German-Japanese conference on

Stochastic Analysis and Applications

Continuity and bounds of the transition density functions of the solutions to path-dependent stochastic differential equations

Seiichiro Kusuoka

(Tohoku University)

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1. Introduction

Recently I study, by means of Stochastic Analysis, the Gaussian two-sided bounds of the fund. solution and the Hölder (Lipschitz) continuity of the solution to the parabolic partial differential equation:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) &= \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} a_{ij}(t,x)\dfrac{\partial^2}{\partial x_i \partial x_j}u(t,x) \\ &\quad + \displaystyle\sum_{i=1}^{d} b_i(t,x)\dfrac{\partial}{\partial x_i}u(t,x) + c(t,x)u(t,x) \\ u(0,x) &= f(x) \end{cases}$$

where a, b, c have bad regularities

and a is uniformly positive definite.

In this talk we consider stochastic differential equations with path-dependent drift terms.

Consider the path-dependent SDE:

$$\begin{cases} dY_t^x &= \sigma(t, Y_t^x)dB_t + b(t, Y_t^x)dt + \tilde{b}(t, Y^x)dt \\ Y_0^x &= x. \end{cases}$$

where

σ: $[0, T] \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$, measurable

b: $[0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, measurable

$\tilde{b}$: $[0, T] \times C([0, T]; \mathbb{R}^d) \to \mathbb{R}^d$ such that

$$\mathscr{B}([0, T]) \otimes \mathscr{B}(C([0, T]; \mathbb{R}^d))/\mathscr{B}(\mathbb{R}^d)\text{-measurable}$$

Let

π_t: $C([0,T];\mathbb{R}^d) \to C([0,T];\mathbb{R}^d)$

defined by $(\pi_t w) = w(\cdot \wedge t)$

$\mathscr{B}_t(C([0,T];\mathbb{R}^d)) := \pi_t^{-1}[\mathscr{B}(C([0,T];\mathbb{R}^d))]$

Assumption

$\tilde{b}(t,\cdot)$: $\mathscr{B}_t(C([0,T];\mathbb{R}^d))/\mathscr{B}(\mathbb{R}^d)$-m'ble $\forall t \in [0,T]$,

$$\|\tilde{b}\|_\infty := \sup_{(t,w)\in[0,T]\times C([0,T];\mathbb{R}^d)} |\tilde{b}(t,w)| < \infty.$$

Let $a(t,x) := \sigma(t,x)^T \sigma(t,x)$ for $(t,x) \in [0,T] \times \mathbb{R}^d$.

Assumption

$$\Lambda^{-1} I \le a(t,x) \le \Lambda I, \quad (t,x) \in [0,T] \times \mathbb{R}^d,$$

$\exists M_a$: a conti. func. on $[0,\infty)$ such that $M_a(0) = 0$,

$$\sup_{t \in [0,T]} |a(t,x) - a(t,y)| \le M_a(|x-y|),$$

$$\int_0^1 \frac{1}{\tilde{r}} \left(\int_0^{\tilde{r}} \frac{1}{r} M_a(r) dr \right) d\tilde{r} < \infty,$$

$\|b\|_\infty := \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |b(t,x)| < \infty$

$\exists M_b$: a conti. func. on $[0,\infty)$ such that $M_b(0) = 0$,

$$\sup_{t \in [0,T]} |b(t,x) - b(t,y)| \le M_b(|x-y|), \ \int_0^1 \frac{1}{r} M_b(r) dr < \infty.$$

Consider the SDE of Markov type:

$$\begin{cases} dX_t^x &= \sigma(t, X_t^x)dB_t + b(t, X_t^x)dt \\ X_0^x &= x. \end{cases}$$

Then, the transition probability density function $q(s, x; t, y)$ of X and the estimates

$$\frac{C_-}{(t-s)^{d/2}} \exp\left(-\frac{\gamma_-|x-y|^2}{t-s}\right) \le q(s, x; t, y) \le \frac{C_+}{(t-s)^{d/2}} \exp\left(-\frac{\gamma_+|x-y|^2}{t-s}\right),$$

$$|\nabla_x q(s, x; t, y)| \le \frac{C_+}{(t-s)^{(d+1)/2}} \exp\left(-\frac{\gamma_+|x-y|^2}{t-s}\right)$$

for $(s, x), (t, y) \in [0, T] \times \mathbb{R}^d$ s.t. $s < t$.

