h}{v + \epsilon h}\right) d\nu. \tag{B.3}$$

Hence, combining (B.2) and (B.3)

$$\inf_{u \in \mathcal{D}_+(\mathcal{H}^{\mu+F})} \int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} u + \epsilon h}{u + \epsilon h}\right) d\nu = \inf_{u \in \mathcal{B}_b^+} \int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} u + \epsilon h}{u + \epsilon h}\right) d\nu,$$

which implies the lemma. □

Lemma B.2. *If $I(\nu) < \infty$, then ν is absolutely continuous with respect to m.*

Proof. By the similar argument in the proof of [16, Lemma 4.1], we obtain this lemma. Indeed, for $a > 0$ and $A \in \mathcal{B}(X)$, set $u(x) = a1_A(x) + 1 \in \mathcal{B}_b^+(X)$. Then

$$\int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} u + \epsilon h}{u + \epsilon h}\right) d\nu = \int_X \log\left(\frac{a\alpha R_\alpha^{\mu+F}(x, A) + \alpha R_\alpha^{\mu+F}(x, X) + \epsilon h}{a1_A(x) + 1 + \epsilon h}\right) d\nu.$$

Define the measure ν_α as in the proof of Lemma B.1. Put

$$c_\alpha = \int_X \alpha R_\alpha^{\mu+F}(x, X) d\nu(x) (= \nu_\alpha(X)).$$

We see from Lemma B.1 and Jensen's inequality that

$$\log(a\nu_\alpha(A) + c_\alpha + \epsilon h) \geq \nu(A) \log(a + 1 + \epsilon h) + \nu(A^c) \log(1 + \epsilon h) - I(\nu)/\alpha,$$

and by letting $\epsilon \to 0$

$$\log\left(a\nu_\alpha(A) + c_\alpha\right) \geq \nu(A)\log(a+1) - I(\nu)/\alpha.$$

Since $\log x \leq x - 1$ for $x > 0$, we have

$$a\nu_\alpha(A) + c_\alpha - 1 \geq \nu(A)\log(a+1) - I(\nu)/\alpha,$$

and so

$$\nu_\alpha(A) - \nu(A) \geq \frac{-I(\nu)/\alpha + \nu(A)(\log(a+1) - a) + 1 - c_\alpha}{a}.$$

Noting that $\log(a+1) - a < 0$, we have

$$\nu_\alpha(A) - \nu(A) \geq \frac{-I(\nu)/\alpha + (\log(a+1) - a) + 1 - c_\alpha}{a}$$

for all $A \in \mathcal{B}(X)$ and

$$\begin{aligned}\nu(A) - \nu_\alpha(A) &= 1 - c_\alpha + (\nu_\alpha(A^c) - \nu(A^c)) \\ &\geq \frac{-I(\nu)/\alpha + (\log(a+1) - a) + (1 - c_\alpha)(a+1)}{a}\end{aligned}$$

for all $A \in \mathcal{B}(X)$. Therefore we can conclude that

$$\sup_{A\in\mathcal{B}(X)} |\nu(A) - \nu_\alpha(A)| \leq \frac{a - \log(a+1) + I(\nu)/\alpha + (1 - c_\alpha)(a+1)}{a}.$$

Note that $c_\alpha \to 1$ as $\alpha \to \infty$. Then since

$$\limsup_{\alpha\to\infty} \sup_{A\in\mathcal{B}(X)} |\nu(A) - \nu_\alpha(A)| \leq \frac{a - \log(a+1)}{a}$$

and the right hand side converges to 0 as $a \to 0$, the lemma follows. □

Proposition B.3. *It holds that for $\nu \in \mathcal{P}(X)$,*

$$I(\nu) = I_{\mathcal{E}^{\mu+F}}(\nu).$$

Proof. We follow the argument of the proof of [16, Theorem 5]. Suppose that $I(\nu) = \ell < \infty$. By Lemma B.2, ν is absolutely continuous with respect to m. Let us denote by f its density and let $f^n = \sqrt{f} \wedge n$. Since $\log(1-x) \leq -x$ for $-\infty < x < 1$ and

$$-\infty < \frac{f^n - \alpha R_\alpha^{\mu+F} f^n}{f^n + \epsilon h} < 1,$$

$$\begin{aligned}\int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} f^n + \epsilon h}{f^n + \epsilon h}\right) f dm &= \int_X \log\left(1 - \frac{f^n - \alpha R_\alpha^{\mu+F} f^n}{f^n + \epsilon h}\right) f dm \\ &\leq -\int_X \frac{f^n - \alpha R_\alpha^{\mu+F} f^n}{f^n + \epsilon h} f dm,\end{aligned}$$

so

$$\int_X \frac{f^n - \alpha R_\alpha^{\mu+F} f^n}{f^n + \epsilon h} f dm \le I_\alpha(f \cdot m).$$

