# Modified Arratia Flow and Wasserstein Diffusion

Max von Renesse

Universität Leipzig

Joint work with

Vitalii Konarovskyi
(MPI Leipzig/U. Chernivtsi (Ukraine))

GER-JAP Meeting
Sendai, 2015

# Outline

Part I: Background

- *Reminder: Brownian Motion on Manifolds*
- *... and on Wasserstein Space (?)*
- *The Model from [v.R/Sturm, 2009]*
  - *Entropic Measure*
  - *Martingale Problem and Varadhan's formula*

Parti II Modified Arratia Flow (2015)

- *Konarovskyi's Construction*
- *SPDE Picture and Martingale Problem*
- *Large Deviations and Wasserstein Metric*

# I) Background

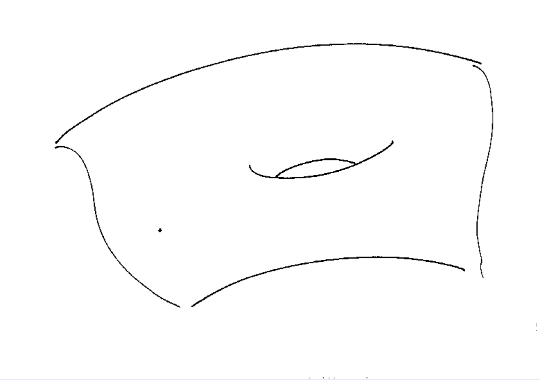
# Motivation

Questions:

- What is the 'most natural' random perturbation of e.g. heat flow, seen as gradient flow for entropy on Wasserstein space?
- Is there a model for reshuffling a fluid where transitions are uniform w.r.t. to dissipated energy?
- Is there a 'Brownian Motion' on the Wasserstein space?

# Reminder: Riemannian Brownian Motion

## Construction via Central Limit Theorem

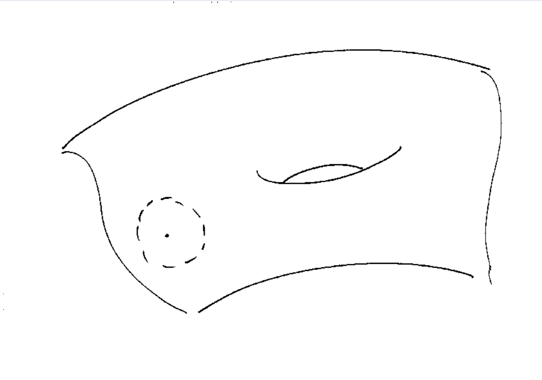


$$X_0 \rightsquigarrow S_r(X_0) \rightsquigarrow X_1^{(r)} \rightsquigarrow S_r(X_1^{(r)}) \rightsquigarrow X_2^{(r)} \rightsquigarrow S_r(X_2^{(r)}) \rightsquigarrow X_3^{(r)} \rightsquigarrow S_r(X_3^{(r)}) \rightsquigarrow X_4^{(r)}$$

$$\boxed{X_t := \text{w-}\lim_{k\to\infty} X^{(\frac{1}{k})}_{\lfloor t\cdot k^2 \rfloor}}$$

# Reminder: Riemannian Brownian Motion

## Construction via Central Limit Theorem

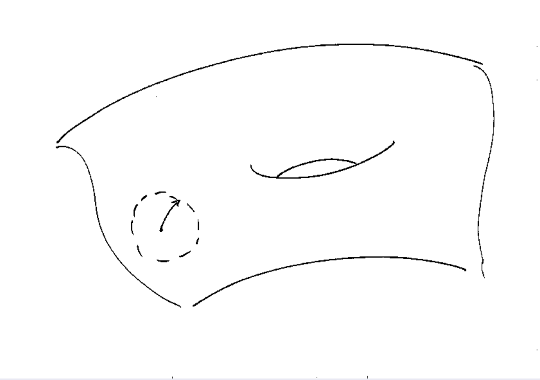


$$X_0 \rightsquigarrow S_r(X_0) \rightsquigarrow X_1^{(r)} \rightsquigarrow S_r(X_1^{(r)}) \rightsquigarrow X_2^{(r)} \rightsquigarrow S_r(X_2^{(r)}) \rightsquigarrow X_3^{(r)} \rightsquigarrow S_r(X_3^{(r)}) \rightsquigarrow X_4^{(r)}$$

$$\boxed{X_t := \text{w-}\lim_{k\to\infty} X^{(\frac{1}{k})}_{\lfloor t\cdot k^2 \rfloor}}$$