These are obtained by parametrix.

2. Perturbation

$$\mathcal{E}(t, X^x) := \exp\left(\int_0^t \langle \tilde{b}_\sigma(s, X^x), dB_s \rangle - \frac{1}{2}\int_0^t |\tilde{b}_\sigma(s, X^x)|^2 ds\right)$$

where $\tilde{b}_\sigma(t, w) := \sigma(t, w_t)^{-1}\tilde{b}(t, w)$.

Then, by the Girsanov transform

$$E\left[f(Y_t^x)\right] = E\left[f(X_t^x)\mathcal{E}(t, X^x)\right].$$

We regard Y as a perturbation of X,

and consider estimates of the density of Y_t.

We have the existence of the density function $p(0, x; t, y)$ of Y_t^x and

$$p(0, x; t, y) = q(0, x; t, y) E^{X_t^x = y} [\mathcal{E}(t, X^x)]$$

(for a.e. y w.r.t. the Lebesgue measure)
where $E^{X_t^x = y}[\cdot]$ is the expectation w.r.t. the regular conditional probability measure given by $X_t^x = y$.

We have known that $q(s, x; t, y)$ satisfies the Gaussian estimates.
Hence, to see the Gaussian estimates of $p(0, x; t, y)$ it is sufficient to have estimates of $E^{X_t^x = y} [\mathcal{E}(t, X^x)]$.

Let $p^s(0,x;t,y) := E\left[\mathcal{E}(s,X^x)q(s,X_s^x;t,y)\right]$

for $s \in [0,t)$ and $x,y \in \mathbb{R}^d$.

By the Markov property we have

$$E\left[\mathcal{E}(s,X^x)q(s,X_s^x;t,y)\right] = q(0,x;t,y)E^{X_t^x=y}\left[\mathcal{E}(s,X^x)\right]$$

for $s \in [0,t)$.

Lemma For each $r \in \mathbb{R}$,

$$\sup_{s\in[0,t)} E\left[\mathcal{E}(s,X^x)^r q(s,X_s^x;t,y)\right] \le Ct^{-d/2}\exp\left(-\frac{\gamma|x-y|^2}{t}\right)$$

Proof. Since

$$
\begin{aligned}
&-\frac{\partial}{\partial s} q(s,x;t,y) = \frac{\partial}{\partial t} q(s,x;t,y) \\
&= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} q(s,x;t,y) + \sum_{i=1}^{d} b_i(t,x)\frac{\partial}{\partial x_i} q(s,x;t,y),
\end{aligned}
$$

by Itô's formula we have

$$
\begin{aligned}
&E[\mathcal{E}(s,X^x)^r q(s,X_s^x;t,y)] - q(0,x;t,y) \\
&= \frac{r(r-1)}{2}\int_0^s E\left[\mathcal{E}(u,X^x)^r q(u,X_u^x;t,y)|\tilde{b}_\sigma(u,X)|^2\right] du \\
&\quad + \frac{r}{2}\int_0^s E\left[\mathcal{E}(u,X^x)^r \langle [\nabla q(u,\cdot;t,y)](X_u^x), \tilde{b}(u,X)\rangle\right] du.
\end{aligned}
$$

□

Hence, we have the following.

Lemma $p(0,x;t,y) = \lim_{s \nearrow t} p^s(0,x;t,y)$

Proposition For each $r \in \mathbb{R}$,

$$q(0,x;t,y)E^{X_t^x=y}\left[\mathcal{E}(t,X^x)^r\right] \le Ct^{-d/2}\exp\left(-\frac{\gamma|x-y|^2}{t}\right)$$

Since $p(0,x;t,y) = q(0,x;t,y)E^{X_t^x=y}\left[\mathcal{E}(t,X^x)\right]$,

we obtain the Gaussian upper bound of $p(0,x;t,y)$.

