# Modelling a random rubber band

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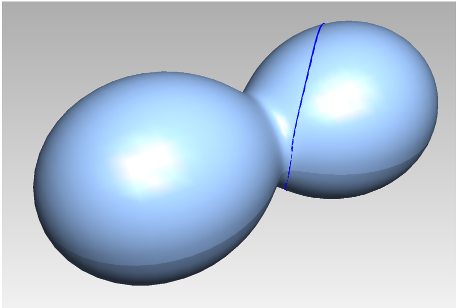
Joint work with L. Zambotti, Y. Bruned (Paris VI)

# Introduction

Rubber band constrained to lie on a Manifold: $u\colon S^1 \to \mathcal{M}$.



Deterministic part of evolution: “length shortening” / heat flow.
Suitable $L^2$-gradient flow for $\int g_u(\dot u, \dot u)\, dt$.

# Local coordinates

In local coordinates, heat flow given by:

$$\partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\,\partial_x u^\beta \partial_x u^\gamma =: (\triangle u)^\alpha \;.$$

Nonlinearity given by Christoffel symbols of Levi-Civita connection $\Gamma$.

Introduced by Eells-Sampson to show that every smooth map $\mathcal{N} \to \mathcal{M}$ is homotopic to a harmonic map.

Makes sense for any torsion-free connection (not just Levi-Civita): equilibria satisfy $\nabla_{\partial_x u} \partial_x u = 0$. Unit speed geodesics.

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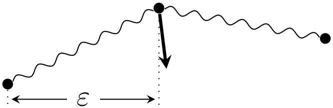
# Adding noise

Consider a discrete version:




Add independent noises to each bead.

Basic question: how strong should the noise be to create an effect of order $1$? Recall that noise of strength $\sigma$ yields displacement $\mathcal{O}(\sigma\sqrt{t})$.

First guess: relevant timescale $\mathcal{O}(\varepsilon^2)$, want displacement $\mathcal{O}(\varepsilon) \Rightarrow \sigma \approx 1$.

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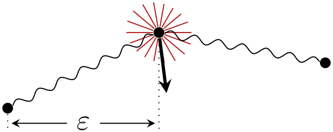


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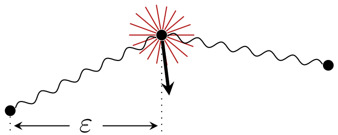


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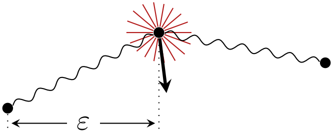


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# A convergence result

Continuous version:

$$\partial_t u_\varepsilon = \triangle u_\varepsilon + \sigma_i(u_\varepsilon)\xi_i^{(\varepsilon)} ,$$

$\xi_i$ centred Gaussian with $\mathbf{E}\xi_i^{(\varepsilon)}(t,x)\xi_j^{(\varepsilon)}(s,y) = \varepsilon^{-2}\delta_{ij}\varrho(\frac{t-s}{\varepsilon^2}, \frac{y-x}{\varepsilon})$ for $\varrho$ smooth compactly supported. (Factor $\varepsilon^{-2}$ correct scaling for white noise in time.)

**Theorem:** As $\varepsilon \to 0$, $u_\varepsilon \to u$ where $u$ solves

$$\partial_t u = \triangle u + c\big(\nabla_{\sigma_i}\sigma_i\big)(u) ,$$

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# Stronger noise

To have any hope for finite limit with stronger noise, need $\nabla_{\sigma_i}\sigma_i = 0$, or need to compensate this term.

Geometric interpretation: If vectors $\sigma_i$ span the whole tangent space at each point, then they define an (inverse) metric by

$$g^{\alpha\beta}(u) = \sigma_i^\alpha(u)\sigma_i^\beta(u) \;.$$

Metric + connection $\Rightarrow$ Laplace-Beltrami operator $\Delta$.

Fact: diffusion $\dot{x} = \sigma_i(x)\xi_i$ with i.i.d. white noises $\xi_i$ interpreted in Stratonovich sense has generator $\Delta$ if and only if $\nabla_{\sigma_i}\sigma_i = 0$.

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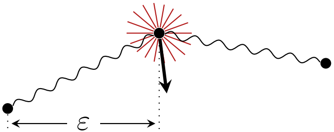


Random limit ⇒ expect solutions to behave like random walks ⇒ each "step" should be of size $\mathcal{O}(\sqrt{\varepsilon})$ to yield effect $\mathcal{O}(1)$ after $\mathcal{O}(1/\varepsilon)$ steps.