By letting $n \to \infty$ and $\epsilon \to 0$, we have

$$\int_X \sqrt{f}(\sqrt{f} - \alpha R_\alpha^{\mu+F}\sqrt{f})dm \le I_\alpha(f \cdot m) \le \frac{I(f \cdot m)}{\alpha},$$

which implies that $\sqrt{f} \in \mathcal{F}$ and $\mathcal{E}^{\mu+F}(\sqrt{f}, \sqrt{f}) \le I(f \cdot m)$.

Let $\phi \in \mathcal{D}_+(\mathcal{H}^{\mu+F})$ and define the semigroup P_t^ϕ by

$$P_t^\phi f(x) = \mathbb{E}_x \left[e^{A_t(\mu+F)} \cdot \frac{(\phi+\epsilon h)(X_t)}{(\phi+\epsilon h)(X_0)} \exp\left(-\int_0^t \frac{\mathcal{H}^{\mu+F}\phi}{\phi+\epsilon h}(X_s)ds \right) f(X_t) \right].$$

Then, P_t^ϕ is $(\phi+\epsilon h)^2 m$-symmetric and satisfies $P_t^\phi 1 \le 1$. Given $\nu = f \cdot m \in \mathcal{F}_1$ with $\sqrt{f} \in \mathcal{F}$, set

$$S_t^\phi \sqrt{f}(x) = \mathbb{E}_x \left[e^{A_t(\mu+F)} \cdot \exp\left(-\int_0^t \frac{\mathcal{H}^{\mu+F}\phi}{\phi+\epsilon h}(X_s)ds \right) \sqrt{f}(X_t) \right]$$

Then

$$\begin{aligned}
\int_X (S_t^\phi \sqrt{f})^2 dm &= \int_X (\phi+\epsilon h)^2 \left(P_t^\phi \left(\frac{\sqrt{f}}{\phi+\epsilon h} \right) \right)^2 dm \\
&\le \int_X (\phi+\epsilon h)^2 P_t^\phi \left(\left(\frac{\sqrt{f}}{\phi+\epsilon h} \right)^2 \right) dm \\
&\le \int_X (\phi+\epsilon h)^2 \left(\frac{\sqrt{f}}{\phi+\epsilon h} \right)^2 dm \\
&= \int_X f dm.
\end{aligned}$$

Hence

$$0 \le \lim_{t\to 0} \frac{1}{t}(\sqrt{f} - S_t^\phi \sqrt{f}, \sqrt{f})_m = \mathcal{E}^{\mu+F}(\sqrt{f}, \sqrt{f}) + \int_X \frac{\mathcal{H}^{\mu+F}\phi}{\phi+\epsilon h} f dm,$$

and thus $\mathcal{E}^{\mu+F}(\sqrt{f}, \sqrt{f}) \ge I(f \cdot m)$. □

Put

$$\lambda_2(\mu+F) = \inf \left\{ \mathcal{E}^{\mu+F}(u,u) : u \in \mathcal{F},\ \|u\|_2 = 1 \right\},$$

and let $\{u_n\}$ be a minimizing sequence of $\mathcal{F}$, that is, $\|u_n\|_2 = 1$ and $\lambda_2(\mu+F) = \lim_{n\to\infty} \mathcal{E}^{\mu+F}(u_n, u_n)$. Put $\mu' = |\mu| + |\mu_{F_1}|$. Since $\mathcal{E}(u_n, u_n) \le c\mathcal{E}_F(u_n, u_n)$ and

$$\int_X u_n^2 d\mu' \le \|G_\alpha \mu'\|_\infty \left(\mathcal{E}(u_n, u_n) + \alpha \right),$$