Since

$$1 = E^{X_t^x = y}\left[\mathcal{E}(t, X^x)^{1/2}\mathcal{E}(t, X^x)^{-1/2}\right]^2$$
$$\leq E^{X_t^x = y}\left[\mathcal{E}(t, X^x)\right] E^{X_T^x = y}\left[\mathcal{E}(t, X^x)^{-1}\right],$$

we have

$$E^{X_t^x = y}\left[\mathcal{E}(t, X^x)^{-1}\right]^{-1} \leq E^{X_t^x = y}\left[\mathcal{E}(t, X^x)\right].$$

Hence,

$$p(0, x; t, y) = q(0, x; t, y) E^{X_t^x = y}\left[\mathcal{E}(t, X^x)\right]$$
$$\geq \frac{q(0, x; t, y)^2}{q(0, x; t, y) E^{X_t^x = y}\left[\mathcal{E}(t, X^x)^{-1}\right]}$$
$$\geq \frac{C_-^2 t^{-d} \exp\left(-2\gamma_- |x - y|^2 / t\right)}{C t^{-d/2} \exp\left(-\gamma |x - y|^2 / t\right)}$$

Theorem

There exist positive constants $\tilde{C}_-$, $\tilde{\gamma}_-$ $\tilde{C}_+$ and $\tilde{\gamma}_+$ depending on T, d, Λ, $\|b\|_\infty$, $\|\tilde{b}\|_\infty$, M_a and M_b such that

$$\tilde{C}_- t^{-d/2} \exp\left(-\frac{\tilde{\gamma}_- |x-y|^2}{t}\right) \leq p(0,x;t,y) \leq \tilde{C}_+ t^{-d/2} \exp\left(-\frac{\tilde{\gamma}_+ |x-y|^2}{t}\right)$$

for $t \in [0,T]$ and $x, y \in \mathbb{R}^d$.

Next we consider the continuity of

the density function.

Theorem Let $t \in (0,T]$ be fixed. Assume that $\exists \beta \in [0,1]$ and a conti. func. ρ s.t. $\rho(0)=0$,

$$|q(s,x;t,y_1)-q(s,x;t,y_2)| \le (t-s)^{-(d+\beta)/2}\rho(|y_1-y_2|)$$

$$|\nabla_x q(s,x;t,y_1)-\nabla_x q(s,x;t,y_2)| \le (t-s)^{-(d+2\beta)/2}\rho(|y_1-y_2|)$$

for $s \in [0,t)$ and $x, y_1, y_2 \in \mathbb{R}^d$. Then, for any $\varepsilon \in (0,1)$,

$$|p(0,x;t,y_1)-p(0,x;t,y_2)| \le \tilde{C} t^{-[d+\beta(1-\varepsilon)]/2}\rho(|y_1-y_2|)^{1-\varepsilon} \times \exp\left(-\frac{\tilde{\gamma}\min\{|x-y_1|^2,|x-y_2|^2\}}{t}\right).$$

Remark: Consider the case that X^x is a d-dimensional standard Brownian motion. In this case,

$$q(s, x; t, y) = (2\pi)^{-d/2}(t-s)^{-d/2} \exp\left(-\frac{|x-y|^2}{2(t-s)}\right).$$

Then,

$$|q(s, x; t, y_1) - q(s, x; t, y_2)| \leq C(t-s)^{-(d+1)/2}|y_1 - y_2|$$

$$|\nabla_x q(s, x; t, y_1) - \nabla_x q(s, x; t, y_1)| \leq C'(t-s)^{-(d+2)/2}|y_1 - y_2|.$$

Hence, we let $\beta := 1$ and $\rho(r) := Cr$.

