Super-Ricci Flows for Metric Measure Spaces

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OT and BM on Manifolds with $\mathrm{Ric} \geq 0$

Nonnegative Ricci curvature implies that – in many respects – optimal transports, heat flows, and Brownian motions behave as nicely as on Euclidean spaces. For instance

- Heat kernel comparison

$$p_t(x,y) \geq (4\pi t)^{-n/2} \exp\left(-\frac{d^2(x,y)}{4t}\right)$$

- Li-Yau estimates
- Gradient estimates

$$|\nabla P_t u| \leq P_t(|\nabla u|)$$

- Transport estimates

$$W(P_t\mu, P_t\nu) \leq W(\mu,\nu)$$

- $\forall x, y :\ \exists$ coupled Brownian motions $(X_s, Y_s)_{s\geq 0}$ starting at (x,y) s.t. $\mathbb{P}$-a.s. for all $s \geq 0$

$$d(X_s, Y_s) \leq d(x,y)$$

Indeed, $\mathrm{Ric} \geq 0$ is necessary and sufficient for each of the latter properties.

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Indeed, $\mathrm{Ric} \geq 0$ is necessary and sufficient for each of the latter properties.

Among the applications:
'Market Fragility, Systemic Risk, and Ricci Curvature' (Sandhu et al. 2015)
'Ricci curvature and robustness of cancer networks' (Tannenbaum et al. 2015)

Super Ricci Flows

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Two main examples

- Static manifolds with $\mathrm{Ric} \geq 0$ ('elliptic case')
- Ricci flows $\mathrm{Ric}_t = -\frac{1}{2}\partial_t g_t$ ('minimal super-Ricci flows')

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Goal.

- Extend Sturm‘06, Lott/Villani’09, Ambrosio/Gigli/Savare’11-’14, Erbar/Kuwada/Sturm’14 (‘Synthetic Ricci bounds for metric measure spaces’) to time-dependent setting

  Extend Bakry/Emery’83 (‘Γ-calculus’) to time-dependent setting
- Extend McCann/Topping’10, Lott’09, Arnaudon/Coulibaly/Thalmaier’08, Kuwada/Philipowski’11, X.-D.Li’14, Kleiner/Lott’14, Haslhofer/Naber’15 (‘OT and BM on time-dependent manifolds’) to singular setting

Super Ricci Flows for Diffusion Operators

Given a family $(\mathrm{L}_t)_{t\in[0,T)}$ of diffusion operators defined on a common algebra $\mathcal{A}$

e.g. $L_t = \Delta_t$ Laplace-Beltrami w.r.t. g_t, $\mathcal{A} = C_c^\infty(M)$, $X =$ Riem.mfd. M

For each t, define

- Square field operator $\Gamma_t(f,g) = \frac{1}{2}[L_t(fg) - f L_t g - g L_t f]$
- Γ_2-operator $\Gamma_{2,t}(f,g) = \frac{1}{2}[L_t \Gamma_t(f,g) - \Gamma_t(f, L_t g) - \Gamma_t(g, L_t f)]$

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We say that $(\mathrm{L}_t)_{t\in[0,T)}$ is a super-Ricci flow if

$$\Gamma_{2,t} \geq \frac{1}{2}\partial_t\Gamma_t.$$

It is a super-N-Ricci flow if

$$\Gamma_{2,t}(f) - \frac{1}{N}(L_tf)^2 \geq \frac{1}{2}\partial_t\Gamma_t(f).$$

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Definition consistent in Riemannian case! Note that $\Gamma_t(f,g) = \nabla_t f\, \nabla_t g$,

$$\Gamma_{2,t}(f,f) = \frac{1}{2}\Delta_t(|\nabla_t f|^2) - \nabla_t f\, \nabla_t \Delta_t f = \mathrm{Ric}_t(\nabla_t f, \nabla_t f) + \|\nabla_t^2 f\|_{HS}^2$$

and recall that (M, g_t), $t \in [0,T]$, is a super Ricci flow iff $\mathrm{Ric}_t \geq -\frac{1}{2}\partial_t g_t$.