we have

$$\begin{aligned}\mathcal{E}^{\mu+F}(u_n,u_n) &\geq \mathcal{E}_F(u_n,u_n) - \|G_\alpha\mu'\|_\infty\left(\mathcal{E}(u_n,u_n)+\alpha\right)\\ &\geq \frac{1}{c}\mathcal{E}(u_n,u_n) - \|G_\alpha\mu'\|_\infty\left(\mathcal{E}(u_n,u_n)+\alpha\right)\\ &= \left(\frac{1-c\|G_\alpha\mu'\|}{c}\right)\mathcal{E}(u_n,u_n) - \alpha\|G_\alpha\mu'\|_\infty\end{aligned}$$

Choosing α so large that $1-c\|G_\alpha\mu'\|_\infty > 0$, we have

$$\mathcal{E}(u_n,u_n) \leq \frac{c}{1-c\|G_\alpha\mu'\|_\infty}\left(\mathcal{E}^{\mu+F}(u_n,u_n)+\alpha\|G_\alpha\mu'\|_\infty\right).$$

We thus see from Assumption (V) that for any $\epsilon > 0$ there exists a compact set K such that

$$\sup_n \int_{K^c} u_n^2 \cdot dm \leq \|R_1 I_{K^c}\|_\infty \cdot \left(\sup_n \mathcal{E}(u_n,u_n)+\alpha\right) < \epsilon,$$

that is, the subset $\{u_n^2 m\}$ of $\mathcal{P}(X)$ is tight. Hence there exists a subsequence $\{u_{n_k}^2 m\}$ weakly converges to a probability measure ν. Since the function $I_{\mathcal{E}^{\mu+F}}$ is lower semi-continuous by Lemma B.2,

$$I_{\mathcal{E}^{\mu+F}}(\nu) \leq \liminf_{k\to\infty} I_{\mathcal{E}^{\mu+F}}(u_{n_k}^2 m) = \liminf_{k\to\infty} \mathcal{E}^{\mu+F}(u_{n_k},u_{n_k}) < \infty.$$

Therefore, Proposition B.3 says that $\nu = u_0^2 m$, $u_0 \in \mathcal{F}$. The function u_0 is the ground state, $\lambda_2(\mu+F) = \mathcal{E}^{\mu+F}(u_0,u_0)$. The uniqueness of the ground state follows from Assumption (I) (Irreducibility) (see [14, Proposition 1.4.3]). Therefore we have:

Proposition B.4. *Under Assumptions* (I), (III) *and* (V), *there exists a unique ground state* $u_0 \in \mathcal{F}$ *of* $\mathcal{H}^{\mu+F}$.

Define the probability measure $Q_{x,t}$ on $\mathcal{P}(X)$ by

$$Q_{x,t}(B) = \frac{\mathbb{E}_x[e^{A_t(\mu+F)}; L_t \in B, t<\zeta]}{\mathbb{E}_x[e^{A_t(\mu+F)}; t<\zeta]}, \quad B \in \mathcal{B}(\mathcal{P}(X)). \tag{B.4}$$

Here, L_t is the occupation distribution defined as in (3.2). Define the function J on $\mathcal{P}(X)$ by

$$J(\nu) = I_{\mathcal{E}^{\mu+F}}(\nu) - \lambda_2(\mu+F). \tag{B.5}$$

Lemma B.5. *The function J satisfies:*

(i) $0 \leq J(\nu) \leq \infty$.

(ii) J *is lower semicontinuous.*

(iii) *For each* $l < \infty$, *the set* $\{\nu \in \mathcal{P}(X) : J(\nu) \leq l\}$ *is compact.*

(iv) $J(u_0^2 \cdot m) = 0$ *and* $J(\nu) > 0$ *for* $\nu \neq u_0^2 \cdot m$.