# Reminder: Riemannian Brownian Motion

## Construction via Central Limit Theorem

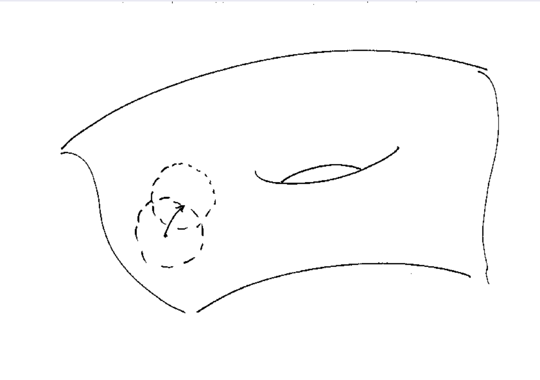


$$X_0 \rightsquigarrow S_r(X_0) \rightsquigarrow X_1^{(r)} \rightsquigarrow S_r(X_1^{(r)}) \rightsquigarrow X_2^{(r)} \rightsquigarrow S_r(X_2^{(r)}) \rightsquigarrow X_3^{(r)} \rightsquigarrow S_r(X_3^{(r)}) \rightsquigarrow X_4^{(r)}$$

$$\boxed{X_t := \operatorname*{w-\lim}_{k\to\infty} X^{(\frac{1}{k})}_{\lfloor t\cdot k^2 \rfloor}}$$

# Reminder: Riemannian Brownian Motion

## Construction via Central Limit Theorem

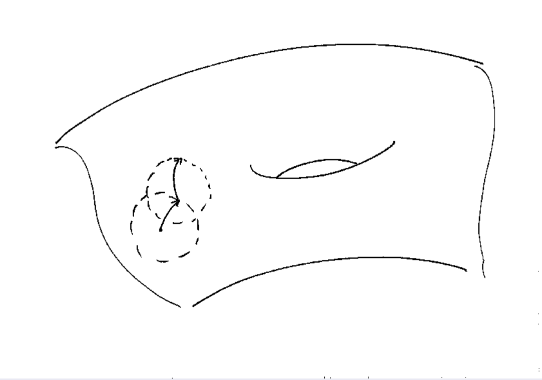


$$X_0 \rightsquigarrow S_r(X_0) \rightsquigarrow X_1^{(r)} \rightsquigarrow S_r(X_1^{(r)}) \rightsquigarrow X_2^{(r)} \rightsquigarrow S_r(X_2^{(r)}) \rightsquigarrow X_3^{(r)} \rightsquigarrow S_r(X_3^{(r)}) \rightsquigarrow X_4^{(r)}$$

$$\boxed{X_t := \operatorname{w-}\lim_{k\to\infty} X^{(\frac{1}{k})}_{\lfloor t\cdot k^2 \rfloor}}$$

# Reminder: Riemannian Brownian Motion

## Construction via Central Limit Theorem

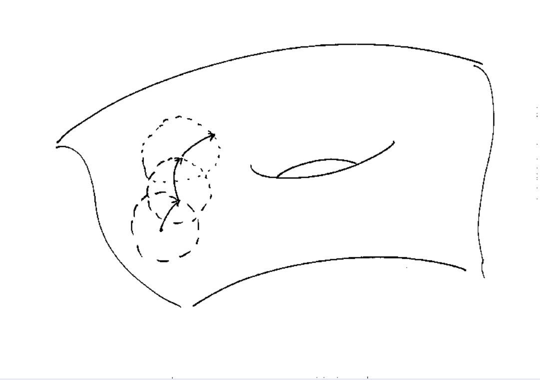


$$X_0 \rightsquigarrow S_r(X_0) \rightsquigarrow X_1^{(r)} \rightsquigarrow S_r(X_1^{(r)}) \rightsquigarrow X_2^{(r)} \rightsquigarrow S_r(X_2^{(r)}) \rightsquigarrow X_3^{(r)} \rightsquigarrow S_r(X_3^{(r)}) \rightsquigarrow X_4^{(r)}$$

$$\boxed{X_t := \operatorname*{w\text{-}lim}_{k\to\infty} X^{(\frac{1}{k})}_{\lfloor t\cdot k^2 \rfloor}}$$