Super Ricci Flows for Diffusion Operators

Theorem. The following are equivalent:

(i) $\Gamma_{2,t}(u) \geq \frac{1}{2}\partial_t \Gamma_t(u)$

(ii) $\Gamma_t(P_t^s u) \leq P_t^s(\Gamma_s(u))$

Here $(P_t^s)_{0\leq s\leq t<T}$ is the propagator for $(\mathrm{L}_t)_t$, i.e. a 2-parameter family of linear operators on $\mathcal{A}$ satisfying for all $s \leq r \leq t$ and all $u \in \mathcal{A}$

- $P_t^r(P_r^s u) = P_t^s u$
- $\partial_t P_t^s u = \mathrm{L}_t P_t^s u$
- $\partial_s P_t^s u = -P_t^s(\mathrm{L}_s u)$

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Proof. Differentiating the function $q_r := P_t^r \Gamma_r(P_r^s u)$ w.r.t. $r \in (s,t)$ yields

$$\begin{aligned}
\partial_r q_r &= P_t^r\Big(-\mathrm{L}_r \Gamma_r(P_r^s u) + (\partial_r \Gamma_r)(P_r^s u) + 2\Gamma_r(\partial_r P_r^s u, P_r^s u)\Big) \\
&= P_t^r\Big(-\mathrm{L}_r \Gamma_r(v) + \partial_r \Gamma_r(v) + 2\Gamma_r(\mathrm{L}_r v, v)\Big) \\
&= P_t^r\Big(-2\Gamma_{2,r}(v) + \partial_r \Gamma_r(v)\Big)
\end{aligned}$$

where $v = P_r^s u$. Thus (i) implies $\partial_r q_r \leq 0$ for all $r \in [s,t]$ which in turn yields $q_t \leq q_s$. This is (ii).

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Theorem. The following are equivalent:

(i) $\Gamma_{2,t}(u) - \frac{1}{N}(\mathrm{L}_t u)^2 \geq \frac{1}{2}\partial_t \Gamma_t(u)$

(ii) $\Gamma_t(P_t^s u) + \frac{2}{N}\int_s^t (P_t^r \mathrm{L}_r P_r^s u)^2 dr \leq P_t^s(\Gamma_s(u))$

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Possible extensions:
L_t discrete Laplacian, general Markov operator

In the sequel:
L_t Laplacian on time-dependent metric measure space (X, d_t, m_t)

Heat Flow on (Static) Metric Measure Spaces

(X, d) complete separable metric space, m locally finite measure

Heat equation on X

- either as gradient flow on $L^2(X, m)$ for the **energy**

$$\mathcal{E}(u) = \frac{1}{2}\int_X |\nabla u|^2 \, dm = \liminf_{v \to u \text{ in } L^2} \frac{1}{2}\int_X (\mathrm{lip}_x v)^2 \, dm(x)$$

  with $|\nabla u|$ = minimal weak upper gradient

- or as gradient flow on $\mathcal{P}_2(X, d)$ for the **relative entropy**

$$\mathrm{Ent}(u) = \int_X u \log u \, dm.$$

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Theorem (Ambrosio/Gigli/Savare).

For arbitrary metric measure spaces (X, d, m) satisfying $CD(K, \infty)$ both approaches coincide.

$\mathbb{R}^n$: Jordan/Kinderlehrer/Otto
Riemann (M, g): Ohta, Savare, Villani, Erbar
Finsler (M, F, m): Ohta/Sturm
Alexandrov spaces: Gigli/Kuwada/Ohta

Neumann Laplacian: Lierl/Sturm
Wiener space: Fang/Shao/Sturm
Heisenberg group: Juillet
Discrete spaces: Maas, Mielke
Levy semigroups: Erbar

The Curvature-Dimension Condition $CD(K,\infty)$

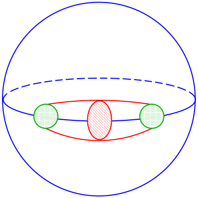
Definition. $CD(K,\infty)$ or $\mathrm{Ric}(X,\mathsf{d},m) \geq K$