We see from Lemma B.5 that the function J satisfies those conditions for the rate function which Varadhan imposed in his formulation. Then we have the next theorem([41]):

Theorem B.6. *Let* $\mathbb{M}$ *be a Hunt process satisfying Assumptions* (I), (III) *and* (V). *Let* μ *be a measure in* $\mathcal{K}_\infty$ *and* F *a function in* $\mathcal{J}_\infty$. *Define by (B.4) a sequence* $\{Q_{x,t}\}_{t>0}$ *of probability measures on* $\mathcal{P}(X)$. *Then the sequence* $\{Q_{x,t}\}_{t>0}$ *obeys the large deviation principle with rate function* J:

(i) *For each open set* $G \subset \mathcal{P}(X)$

$$\liminf_{t\to\infty} \frac{1}{t} \log Q_{x,t}(G) \geq - \inf_{\nu\in G} J(\nu).$$

(ii) *For each closed set* $K \subset \mathcal{P}(X)$

$$\limsup_{t\to\infty} \frac{1}{t} \log Q_{x,t}(K) \leq - \inf_{\nu\in K} J(\nu).$$

Corollary B.7. *The measure* $Q_{x,t}$ *weakly converges to* $\delta_{u_0^2\cdot m}$ *as* $t \to \infty$.

Proof. If a closed set K does not contain $u_0^2 \cdot m$, then $\inf_{x\in K} J(x) > 0$ by Lemma B.5 (iv). Hence Theorem B.6 (ii) says that $\lim_{t\to\infty} Q_{x,t}(K) = 0$ and $\lim_{t\to\infty} Q_{x,t}(K^c) = 1$. For a positive constant δ and a bounded continuous function f on the set of $\mathcal{P}(X)$, define the closed set $K \subset \mathcal{P}(X)$ by $K = \{\nu \in \mathcal{P}(X) : |f(\nu) - f(u_0^2 \cdot m)| \geq \delta\}$. Then we have

$$\begin{aligned}
&\left|\int_{\mathcal{P}(X)} f(\nu)Q_{x,t}(d\nu) - f(u_0^2 \cdot m)\right| \leq \int_{\mathcal{P}(X)} |f(\nu) - f(u_0^2 \cdot m)|Q_{x,t}(d\nu) \\
&= \int_K |f(\nu) - f(u_0^2 \cdot m)|Q_{x,t}(d\nu) + \int_{K^c} |f(\nu) - f(u_0^2 \cdot m)|Q_{x,t}(d\nu) \\
&\leq \delta Q_{x,t}(K^c) + 2\|f\|_\infty Q_{x,t}(K) \longrightarrow \delta
\end{aligned}$$

as $t \to \infty$. Since δ is arbitrary, we can conclude the weak convergence. □

On account of Corollary B.7, we realize that Theorem B.6 implies a large deviation from the ground state.

Appendix C

Time Change and Subordination

C.1 Time Changed Hunt Processes

In this section, we study some properties of time changed Hunt processes by PCAF. Let (X, m), $\mathbb{M}$ and $(\mathcal{E}, \mathcal{F})$ be as in the preceding chapter. Given a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$, we denote by $\mathcal{F}_e$ the family of m-measurable functions u on X such that $|u| < \infty$ m-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of functions $\mathcal{F}$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. We call $(\mathcal{F}_e, \mathcal{E})$ the *extended Dirichlet space* of $(\mathcal{E}, \mathcal{F})$.

Lemma C.1 ([20, Lemma 1.5.5])**.** *If a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$ is transient then its extended Dirichlet space $\mathcal{F}_e$ is Hilbert space with respect to inner product $\mathcal{E}(\cdot, \cdot)$.*

Let us fix a positive Radon measure $\mu \in \mathcal{K}$ and $A_t(\mu)$ the PCAF with Revuz measure μ. We denote by Y the topological support of μ, i.e. Y is the smallest closed set outside which μ vanishes. Let $\{\tau_t\}_{t\geq 0}$ be the right continuous inverse function of $A_t(\mu)$, $\tau_t = \inf\{s > 0 : A_s(\mu) > t\}$. We assume that support Y equals the *quasi-support* $\tilde{Y}$ of μ, $\tilde{Y} = \{x \in X : \mathbb{P}_x(\tau_0 = 0) = 1\}$.