Proof. By the interpolation inequality,

$$\left|\nabla_\xi q(s,\xi;t,y_1) - \nabla_\xi q(s,\xi;t,y_2)\right|$$
$$\le C(t-s)^{-d/2-\beta(1-\varepsilon)}\rho(|y_1-y_2|)^{1-\varepsilon}$$
$$\times \max_{\eta=y_1,y_2} \exp\left(-\frac{\varepsilon\gamma_+|\xi-\eta|^2}{t-s}\right)$$
$$|q(0,x;t,y_1) - q(0,x;t,y_2)|$$
$$\le Ct^{-d/2-\beta(1-\varepsilon)/2}\rho(|y_1-y_2|)^{1-\varepsilon} \max_{\eta=y_1,y_2} \exp\left(-\frac{\varepsilon\gamma_+|\xi-\eta|^2}{t}\right)$$

$$
\begin{aligned}
&|p^s(0,x;t,y_1) - p^s(0,x;t,y_2)| \\
&= |E\left[\mathcal{E}(s,X^x)q(s,X^x_s;t,y_1)\right] - E\left[\mathcal{E}(s,X^x)q(s,X^x_t;t,y_2)\right]| \\
&\leq |q(0,x;t,y_1) - q(0,x;t,y_2)| \\
&\quad + \frac{1}{2}\left|\int_0^s E\left[\mathcal{E}(u,X^x)\langle[\nabla q(u,\cdot;t,y_1)(X^x_u)\right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left. - \nabla q(u,\cdot;t,y_2)(X^x_u)], b(u,X)\rangle\right] du\right| \\
&\leq C t^{-d/2-\beta(1-\varepsilon)/2} \rho(|y_1 - y_2|)^{1-\varepsilon} \max_{\eta=y_1,y_2} \exp\left(-\frac{\varepsilon\gamma_+|x-\eta|^2}{t}\right) \\
&\quad + C\rho(|y_1-y_2|)^{1-\varepsilon} \int_0^s (t-u)^{-d/2-\beta(1-\varepsilon)} \\
&\qquad\qquad \times E\left[\mathcal{E}(u,X^x) \max_{\eta=y_1,y_2} \exp\left(-\frac{\varepsilon\gamma_+|X^x_u-\eta|^2}{t-u}\right)\right] du
\end{aligned}
$$

□

3. Path-dependent drift perturbation of symmetric diffusion processes

$\sigma \in C_b^1(\mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d)$ s.t.

$\exists M_\sigma \in C([0,\infty); \mathbb{R}_+)$ satisfying $M_\sigma(0) = 0$,

$$|\nabla\sigma(x) - \nabla\sigma(y)| \le M_\sigma'(|x-y|), \text{ and } \int_0^1 \frac{M_\sigma'(r)}{r} dr < \infty.$$

Assume $\Lambda^{-1} I \le \sigma(x)\sigma(x)^T \le \Lambda I$.

Define $b \in C_b(\mathbb{R}^d; \mathbb{R}^d)$ by

$$\begin{aligned} b_i(x) &:= \frac{1}{2} \sum_{j=1}^d \frac{\partial}{\partial x_j} \left(\sum_{k=1}^d \sigma_{jk}(x)\sigma_{ik}(x) \right) \\ &= \frac{1}{2} \sum_{j,k=1}^d \left(\frac{\partial \sigma_{jk}}{\partial x_j}(x)\sigma_{ik}(x) + \sigma_{jk}(x) \frac{\partial \sigma_{ik}}{\partial x_j}(x) \right). \end{aligned}$$

Consider

$$\begin{cases} dX_t^x &= \sigma(X_t^x)dB_t + b(X_t^x)dt \\ X_0^x &= x. \end{cases}$$

The results by parametrix imply that the transition probability density function $q(t,x,y)$ exists and

$$\frac{C_-}{t^{d/2}} \exp\left(-\frac{\gamma_-|x-y|^2}{t}\right) \le q(t,x,y) \le \frac{C_+}{t^{d/2}} \exp\left(-\frac{\gamma_+|x-y|^2}{t}\right)$$

$$|\nabla_x q(t,x,y)| \le Ct^{-(d+1)/2} \exp\left(-\frac{\gamma|x-y|^2}{t}\right)$$

$$\left|\nabla_x^2 q(t,x,y)\right| \le Ct^{-(d+2)/2} \exp\left(-\frac{\gamma|x-y|^2}{t}\right)$$

for $t \in [0,T]$ and $x, y \in \mathbb{R}^d$.