$\iff \quad \forall \mu_0,\mu_1 \in \mathcal{P}_2(X)$: $\exists$ geodesic $(\mu_t)_t$ s.t. $\forall t \in [0,1]$:

$$\mathrm{Ent}(\mu_t|m) \leq (1-t)\mathrm{Ent}(\mu_0|m) + t\,\mathrm{Ent}(\mu_1|m) - \frac{K}{2}\,t(1-t)\,W_2^2(\mu_0,\mu_1)$$

$$\mathrm{Ent}(\nu|m) = \begin{cases} \int_X \rho\log\rho\,dm & \text{, if } \nu = \rho\cdot m \\ +\infty & \text{, if } \nu \not\ll m \end{cases}$$

$$W_2(\mu_0,\mu_1) = \inf_q \left[\int_{X\times X} d^2(x,y)\,d\,q(x,y)\right]^{1/2}$$



The Curvature-Dimension Condition $CD(K, N)$

Def. A metric measure space (X, d, m) satisfies $CD(K, N)$

$\Longleftrightarrow$ S:= Ent(.) is (K,N)-convex on $\mathcal{P}_2(X, d)$

$\Longleftrightarrow$ $\text{Hess}\, S - \frac{1}{N} (\nabla S \otimes \nabla S) \geq K$

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Riemannian manifolds:

$$CD(K, N) \quad \Longleftrightarrow \quad \mathrm{Ric}_M \geq K \quad \text{and} \quad \dim_M \leq N$$

Weighted Riemannian spaces (M, d, m) with $dm = e^{-V} dvol$:

$$\mathrm{Ric}_M + \mathrm{Hess}\, V - \frac{1}{N-n} DV \otimes DV \;\geq\; K \quad \text{and} \quad \dim_M \leq N$$

Further examples: Ricci limit spaces, Alexandrov spaces, Wiener space $(K = 1, N = \infty)$.

Constructions: Products, cones, suspensions, warped products.

Time-dependent Metric Measure Spaces

For the sequel: $(X, d_t, m_t)_{t\in I}$ with $\frac{d_t(x,y)}{d_s(x,y)} \le C$ and $m_t(dx) = e^{-f_t(x)} m_0(dx)$ where

$$f_t(x) - f_s(x) \le C, \qquad f_t(x) - f_t(y) \le C \cdot d_t(x,y)$$

Assume $\forall t \in I$: the metric measure space (X, d_t, m_t) is infinitesimally Hilbertian (i.e. the energy $\mathcal{E}_t$ is quadratic) and satisfies $\mathrm{CD}(K,N)$

Thus $\forall t \in I$: Dirichlet form $\mathcal{E}_t$, Laplacian Δ_t, squared gradient $\Gamma_t(u) = |\nabla_t u|^2$.

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Theorem $\forall (s, T) \subset I$

- $\forall h \in L^2 : \exists! u_t = P_t^s h$ which solves $\partial_t u_t = \Delta_t u_t$ on $(s, T) \times X$ and $u_s = h$

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- $\exists$ kernel $p_t^s(x,y)$ s.t. $P_t^s h(x) = \int p_t^s(x,y) h(y)\, m_s(dy)$
- All solutions to heat equation are Holder continuous, nonnegative solutions satisfy parabolic Harnack inequality

Lions/Magenes, Renardy/Rogers, Lierl/Saloff-Coste

The Dual Propagator

Def $\hat{P}_t^s : \mathcal{P} \to \mathcal{P}$ by duality

$$\int ud\left(\hat{P}_t^s \mu\right) = \int \left(P_t^s u\right) d\mu \qquad (\forall u \in \mathcal{C}_b, \forall \mu \in \mathcal{P})$$

Then $\nu_s = \hat{P}_t^s \mu$ solves

$$-\partial_s \nu_s = \hat{\Delta}_s \nu_s, \qquad \nu_t = \mu$$

where $\int ud(\hat{\Delta}_s \nu_s) = \int (\Delta_s u) d\nu_s$. Moreover, it is the upward gradient flow for the Boltzmann entropy in the time-dependent Wasserstein space $(\mathcal{P}, W_s)$:

$$\dot{\nu}_s = \nabla_s Ent(\nu_s | m_s)$$

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Assume $|\partial_s f_s| \leq C$. Then