# Reminder: Riemannian Brownian Motion

## Construction via Central Limit Theorem

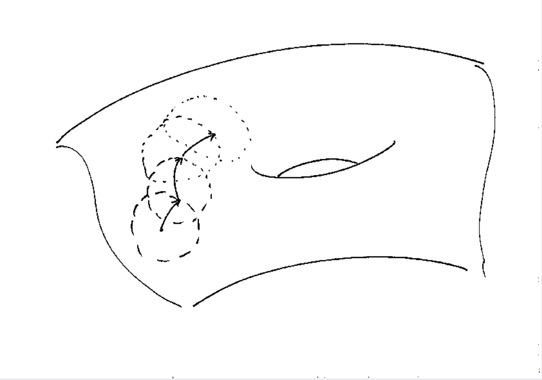


$$X_0 \rightsquigarrow S_r(X_0) \rightsquigarrow X_1^{(r)} \rightsquigarrow S_r(X_1^{(r)}) \rightsquigarrow X_2^{(r)} \rightsquigarrow S_r(X_2^{(r)}) \rightsquigarrow X_3^{(r)} \rightsquigarrow S_r(X_3^{(r)}) \rightsquigarrow X_4^{(r)}$$

$$\boxed{X_t := \underset{k\to\infty}{\text{w-}\lim}\, X^{(\frac{1}{k})}_{\lfloor t\cdot k^2 \rfloor}}$$

# Reminder: Riemannian Brownian Motion

## Construction via Central Limit Theorem

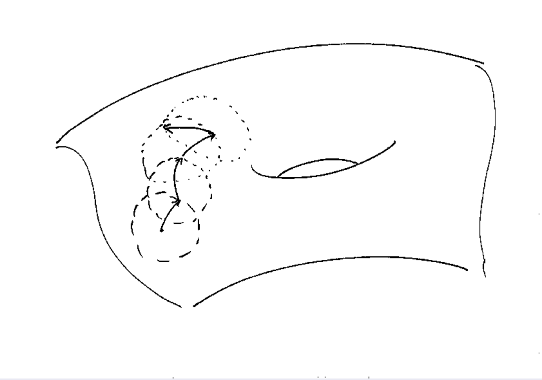


$$X_0 \rightsquigarrow S_r(X_0) \rightsquigarrow X_1^{(r)} \rightsquigarrow S_r(X_1^{(r)}) \rightsquigarrow X_2^{(r)} \rightsquigarrow S_r(X_2^{(r)}) \rightsquigarrow X_3^{(r)} \rightsquigarrow S_r(X_3^{(r)}) \rightsquigarrow X_4^{(r)}$$

$$\boxed{X_t := \text{w-}\lim_{k\to\infty} X^{(\frac{1}{k})}_{\lfloor t\cdot k^2 \rfloor}}$$

# Reminder: Riemannian Brownian Motion

## Construction via Central Limit Theorem

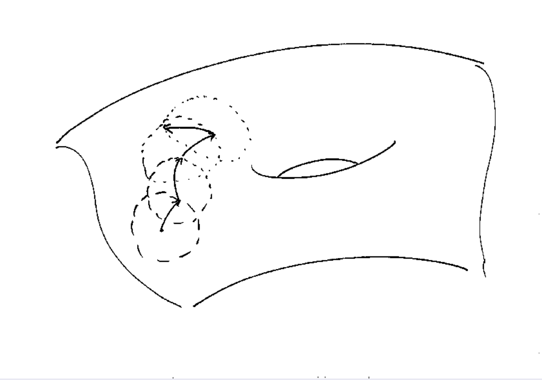


$$X_0 \rightsquigarrow S_r(X_0) \rightsquigarrow X_1^{(r)} \rightsquigarrow S_r(X_1^{(r)}) \rightsquigarrow X_2^{(r)} \rightsquigarrow S_r(X_2^{(r)}) \rightsquigarrow X_3^{(r)} \rightsquigarrow S_r(X_3^{(r)}) \rightsquigarrow X_4^{(r)}$$

$$\boxed{X_t := \underset{k\to\infty}{\text{w-}\lim} X^{(\frac{1}{k})}_{\lfloor t\cdot k^2 \rfloor}}$$

