Stochastic Homogenization in Geometry

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3rd September 2015
German-Japanese conference
on Stochastic Analysis and Applications
Sendai

Scaling of Riemannian Metrics

G is a smooth connected Lie group with left invariant Riemannian metric, H a proper compact connected subgroup, $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras.

$\{X_1, \dots, X_p, X_{p+1}, \dots, X_n\}$ is an orthonormal basis of $\mathfrak{g}$, extending an o.n.b. $\{X_1, \dots, X_p\}$ of $\mathfrak{h}$.

For $\epsilon > 0$, we rescale the scalar product by declaring that

$$\{\frac{1}{\sqrt{\epsilon}} X_1^*, \dots, \frac{1}{\sqrt{\epsilon}} X_p^*, X_{p+1}^*, \dots, X_n^*\}$$

a new orthonormal frame. Denoted it by m^ϵ.

Our Model: The Interpolation Equation

$$dg_t = \sum_{k=1}^{p} \gamma A_k^*(g_t) \circ db_t^k + \frac{1}{\epsilon} A_0^*(g_t)dt + \delta Y_0^*(g_t)dt.$$

1. If $\delta = 0$, $g_0 = 1$, $A_0 = 0$, $\{A_k\}$ an o.n.b. of $\mathfrak{h}$, then (g_t) is a scaled Brownian motion on H. $\pi(g_t) = \pi(g_0)$.
2. If $\gamma = 0$, $g_t = g_0 \exp(\delta t Y_0)$.

Problem. $\gamma = \frac{1}{\sqrt{\epsilon}}$ and $\delta = 1$ and take $\epsilon \to 0$.

Examples

Example

- H is a maximal torus group of a semi-simple group G.
- Stiefel Manifold.
- Oriented Grassmannian manifold of k-planes.
- Non-compact case: The Hyperbolic space.
- $S^n = SO(n+1)/SO(n)$.

Example 1

$A_{i,j} = \frac{1}{\sqrt{2}}(E_{ij} - E_{ji})$
E_{ij} is the (n+1)x (n+1) elementary matrix.

$$dg_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \sum_{1 \le i < j \le n} g_t^\epsilon A_{i,j} \circ db_t^{i,j} + \frac{1}{8} n(n-1) g_t^\epsilon Y_0^* dt.$$

Y_0 is skew symmetric orthogonal to $A_{i,j}$, $i, j \neq n+1$.

Theorem. There is an 'effective limit' which are Brownian motions on S^n, horizontal Brownian motions on $SO(n+1)$.

Example 2

$$dg_t^\epsilon = g_t^\epsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} dt + \frac{1}{\sqrt{\epsilon}} g_t^\epsilon \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} db_t.$$

$$x_t^\epsilon = \pi(g_t^\epsilon), \quad \pi(z, w) = (\frac{1}{2}(|w|^2 - |z|^2), z\bar{w}).$$

Example

$\pi(g_{\frac{t}{\epsilon}}^\epsilon)$ converges to Markov process with generator $\frac{1}{4}\Delta_{S^2(\frac{1}{2})}$.

$4 = 2\lambda$, $\lambda = 2$ is an eigenvalue of $\mathcal{L}_0 = \frac{1}{2}\Delta$.

$(\tilde{x}_t^\epsilon)$ converges weakly to a hypoelliptic diffusion.

$$\sup_{0 \le t \le T} W_1(Law(\tilde{x}_{\frac{t}{\epsilon}}^\epsilon), \bar{\mu}_t) \le C\epsilon^r.$$

As Random Perturbation

$\{A_k\} \in \mathfrak{h}$,

$$dg_t = \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^{N} A_k^*(g_t) \circ db_t^k + \frac{1}{\epsilon} A_0^*(g_t) dt + Y_0^*(g_t) dt. \qquad (\star)$$

$(\star)$ is a random perturbation to $(\star\star)$,

$$dg_t = \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^{p} A_k^*(g_t) \circ db_t^k + \frac{1}{\epsilon} A_0^*(g_t) dt. \qquad (\star\star)$$

We see a singular perturbation problem:

$$L^\epsilon = \frac{1}{\epsilon} \mathcal{L}_0 + L_{Y_0^*}, \qquad \mathcal{L}_0 = \frac{1}{2} \sum_k L_{A_k^*} L_{A_k^*} + L_{A_0^*}.$$