Since

$$\frac{1}{2}\sum_{i,j,k=1}^{d}\sigma_{ik}(x)\sigma_{jk}(x)\frac{\partial^2}{\partial x_i \partial x_j}+\frac{1}{2}\sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$$
$$=\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{d}\sigma_{jk}(x)\sigma_{ik}(x)\right)\frac{\partial}{\partial x_i},$$

the generator is symmetric on $L^2(\mathbb{R}^d, dx)$. Hence,

$$q(t,x,y)=q(t,y,x).$$

Therefore,

$$|\nabla_x \nabla_y q(t,x,y)| \le C t^{-(d+2)/2} \exp\left(-\frac{\gamma|x-y|^2}{t}\right).$$

Let $\tilde{b}$ be an $[0,T] \times C([0,T];\mathbb{R}^d) \to \mathbb{R}^d$ s.t.
$\mathscr{B}([0,T]) \otimes \mathscr{B}(C([0,T];\mathbb{R}^d))/\mathscr{B}(\mathbb{R}^d)$-measurable
and $\tilde{b}(t,\cdot)$ is $\mathscr{B}_t(C([0,T];\mathbb{R}^d))/\mathscr{B}(\mathbb{R}^d)$-measurable.

Consider the path-dependent stochastic differential equation

$$\begin{cases} dY_t^x & = \sigma(t, Y_t^x) dB_t + \tilde{b}(t, Y^x) dt \\ Y_0^x & = x. \end{cases}$$

We assume

$$\|\tilde{b}\|_\infty := \sup_{(t,w) \in [0,T] \times C([0,T];\mathbb{R}^d)} |\tilde{b}(t,w)| < \infty.$$

Corollary

Then, the distribution of Y_t^x has the density function $p(0,x;t,\cdot)$ which satisfies

$$\tilde{C}_- t^{-d/2} \exp\left(-\frac{\tilde{\gamma}_-|x-y|^2}{t}\right) \le p(0,x;t,y) \le \tilde{C}_+ t^{-d/2} \exp\left(-\frac{\tilde{\gamma}_+|x-y|^2}{t}\right)$$

and for any $\varepsilon \in (0,1]$

$$|p(0,x;t,y_1) - p(0,x;t,y_2)| \le \tilde{C} t^{-(d+1)/2} |y_1 - y_2|^{1-\varepsilon} \exp\left(-\frac{\tilde{\gamma}\min\{|x-y_1|^2, |x-y_2|^2\}}{t}\right).$$

4. Further consideration to less regular diffusion coefficients

$$\begin{cases} dX_t^x &= \sigma(t, X_t^x)dB_t, \quad t \in [0, T] \\ X_0^x &= x. \end{cases}$$

where $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$.

Assume σ is smooth and consider a priori estimates.

Denote the transition probability density function of $X^\cdot$ by $q(s, x; t, y)$.

Let $a(t, x) := \sigma(t, x)\sigma(t, x)^T$ for $(t, x) \in [0, T] \times \mathbb{R}^d$.

We assume

$$\Lambda^{-1} I \leq a(t,x) \leq \Lambda I, \quad (t,x) \in [0,\infty) \times \mathbb{R}^d,$$

$$\frac{C_-}{(t-s)^{d/2}} \exp\left(-\frac{\gamma_-|x-y|^2}{t-s}\right) \leq q(s,x;t,y) \leq \frac{C_+}{(t-s)^{d/2}} \exp\left(-\frac{\gamma_+|x-y|^2}{t-s}\right).$$

Moreover, we assume

$$\sum_{i,j=1}^{d} \sup_{t\in[0,T]} \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} a_{ij}(s,x) \right|^{\theta} e^{-m|x|} dx \leq M$$

where θ is a constant in $[d,\infty) \cap (2,\infty)$, m and M are nonnegative constants.