We consider the following orthogonal decomposition of the Hilbert space $(\mathcal{F}_e, \mathcal{E})$:

$$\mathcal{F}_e = \mathcal{F}_{e,X\setminus Y} \oplus \mathcal{H}_Y$$
$$\mathcal{F}_{e,X\setminus Y} = \{u \in \mathcal{F}_e : u = 0 \text{ q.e. on } Y\},$$

and denote by H_Y the orthogonal projection from $\mathcal{F}_e$ to $\mathcal{H}_Y$. It is known in [20, Theorem 4.3.2] that $H_Y u(x) = \mathbb{E}_x[u(X_{\sigma_Y})]$ for any $u \in \mathcal{F}_e$ and $x \in X$.

Define a symmetric form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ on $L^2(Y;\mu)$ by

$$\begin{cases} \check{\mathcal{F}} = \{\phi \in L^2(Y;\mu) : \phi = u \ \mu\text{-a.e. on } \ Y \text{ for some } u \in \mathcal{F}_e\}, \\ \check{\mathcal{E}}(\phi,\phi) = \mathcal{E}(H_Y u, H_Y u), \quad \phi \in \check{\mathcal{F}}, \ \phi = u \ \mu\text{-a.e. on } Y, \ u \in \mathcal{F}_e. \end{cases}$$

Let us also define the *time changed process* $\check{\mathbb{M}} = (\check{X}_t, \mathbb{P}_x)_{x\in Y}$ of $\mathbb{M}$ with respect to the PCAF $A_t(\mu)$ by

$$\check{X}_t = X_{\tau_t}.$$

Then the time changed process $\check{\mathbb{M}}$ is a strong Markov process on Y. In particular, the transition function and the resolvent of $\check{\mathbb{M}}$ is respectively given by

$$\begin{aligned} &\check{p}_t\phi(x) = \mathbb{E}_x[\phi(X_{\tau_t})], \quad x \in Y, \\ &\check{R}_\alpha\phi(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\alpha t}\phi(X_{\tau_t})dt\right] = \mathbb{E}_x\left[\int_0^\infty e^{-\alpha A_t(\mu)}\phi(X_t)dA_t(\mu)\right]. \end{aligned}$$

Theorem C.2 ([20, Theorem 6.2.1])**.** *Let $\mathbb{M}$ be a Hunt process associated with a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(X;m)$. Then the time changed process $\check{\mathbb{M}}$ is the Hunt process associated with the Dirichlet form $(\check{\mathcal{E}},\check{\mathcal{F}})$. Furthermore, $(\check{\mathcal{E}},\check{\mathcal{F}})$ is regular.*

For a measure $\mu \in \mathcal{K}$, define

$$\lambda(\mu) = \inf\left\{\mathcal{E}(u,u) : u \in \mathcal{F}, \int_X u(x)^2\mu(dx) = 1\right\}.$$

On account of Lemma 3.1 in [38], we see that $\lambda(\mu)$ is the bottom of the spectrum of the time changed process $\check{\mathbb{M}}$.

Definition C.1. If a measure $\mu \in \mathcal{K}$ satisfies

$$\sup_{x\in X}\mathbb{E}_x[\exp(A_\infty(\mu))] < \infty,$$

then μ is said to be *gaugeable*.

Theorem C.3 (Chen [9, Theorem 5.1])**.** *Let μ be a positive measure in $\mathcal{K}_{\infty,0}$. Then μ is gaugeable if and only if $\lambda(\mu) > 1$.*

Lemma C.4. *Let μ be a positive measure in $\mathcal{K}_{\infty,0}$. Then $\lambda(\mu) \le 1$ if and only if*

$$\lambda_2(\mu) = \inf\{\mathcal{E}^\mu(u,u) : u \in \mathcal{F}, \|u\|_2 = 1\} \le 0.$$

Proof. Suppose that $\lambda(\mu) \le 1$. Then there exists a function $\phi \in \mathcal{F}$ with $\mathcal{E}(\phi,\phi) \le 1$ and $\int_X \phi(x)^2 \mu(dx) = 1$. Let $\psi = \phi/\|\phi\|_2$, then we have

$$\begin{aligned}\mathcal{E}^\mu(\psi,\psi) &= \mathcal{E}(\psi,\psi) - \int_X \psi(x)^2 \mu(dx) \\ &= \frac{1}{\|\phi\|_2^2}\left(\mathcal{E}(\phi,\phi) - \int_X \phi(x)^2\mu(dx)\right) \\ &\le 0.\end{aligned}$$