Conclusion: want $\sigma \approx \varepsilon^{-1/2}$ to have displacement $\mathcal{O}(\sqrt{\varepsilon})$ over timescale $\mathcal{O}(\varepsilon^2)$ ⇒ want $\mathbf{E}\xi_i^{(\varepsilon)}(t,x)\xi_i^{(\varepsilon)}(s,y) = \varepsilon^{-3}\varrho(\frac{t-s}{\varepsilon^2}, \frac{y-x}{\varepsilon})$. Yields space-time white noise in the limit.

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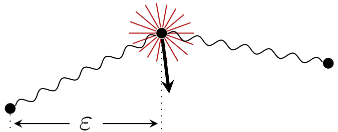


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# Main result

Consider similarly to before

$$\partial_t u_\varepsilon = \triangle u_\varepsilon + \sigma_i(u_\varepsilon)\xi_i^{(\varepsilon)} + h(u_\varepsilon) - \frac{\log \varepsilon}{4\pi\sqrt{3}}\hat{h}(u_\varepsilon)\;,$$

but now $\mathbf{E}\xi_i^{(\varepsilon)}(t,x)\xi_i^{(\varepsilon)}(s,y) = \varepsilon^{-3}\varrho(\frac{t-s}{\varepsilon^2}, \frac{y-x}{\varepsilon})$ and $\nabla_{\sigma_i}\sigma_i = 0$. For $R$ curvature tensor built from $\Gamma$, set $\hat{h}^\alpha = R^\alpha_{\beta\gamma\delta}g^{\gamma\eta}\nabla_\eta g^{\delta\beta}$.

**Theorem:** There is a choice of vector field $h$ depending on $\varrho$ such that, as $\varepsilon \to 0$, $u_\varepsilon \to u$ for a limiting process $u$ independent of $\varrho$. Comes with intrinsic characterisation of limiting process, morally

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# A few existing results

1. Very large literature on SPDEs in flat space. Even there, approximation is highly non-trivial if martingale structure is destroyed (cf. H. & Pardoux, '14).

2. Model of random string considered in (Funaki, '92), requires noise smooth in space.

3. Static case (i.e. make sense of the measure formally described by $\exp(-\int g_u(\dot{u},\dot{u})\,dt)$ "$Du$") quite well understood, see Inoue & Maeda '85, Andersson & Driver '99. Different interpretations of "$Du$" yield slightly different measures.

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# Main idea

One can "guess" the local behaviour of solutions:

$$\partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u) \bullet \partial_x u^\beta \bullet \partial_x u^\gamma + \sigma_i^\alpha(u) \bullet \xi_i \ .$$

Near $(x,t)$, one would expect

$$u^\alpha(y,s) \approx u^\alpha(x,t) + \sigma_i^\alpha(u(x,t))\big(v_i(y,s) - v_i(x,t)\big) \ ,$$

with $v_i$ solving $\partial_t v_i = \partial_x^2 v_i + \xi_i$.

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# Higher order expansion

Graphical notations for symbols: ∘ for noise, — for heat kernel, **—** for derivative of heat kernel. For example $(\Pi_{z_0}\,)(z) = v(z) - v(z_0)$. (! Suitable recentering required !) One has



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# Steps

1. Given function $U$, when does there exist a unique function / distribution $u$ described by $\Pi_z U(z)$ locally "near $z$" for every $z$? Leads to analogues of Hölder / Sobolev / Besov spaces + "reconstruction theorem".

2. Reformulate required operations for $U$ and obtain analogues to classical embedding theorems / Schauder estimates.

3. Derive fixed point problem for $U$. Solution map continuous function of "model" $\Pi$.

4. Show that suitable "renormalised" ways of lifting $\xi^{(\varepsilon)}$ to model $\Pi^{(\varepsilon)}$ converge to limit and characterise limit. Derive "renormalised equation".

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# Algebraic structure

Need to understand two things:

- How does $\Pi_{z_0}\tau$ relate to $\Pi_{z_1}\tau$ (reexpansion)?
- Which renormalisation procedures maintain these relations?

Structure: Two combinatorial Hopf algebras acting on symbols, basis vectors indexed by decorated trees / forests. Reexpansion: generalisation of Connes-Kreimer Hopf algebra / Butcher group. Renormalisation: generalisation of substitution Hopf algebra (analysis of B-series in numerical analysis).

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