We consider the perturbation by path-dependent drift.

Let $\tilde{b} : [0,T] \times C([0,T];\mathbb{R}^d) \to \mathbb{R}^d$ s.t.

$\tilde{b}$ is $\mathscr{B}([0,T]) \otimes \mathscr{B}(C([0,T];\mathbb{R}^d))/\mathscr{B}(\mathbb{R}^d)$-measurable and $\tilde{b}(t,\cdot)$ is $\mathscr{B}_t(C([0,T];\mathbb{R}^d))/\mathscr{B}(\mathbb{R}^d)$-measurable.

Consider the path-dependent stochastic differential equation

$$\begin{cases} dY_t^x &= \sigma(t, Y_t^x)dB_t + \tilde{b}(t, Y^x)dt \\ Y_0^x &= x. \end{cases}$$

We assume the boundedness of $\tilde{b}$. As we have seen, there exists a density function $p(0,x;t,\cdot)$ of Y_t^x.

Define $\mathcal{E}(t, X^x)$ and $p^s(0,x;t,y)$ as before.

Now we see the Gaussian two-sided bounds of Y_t^x. The disadvantage of the setting in this section is that we do not have the estimate

$$|\nabla_x q(s,x;t,y)| \le \frac{C_+}{(t-s)^{(d+1)/2}} \exp\left(-\frac{\gamma_+|x-y|^2}{t-s}\right).$$

A problem appears, when we estimate

$$\left|\int_0^t E\left[\mathcal{E}(s,X^x)^r \langle [\nabla q(s,\cdot;t,y)](X_s^x), \tilde{b}(s,X)\rangle\right] ds\right|.$$

$$
\begin{aligned}
&\int_{t_1}^{t_2} E\left[\mathcal{E}(s,X^x)^r \left|\langle[\nabla q(s,\cdot;t,y)](X_s^x),\tilde{b}(s,X)\rangle\right|\right] ds \\
&\leq \|\tilde{b}\|_\infty \int_{t_1}^{t_2} E\Big[\mathcal{E}(s,X^x)^r q(s,X_s^x;t,y)^{1/4} \\
&\qquad\qquad \times \left([s(t-s)]^{-1/2} q(s,X_s^x;t,y)\right)^{1/4} \\
&\qquad\qquad \times [s(t-s)]^{1/8} \frac{|[\nabla q(s,\cdot;t,y)](X_s^x)|}{q(s,X_s^x;t,y)^{1/2}}\Big] ds \\
&\leq \|\tilde{b}\|_\infty \left(\int_{t_1}^{t_2} E\left[\mathcal{E}(s,X^x)^{4r} q(s,X_s^x;t,y)\right] ds\right)^{1/4} \\
&\qquad \times \left(\int_{t_1}^{t_2} [s(t-s)]^{-1/2} E\left[q(s,X_s^x;t,y)\right] ds\right)^{1/4} \\
&\qquad \times \left(\int_{t_1}^{t_2} [s(t-s)]^{1/4} E\left[\frac{|[\nabla q(s,\cdot;t,y)](X_s^x)|^2}{q(s,X_s^x;t,y)}\right] ds\right)^{1/2}
\end{aligned}
$$

To estimate

$$\int_0^t [s(t-s)]^{1/4} \int_{\mathbb{R}^d} \frac{\left|\nabla_\xi q(s,\xi;t,y)\right|^2}{q(s,\xi;t,y)} q(0,x;s,\xi) d\xi ds$$

we use the estimate

$$\int_0^t [s(t-s)]^{1/4} \int_{\mathbb{R}^d} \frac{\left|\nabla_\xi q(s,\xi;t,y)\right|^2}{q(s,\xi;t,y)} q(0,x;s,\xi) d\xi ds \\ \leq C t^{-d/2+1/2}(1+|\log t|).$$