- density $w_s = \frac{d\nu_s}{dm_s}$ solves $-\partial_s w_s = \Delta_s w_s - (\partial_s f_s) \cdot w_s$
- all solutions to this equation are Holder continuous, nonnegative solutions satisfy parabolic Harnack inequality
- $\forall (t, x)$: the function $(s, y) \mapsto p_t^s(x, y)$ solves this equation

Super Ricci Flows for Metric Measure Spaces

Given a 1-parameter family of metric measure spaces (X, d_t, m_t). Consider the function

$$S: \ (0, T) \times \mathcal{P}(X) \to (-\infty, \infty], \quad (t, \mu) \mapsto S_t(\mu) = \mathrm{Ent}(\mu | m_t)$$

where $\mathcal{P}(X)$ is equipped with the 1-parameter family of metrics W_t (= L^2-Wasserstein metrics w.r.t. d_t).

Definition.

$(X, d_t, m_t)_{t \in (0,T)}$ is a super-Ricci flow iff for all μ^0, μ^1 and a.e. t there exists a W_t-geodesic $(\mu^a)_{a \in [0,1]}$ s.t.

$$\partial_a S_t(\mu^0) - \partial_a S_t(\mu^1) \leq \frac{1}{2} \partial_t W_t^2(\mu^0, \mu^1).$$

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$$\partial_a S_t(\mu^0) - \partial_a S_t(\mu^1) \le \frac{1}{2}\partial_t W_t^2(\mu^0, \mu^1).$$

Consistent with the Riemannian definition: a family of Riemannian manifolds (M, g_t), $t \in (0, T)$, evolves according to super-Ricci flow iff

$$\mathrm{Ric}_t \ge -\frac{1}{2}\partial_t g_t.$$

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$$\partial_a S_t(\mu^0) - \partial_a S_t(\mu^1) \leq \frac{1}{2}\partial_t W_t^2(\mu^0,\mu^1) - \frac{1}{N}\Big(S_t(mu^0) - S_t(\mu^1)\Big)^2.$$

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Theorem.

The family of all super-N-Ricci flows $(X, d_t, m_t)_{t\in(0,T)}$ with uniform bounds for the diameter and for the growth of d_t and m_t is compact.

Super Ricci Flows for Metric Measure Spaces

The following are equivalent:

- $(X, d_t, m_t)_{t\in(0,T)}$ is a super-Ricci flow
- $W_s(\hat{P}^s_t\mu, \hat{P}^s_t\nu) \leq W_t(\mu,\nu)$
- $|\nabla_t(P^s_t u)|^2 \leq P^s_t(|\nabla_s u|^2)$
- $\Gamma_{2,t} \geq \frac{1}{2}\partial_t\Gamma_t$

Here and in the sequel we assume

- $\log \frac{d_t(x,y)}{d_s(x,y)}$ uniformly bounded and Lip in t
- $\log \frac{m_t(dx)}{m_s(dx)}$ uniformly bounded and Lip in x,
- (X, d_t, m_t) is infinitesimally Hilbertian for each t (i.e. Cheeger energy is quadratic).

Super Ricci Flows for Metric Measure Spaces

The following are equivalent:

- $(X, d_t, m_t)_{t\in(0,T)}$ is a super-Ricci flow
- $W_s^\infty(\hat{P}_t^s\mu, \hat{P}_t^s\nu) \le W_t^\infty(\mu,\nu)$
- $\forall x, y$ there exist coupled backward Brownian motions $(X_s)_{s\le t}$, $(Y_s)_{s\le t}$ starting at x, y at time t s.t. $\mathbb{P}$-a.s. for all $s \le t$

$$d_s(X_s, Y_s) \le d_t(x, y)$$

- $|\nabla_t P_t^s u| \le P_t^s(|\nabla_s u|)$
- $\Gamma_{2,t} \ge \frac{1}{2}\partial_t \Gamma_t$

Thank You For Your Attention!