# Reminder: Riemannian Brownian Motion

## Construction via Central Limit Theorem



$$X_0 \rightsquigarrow S_r(X_0) \rightsquigarrow X_1^{(r)} \rightsquigarrow S_r(X_1^{(r)}) \rightsquigarrow X_2^{(r)} \rightsquigarrow S_r(X_2^{(r)}) \rightsquigarrow X_3^{(r)} \rightsquigarrow S_r(X_3^{(r)}) \rightsquigarrow X_4^{(r)}$$

$$\boxed{X_t := \text{w-}\lim_{k\to\infty} X^{(\frac{1}{k})}_{\lfloor t\cdot k^2 \rfloor}}$$

# Reminder: Laplace-Beltrami Operator in a Local Chart

- Let $M \simeq^{homeo} \mathbb{R}^d$ Manifold
- Riem. structure

$$\mathbb{R}^d \ni x \to (g_{ij}(x)) =: G^{-1}(x) \in L(\mathbb{R}^d, \mathbb{R}^d)$$

- Riem. Gradient for $f : \mathbb{R}^d \to \mathbb{R}$

$$\nabla^g f(x) = G(x) Df(x)$$

- Associated norm on tangent space

$$\|\nabla^g f\|^2_{T_x} = \langle DF(x), G(x) DF(x) \rangle_{\mathbb{R}^d}$$

- Laplace-Beltrami Operator

$$\begin{aligned} \Delta^g f(x) &= \frac{1}{\sqrt{\det G}} \operatorname{div}\left(\sqrt{\det G}\, G \cdot Df\right)(x) \\ &= U(x) \cdot DF(x) + g_{ij}(x) D_{ij} f(x) \end{aligned}$$

# SDE Construction of Riemannian BM

- Laplace-Beltrami operator in chart

$$\Delta^g f(x) = U(x) \cdot DF(x) + g_{ij}(x) D_{ij} f(x).$$

- Associated diffusion $\rightsquigarrow$ SDE

$$dx_t = U(x_t)dt + \sqrt{2}\sqrt{G(x_t)}db_t$$

- $\rightsquigarrow$ solves *Martingale Problem*: For all smooth $f : M \to \mathbb{R}$

$$f(x_t) - \int_0^t \Delta^g f(x_s) ds = M_t^f \quad \text{martingale with}$$

$$[M^f]_t = \int_0^t \|\nabla^g f\|^2_{T_{x_s}} ds \quad \text{quadratic variation process.}$$

# Infinite Dimensional Example – 'Otto Calculus'

- Let $M = \mathcal{P}(\mathbb{R}^d)$: space of (smooth) probability measures on $\mathbb{R}^d$.
- Flat Parametrization over $L^2(\mathbb{R}^d)$

$$L^2(\mathbb{R}^d) \ni m \to \mu(dx) := m(x)dx \in \mathcal{P}(\mathbb{R}^d)$$

- Riem. structure [Otto, 2001]

$$\mathcal{P} \ni \mu \to G(\mu) = -\Delta^\mu : L^2(\mathbb{R}^d, dx) \to L^2(\mathbb{R}^d, dx)$$

$$\Delta^\mu f(x) = -\operatorname{div}(\mu \nabla f)(x)$$

- Associated norm on tangent space

$$\|\nabla^{\mathcal{W}} F\|^2_{T_\mu} = \langle DF, G(\mu)DF\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\nabla_x DF_{|\mu}|^2(x)\mu(dx).$$

- Intrinsic metric = Wasserstein-distance [Benamou-Brenier, 2000]

$$d^2_{\mathcal{W}}(\mu,\nu) = \inf_{\substack{\Pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \\ \Pi^1_* = \mu; \Pi^2_* = \nu}} \iint_{M \times M} d^2(x,y)\Pi(dx,dy).$$

# A Candidate Generator for the "Wasserstein Diffusion"

- Formal ('Wasserstein-Otto'-)Laplace-Beltrami Operator

$$\Delta^{\mathcal{W}} F(\mu) = \frac{1}{\sqrt{\det_{L^2} G(\mu)}} \mathrm{div}_{L^2}\left(\sqrt{\det G}\, G \cdot DF\right)(\mu)$$
$$= U(\mu) \cdot DF(\mu) + \mathrm{trace}_{L^2}\left(-\Delta^{\mu} D^2 F_{|\mu}\right)$$

- Associated SPDE for Wasserstein-Brownian Motion on $\mathcal{P}(\mathbb{R}^d)$

$$d\mu_t = U(\mu_t)dt + \sqrt{-\Delta^{\mu}}\, dB_t,$$

where $U =$ (some) drift, $dB_t = L^2(\mathbb{R}^d)$-space-time white noise.