By this estimate we have

$$\int_0^t E\left[\mathcal{E}(s,X^x)^r \left|\langle [\nabla q(s,\cdot;t,y)](X_s^x), \tilde{b}(s,X)\rangle\right|\right] ds \\ \leq C t^{-d/2} \exp\left(-\frac{\gamma|x-y|^2}{t}\right).$$

Proposition

There exist positive constants $\tilde{C}_-$, $\tilde{\gamma}_-$, $\tilde{C}_+$ and $\tilde{\gamma}_+$ depending on T, r, d, Λ, $\|\tilde{b}\|_\infty$, C_-, γ_-, C_+, γ_+, θ, m and M, such that

$$\tilde{C}_- t^{-d/2} \exp\left(-\frac{\tilde{\gamma}_-|x-y|^2}{t}\right) \leq p(0,x;t,y) \leq \tilde{C}_+ t^{-d/2} \exp\left(-\frac{\tilde{\gamma}_+|x-y|^2}{t}\right)$$

for $t \in (0,T]$ and $x, y \in \mathbb{R}^d$.

To obtain the Gaussian two-sided bound of the limit process, we perhaps need the continuity of $\tilde{b}$.

Thank you for your attention!

5. Remark in the case of starting at t_0

Let $t_0 \in [0,T)$ and $w \in C([0,t_0];\mathbb{R}^d)$. Consider

$$\begin{cases} dZ_t^{(t_0,w)} & = \sigma(t, Z_t^{(t_0,w)})dB_t + b(t, Z_t^{(t_0,w)})dt + \tilde{b}(t, Z^{(t_0,w)})dt \\ Z_s^{(t_0,w)} & = w_s, \quad s \in [0,t_0]. \end{cases}$$

We consider a reduction to the previous case. For $w \in C([0,t_0];\mathbb{R}^d)$ and $\tilde{w} \in C([0,T-t_0];\mathbb{R}^d)$ define $w \bowtie_{t_0} \tilde{w} \in C([0,T];\mathbb{R}^d)$ by

$$(w \bowtie_{t_0} \tilde{w})_t := \begin{cases} w_t, & t \in [0,t_0] \\ \tilde{w}_{t_0+t}, & t \in (t_0,T]. \end{cases}$$

Let $\tilde{b}^{(t_0,w)}(t,\tilde{w}) := \tilde{b}(t+t_0, w \bowtie_{t_0} \tilde{w})$

for $(t,\tilde{w}) \in [t_0,T]\times C([0,T-t_0];\mathbb{R}^d)$ such that $\tilde{w}_0 = w_{t_0}$.

Let $\sigma^{t_0}(t,x) := \sigma(t_0+t,x)$ and $b^{t_0}(t,x) := b(t_0+t,x)$.

Let $Y^{w_{t_0}}$ be the solution to

$$\begin{cases} dY_t^{w_{t_0}} &= \sigma^{t_0}(t,Y_t^{w_{t_0}})dB_t^{t_0} + b^{t_0}(t,Y_t^{w_{t_0}})dt \\ &\qquad\qquad +\tilde{b}^{(t_0,w)}(t,Y^{w_{t_0}})dt, \quad t \in [0,T-t_0] \\ Y_0^{w_{t_0}} &= w_{t_0}. \end{cases}$$

where $B^{t_0} = B_{t_0+\cdot}$.

Then, we have

$$Z^{(t_0,w)}(t_0+t) = Y^{w_{t_0}}(t) \text{ for } t \in [0,T-t_0].$$

Therefore, we obtain the same conclusion in this case.