Suppose that $\lambda_2(\mu) \le 0$. Then there exists a function $\varphi \in \mathcal{F}$ with $\mathcal{E}^\mu(\varphi,\varphi) \le 0$ and $\|\varphi\|_2 = 1$. Let $\psi_1 = \varphi/(\int \varphi^2 d\mu)^{1/2}$, then we have

$$\begin{aligned}\mathcal{E}(\psi_1,\psi_1) &= \mathcal{E}^\mu(\psi_1,\psi_1) + \int \psi_1(x)^2\mu(dx) \\ &\le 1.\end{aligned}$$

□

C.2 Subordination

Let $\mathbb{M} = (X_t, \mathbb{P}_x)$ be an m-symmetric Hunt process on X satisfying Assumptions (I)–(IV). Let $(\mathcal{E},\mathcal{F})$ be the Dirichlet form generated by $\mathbb{M}$. In this section, we use the spectral representation of the Dirichlet form:

$$\begin{cases}\mathcal{F} = \left\{u \in L^2(X;m) : \displaystyle\int_0^\infty \lambda(dE_\lambda u, u) < \infty\right\}, \\ \mathcal{E}(u,v) = \displaystyle\int_0^\infty \lambda(dE_\lambda u, v) \quad u, v \in \mathcal{F}.\end{cases}$$

Let $\gamma_t^\alpha(s)$ $(s > 0, 0 < \alpha < 2)$ be a unique continuous function satisfying

$$e^{-ta^{\alpha/2}} = \int_0^\infty e^{-as}\gamma_t^{(\alpha)}(s)ds, \quad a, t > 0$$

(see Yosida [49, Chapter IX §11] for more details). Define

$$p_t^{(\alpha)} f(x) = \int_0^\infty \mathbb{E}_x[f(X_s)]\gamma_t^{(\alpha)}(s)ds, \quad t > 0.$$

Then $\{p_t^{(\alpha)}\}_{t>0}$ is a strongly continuous sub-Markovian semigroup on $L^2(X;m)$. We have the corresponding Dirichlet form is expressed by

$$\begin{cases}\mathcal{E}^{(\alpha)}(u,u) = \displaystyle\int_0^\infty \lambda^{\alpha/2} d(E_\lambda u, u), \quad u \in \mathcal{F}^{(\alpha)}, \\ \mathcal{F}^{(\alpha)} = \left\{u \in L^2(X;m) : \displaystyle\int_0^\infty \lambda^{\alpha/2} d(E_\lambda u, u) < \infty\right\}.\end{cases} \tag{C.1}$$

Furthermore, there exists a Hunt process $\mathbb{M}^{(\alpha)}$ generated by $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ ([29, Theorem 3.2]).

Proposition C.5. *If a Hunt process* $\mathbb{M}$ *satisfies Assumptions* (I)–(IV)*, then so does* $\mathbb{M}^{(\alpha)}$.

Proof. (I): Take any $p_t^{(\alpha)}$-invariant set A and a positive function $f \in L^2(X; m)$. Then we have

$$\begin{aligned} 1_A(x)(p_t^{(\alpha)} f(x)) &= 1_A(x) \int_0^\infty \mathbb{E}_x[f(X_s)]\gamma_t^{(\alpha)}(s)ds \\ &= \int_0^\infty 1_A(x)\mathbb{E}_x[f(X_s)]\gamma_t^{(\alpha)}(s)ds \\ &= \int_0^\infty 1_A(x)p_s f(x)\gamma_t^{(\alpha)}(s)ds. \end{aligned}$$

Furthermore,

$$p_t^{(\alpha)}(1_A f(x)) = \int_0^\infty p_s(1_A f(x))\gamma_t^{(\alpha)}(s)ds.$$

Since $\gamma_t^{(\alpha)}(s) > 0$, $p_s(1_A f(x)) = 1_A p_s f(x)$ and thus $m(A) = 0$ or $m(X \setminus A) = 0$ by the irreducibility of p_t.
(II): It is obvious since $p_t = 1$ and $\int_0^\infty \gamma_t^{(\alpha)}(s)ds = 1$.
(III) and (IV): Noting that $\gamma_t^{(\alpha)}(s)ds$ is a bounded measure, we have (III) and (IV) by the dominated convergence theorem. □

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