- Equivalent SPDE (in sense of distribution)

$$\boxed{d\mu_t = U(\mu_t)dt + \mathrm{div}(\sqrt{\mu}\, dW_t),}$$

$dW_t = L^2(\mathbb{R}^d; \mathbb{R}^d)$-space-time white noise.

# Associated Martingale Problem

- Let $t \to \mu_t$ solve SPDE

$$d\mu_t = U(\mu_t)dt + \sqrt{-\Delta^\mu}db_t$$

then for $f : \mathbb{R}^d \to \mathbb{R}$ smooth

$$t \to \langle f, \mu_t \rangle - \int_0^t \langle f, U(\mu_s)\rangle ds =: M_t^f \quad \text{martingale}$$

with quadratic variation process

$$[M^f]_t = \int_0^t \int_{\mathbb{R}^d} |\nabla^{\mathbb{R}^d} f|^2(x)\mu_s(dx)ds = \int_0^t \langle |\nabla f|^2, \mu_s\rangle ds.$$

# Rigorous Construction on $\mathcal{P}([0,1])$ from [v.R/Sturm 2009]

- Idea: Include viscosity/heat flow in drift for better regularity.

$$d\mu = -\beta \nabla^{\mathcal{W}} \operatorname{Ent}(\mu) dt + dB_t^{\mathcal{W}}$$

- Candidate invariant measure ('Entropic Measure')

$$\mathfrak{P}^{\beta}(d\mu) = \frac{1}{Z} e^{-\beta \operatorname{Ent}(\mu)} vol(d\mu)$$

- Apply Dirichlet Form methods $\rightsquigarrow$ weak solution of the SPDE

$$\boxed{d\mu = \beta \Delta \mu dt + \Gamma(\mu) dt + \operatorname{div}(\sqrt{\mu} dW),}$$

where

$$\langle f, \Gamma(\mu) \rangle = \sum_{I \in \operatorname{gaps}(\mu)} \left[ \frac{f''(I_+) + f''(I-)}{2} - \frac{f'(I_+) - f'(I_+)}{|I|} \right]$$

# Martingale Problem and Varadhan's Formula for $(\mu_t)$

## Theorem (v.R./Sturm, AOP 2009)

*For $\beta > 0$ there exists a $\mathfrak{P}^\beta$-symmetric diffusion process on $\mathcal{P}[0,1]$ s.t.*

- *For $f \in C^\infty([0,1])$ with $f'_{\partial[0,1]} = 0$*

$$M_t^f := \langle \mu_t, f \rangle - \beta \cdot \int_0^t \langle f'', \mu_s \rangle ds - \int_0^t \langle \Gamma(\mu_s), f \rangle ds$$

  *is a martingale with quadratic variation process*

$$[M^f]_t = 2 \int_0^t \langle (f')^2, \mu_s \rangle ds.$$

- *For Borel $A, B \subset \mathcal{P}([0,1])$*

$$\lim_{t \to 0} t \log p_t(A,B) = -\frac{(d_{\mathcal{W}}(A,B))^2}{2}.$$

# II) Modified Arratia Flow

## Theorem (Konarovskyi, 2014)

*There is a unique in law process* $y \in D([0,1], C([0,T]))$ *process s.t.*

(C1) *for all* $u \in [0,1]$*, the process* $y(u,\cdot)$ *is a continuous square integrable martingale with respect to the filtration*

$$\mathcal{F}_t = \sigma(y(u,s),\ u \in [0,1],\ s \le t), \quad t \in [0,T];$$

(C2) *for all* $u \in [0,1]$*,* $y(u,0) = u$*;*

(C3) *for all* $u < v$ *from* $[0,1]$ *and* $t \in [0,T]$*,* $y(u,t) \le y(v,t)$*;*

(C4) *for all* $u, v \in [0,1]$*,*

$$[y(u,\cdot), y(v,\cdot)]_t = \int_0^t \frac{\mathbb{I}_{\{\tau_{u,v} \le s\}} ds}{m(u,s)},$$