Asymptotic Expansion and Multi Scale Analysis

$$dg_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^N A_k^*(g_t^\epsilon) \circ db_t^k + \frac{1}{\epsilon} A_0^*(g_t^\epsilon) dt + Y_0^*(g_t^\epsilon) dt$$

$$L^\epsilon = \frac{1}{\epsilon} \mathcal{L}_0 + L_{Y_0^*}, \quad \mathcal{L}_0 = \frac{1}{2} \sum_k L_{A_k^*} L_{A_k^*} + L_{A_0}^*.$$

$$\frac{\partial u_t^\epsilon}{\partial t} = (\mathcal{L}_1 + \frac{1}{\epsilon} \mathcal{L}_2)(u_t^\epsilon).$$

Expand solution in ϵ, for the singular perturbation problem:

$$u_t^\epsilon = u_t^0 + \epsilon u_t^1 + \epsilon^2 u_t^2 + \dots.$$

Seek a slow variable and suitable scale, and an equation for u_t^0 (possibly u_t^1,...) Book: A. Bensoussan, J.-L. Lions, G. C. Papanicolaou

Example

A physical equation is approximated by an ideal equation, e.g. a Hamiltonian system $\dot{x}_t = X_H(x_t)$. The difference is, say, of order ϵ. After a time of order $\frac{1}{\epsilon}$, the orbits deviate visiably from the Hamiltonian orbits unless there is fast oscillation.

$$\dot{x}_t^\epsilon = F(x_t, \epsilon, \zeta_{\frac{t}{\epsilon}}), \qquad \text{if} \quad \frac{1}{T}\int_0^T \mathbf{E}F(x, \zeta_s)ds \to \bar{F}(x).$$

$$\text{then} \quad \int_0^t F(x_s^\epsilon, \epsilon, \zeta_{\frac{s}{\epsilon}})ds - \int_0^t \bar{F}^{(0)}(x_s^\epsilon)ds = o(\epsilon).$$

$$x_t^\epsilon \to \bar{x}_t, \qquad \bar{x}_t = \bar{F}(\bar{x}_t).$$

If $\bar{F}_0 = X_H$, $H(x_{\frac{t}{\epsilon}}^\epsilon)$ is small. The orbits $H(x_{\frac{t}{\epsilon}}^\epsilon) \to ?$

R.Z. Khasminskii, M. Freidlin, A.N. Borodin, D. Wentzell, G. C. Papanicolaou, S.R. S.Varadhan, W. Kohler, V. I. Arnold, ...

Birkhoff's Ergodic Theorem

Consider an ergodic stationary process $(z_t, t \geq 0)$ with one-time marginal μ.

Theorem (Birkhoff's Ergodic Theorem)

If $f \in L^1$, there exists an invariant function $\bar{f} \in L^1$,

$$\frac{1}{t}\int_0^t f(z_r)dr \to \bar{f}, \qquad (a.e.)$$

If θ is ergodic, $\bar{f} = \int f(z)\mu(dz)$.

Sub-elliptic estimates + regularity of $f \Rightarrow$ rate of convergence.

Homogeneous Manifold

H is a connected proper subgroup of a connected Lie group G. Denote G/H the left coset space (homogeneous space/orbit space).

$\pi : G \to G/H$ is the projection:
$\pi(g)$ is the coset containing H.
$o := \pi(1)$.
$(d\pi)_1 : \mathfrak{g} \to T_o M$ has kernel $\ker(d\pi)_1 = \mathfrak{h}$.

If $\mathfrak{m}$ be a complement to $\mathfrak{h}$: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$
$(d\pi)_1 : \mathfrak{m} \to T_o M$ is an isometry.

Program

If $Y_0 = 0$, $\pi(g_t^\epsilon) = \pi(g_0^\epsilon)$. Let $x_t^\epsilon = \pi(g_t^\epsilon)$.

$$dg_t = \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^{N} A_k^*(g_t) \circ db_t^k + \frac{1}{\epsilon} A_0^*(g_t)dt + Y_0^*(g_t)dt.$$

If we take $\epsilon \to 0$, what becomes of x_t^ϵ? [arXiv:1505.06772]

(1) Determine the geometric conditions under which and the scale for which x_t^ϵ converge.