6. Remark on unbounded drift terms.

Consider the one-dimensional SDE

$$\begin{cases} dX_t^x &= dB_t + \kappa X_t^x dt, \quad t \in [0,\infty) \\ X_0^x &= x. \end{cases}$$

where $\kappa \in \mathbb{R} \setminus \{0\}$. The solution X^x is explicitly written as

$$X_t^x = e^{\kappa t}\left(x + \int_0^t e^{-\kappa s} dB_s\right), \quad t \in [0,\infty).$$

Hence, the distribution of X_t^x is

$$\sqrt{\frac{e^{2\kappa t}-1}{4\pi\kappa}} \exp\left(-\frac{\kappa(y - xe^{\kappa t})^2}{e^{2\kappa t}-1}\right) dy.$$

Since

$$\exp\left(-\frac{\kappa(y - xe^{\kappa t})^2}{e^{2\kappa t} - 1}\right)\bigg|_{y=x} = \exp\left(-\frac{\kappa x^2(1 - e^{\kappa t})^2}{e^{2\kappa t} - 1}\right) \searrow 0$$

as $|x| \to \infty$,

the Gaussian lower estimate does NOT hold.

It follows

$$\exp\left(-\frac{\kappa(y - xe^{\kappa t})^2}{e^{2\kappa t} - 1}\right)\bigg|_{y=e^{\kappa t}x} = 1,$$

On the other hand, when $y = e^{\kappa t}x$,

$|y - x| = |e^{\kappa t} - 1| \cdot |x| \to \infty$ as $|x| \to \infty$.

Hence, the Gaussian upper estimate does NOT hold.

7. A remark of Hölder continuity in time

Let $T > 0$ and fix T.

Let $p(s, x; t, y)$ be a non-negative Borel-measurable function s.t.

$$\int_{\mathbb{R}^d} p(s, x; t, \xi) d\xi = 1$$
$$\int_{\mathbb{R}^d} p(s, x; t, \xi) p(t, \xi; T, y) d\xi = p(s, x; T, y).$$

If $p(s, x; t, y)$ is a transition probability density function of a Markov process, then the equations above hold.

Proposition

Assume that

$$|p(t,x_1;T,y) - p(t,x_2;T,y)| \le C(T-t)^{-d/2-\beta}|x_1 - x_2|^{\alpha}$$

$$p(s,x;t,y) \le \frac{C_+}{(t-s)^{d/2}} \exp\left(-\frac{\gamma_+|x-y|^2}{t-s}\right).$$

Then, for $s < t$

$$|p(t,x;T,y) - p(s,x;T,y)| \le \tilde{C}(T-t)^{-d/2-\beta}(t-s)^{\alpha/2}$$

where $\tilde{C}$ is a constant depending on

d, C, α, C_+ and γ_+.

Proof.

$$
\begin{aligned}
&|p(t,x;T,y) - p(s,x;T,y)| \\
&= \left| p(t,x;T,y) \int_{\mathbb{R}^d} p(s,x;t,\xi) d\xi - \int_{\mathbb{R}^d} p(s,x;t,\xi) p(t,\xi;T,y) d\xi \right| \\
&\leq \int_{\mathbb{R}^d} |p(t,x;T,y) - p(t,\xi;T,y)| p(s,x;t,\xi) d\xi \\
&\leq \frac{C_+ C(T-t)^{-d/2-\beta}}{(t-s)^{d/2}} \int_{\mathbb{R}^d} |\xi - x|^{\alpha} \exp\left(-\frac{\gamma_+ |\xi - x|^2}{t-s} \right) d\xi \\
&= \frac{C_+ C(T-t)^{-d/2-\beta}}{(t-s)^{d/2}} \\
&\qquad \times \int_{\mathbb{R}^d} \left(\frac{t-s}{\gamma_+} \right)^{\alpha/2} |\tilde{\xi}|^{\alpha} \exp\left(-|\tilde{\xi}|^2 \right) \left(\frac{t-s}{\gamma_+} \right)^{d/2} d\tilde{\xi} \\
&= \tilde{C}(T-t)^{-d/2-\beta} (t-s)^{\alpha/2}.
\end{aligned}
$$