*where* $m(u,t) = |\{v : \ y(v,t) = y(u,t)\}|$*,*
$\tau_{u,v} = \inf\{t : \ y(u,t) = y(v,t)\} \wedge T$*.*

# Properties

- $(y(.,t))_{t\geq 0}$ is a $D^{\uparrow} \cap L^2([0,1],\mu)$-valued martingale,

  where $\mu(dx) = \kappa(x)dx$ with $\kappa : [0,1] \to [0,1]$

$$\kappa(u) = \begin{cases} u, & u \in [0,1/2] \\ (1-u), & u \in (1/2,1], \end{cases}$$

  and

$$D^{\uparrow} = \{h \in D[0,1] : \ h \text{ is non-decreasing}\}.$$

- For each $g \in C_c(0,1)$, $s \mapsto I(g)(s) := \langle g, y(.,s)\rangle_{L^2(0,1)}$ is a continuous square integrable martingale with quadratic variation

$$[I(g)]_t = \int_0^t \|\mathrm{pr}_{y(s)} g(s)\|_{L_2}^2 ds, \quad t \in [0,T],$$

  where $\mathrm{pr}_{y(s)} g := \mathbb{E}_{dx}[g|\sigma(y_s)]$

# SDE Representation and Ito Formula

**Corollary**

*The process $s \to y_s(\cdot)$ is a solution to*

$$dy_s = \mathrm{pr}_{y_s} dW_s, \quad y_s \in D^{\uparrow}, \quad y_0(x) = x,$$

*where $(dWs)_{s \geq 0}$ is $L^2([0,1])$-space-time white noise.*

**Theorem**

*For all $f \in C_b^2(\mathbb{R})$*

$$\int_0^1 f(y(u,t))du = \int_0^1 f(u)du + \int_0^1 \int_0^t f'(y(u,s))dy(u,s)du + \frac{1}{2}\int_0^t \int_0^1 \frac{f''(y(u,s))}{m(u,s)} duds, \quad t \in [0,T].$$

# Induced Random Flow on $\mathcal{P}(\mathbb{R})$

## Corollary

*Let* $\mu_t := y(.,t)_{\#}\lambda_{\llcorner[0,1]}$, *then for all* $f \in C_b^2(\mathbb{R})$

$$M_t^f := \langle f, \mu_t \rangle - \int_o^t \langle f, \hat{\Gamma}(\mu_t) \rangle \, dt$$

*is a continuous martingale with quadratic variation process*

$$\left[ M^f \right]_t = \int_0^t \int_{\mathbb{R}} (f')^2(x) \mu_s(dx) ds,$$

*where*

$$\langle f, \hat{\Gamma}(\mu) \rangle = \sum_{x \in supp(\mu)} f''(x).$$

# Large Deviation Result for $y^\epsilon_\cdot = y_{\cdot\epsilon}$

Let

$$\mathcal{H} = \{\varphi \in C([0,T]; L_2([0,1],dx) \cap D^\uparrow) : \ \varphi(0) = \text{id and } t \to \varphi_t \in L^2([0,1],dx) \text{ is absolutely continuous}\}$$

and

$$I(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \|\dot{\varphi}(t)\|^2_{L_2(du)} dt, & \varphi \in \mathcal{H}, \\ +\infty, & \text{otherwise.} \end{cases}$$

**Theorem (Konarovskyi/v.R., 2015)**

*For $\epsilon > 0$ let $y^\epsilon_t := y_{\epsilon \cdot t}$, then the family of processes $\{y^\epsilon_\cdot\}_{\epsilon>0}$ satisfies a LDP in the space $C([0,T], L_2(\mu))$ with the good rate function $I$.*

# Varadhan Formula for Measure Valued Flow

**Corollary**

*For measurable* $\Phi \subset L_2([0,1], dx) \cap D^{\uparrow}$

$$\lim_{\epsilon \to 0} \epsilon \log P(y_\epsilon \in \Phi) = -\frac{\left(d_{L^2}(x, \Phi)\right)^2}{2}.$$

**Corollary**

*For measurable* $A \subset \mathcal{P}(\mathbb{R})$

$$\lim_{\epsilon \to 0} \epsilon \log P(\mu_\epsilon \in A) = -\frac{\left(d_{\mathcal{W}}(\lambda_{\llcorner [0,1]}, A)\right)^2}{2}.$$