(2) Reduction of complexity. Equation for x_t^ϵ.

(3) Limit Theorems. [arXiv:1402.5861]

(4) Describe the limit (x_t). Is (x_t) necessarily Markovian? Are they Brownian motions?

Further Assumptions

Assume $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition , i.e. $\mathrm{Ad}_H(\mathfrak{m}) \subset \mathfrak{m}$. Take $Y_0 \in \mathfrak{m}$.

Assumptions.

- H is compact,
- The scalar product on $\mathfrak{g}$ is Ad_H invariant,
- $\{A_1, \dots, A_N\}$ is Lie algebra generating.

$\mathfrak{m}_0$ denotes the set of invariant vectors of Ad_H.
If $Y_0 \in \mathfrak{m}_0$, $u_t^\epsilon = u_0 \exp(tY_0)$.

A Limit Theorem

$\tilde{\mathfrak{m}}$ is Ad_H-invariant and $\tilde{\mathfrak{m}} \cap \mathfrak{m}_0 = \{0\}$.

Theorem (arXiv:1505.06772, c.f. arXiv:1402.5861: Thm5.4)

There exists a horizontal stochastic process (u_t^ϵ) s.t. $\pi(u_t^\epsilon) = \pi(g_t^\epsilon)$. If $Y_0 \in \tilde{\mathfrak{m}}$,

(1) *As $\epsilon \to 0$, $(u^\epsilon_{\frac{s}{\epsilon}}, s \le T)$ converges weakly to a Markov process $(\bar{u}_s, s \le T)$.*

$$\bar{\mathcal{L}} = -\sum_{i,j=1}^{m} \overline{\alpha_i(Y_0)\,(\mathcal{L}_0^{-1}\alpha_j(Y_0))}\, L_{Y_i^*} L_{Y_j^*}.$$

(2) *$(x^\epsilon_{\frac{\cdot}{\epsilon}})$ converges weakly to $(\bar{x}_t)$, where $x_t^\epsilon = \pi(u_t^\epsilon)$.*

Classification of Limits

Is the Markov limit on M necessarily a Scaled Brownian motion?

For the classifications we assume $A_0 = 0$.

Example

Let $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ and $Y_0 \in \mathfrak{m}_1$, randomness cannot be transformed to $\mathfrak{m}_2$ directions, (x_t) is not even elliptic.

Laplacian Like operator: $\Delta_{\mathfrak{m}_l} = \mathrm{trace}_{\mathfrak{m}_l} \nabla^L d$.

M. Freidlin, A. D. Wentzell: Limits on graphs.

$\mathcal{L}_0 = \frac{1}{2}\Delta_H$ case : (1)

$\{A_k, k = 1, \ldots, N\}$ is an orthonormal basis of $\mathfrak{h}$.

Theorem

Then there exists a number $\lambda_l > 0$, independent of $Y_0 \in \mathfrak{m}_l$, such that for any $f \in C_K^2(G; \mathbf{R})$,

$$\bar{\mathcal{L}} f = \frac{1}{\lambda_l} \int_H \nabla^L df \left((\mathrm{Ad}(h)(Y_0))^*, (\mathrm{Ad}(h)(Y_0))^* \right) dh.$$

(x_t) is a Markov process.

$$\frac{1}{2} \sum_{k=1}^{p} \mathrm{ad}^2(A_k)|_{\mathfrak{m}_l} = -\lambda_l id_{\mathfrak{m}_l}.$$

$\frac{1}{2}\Delta_H$ case: (2)

A Lie algebra is simple if it is non Abelian, it has no ideals except $\{0\}$ and $\mathfrak{g}$.

Example

Let H be a maximal torus group of a semi-simple group G. If $Y_0 \in \mathfrak{m}_l$, $l \neq 0$, then $\bar{\mathcal{L}} = \frac{|Y_0|^2}{2\lambda_l}\Delta_{\mathfrak{m}_l}$ where $\Delta_{\mathfrak{m}_l} := \sum_i L_{Y_i} L_{Y_i}$.

Moreover, λ_l is independent of Y_0.

Proof. Complexify $\mathfrak{g}$ and use root space decomposition.

Degenrate Case

$Y_0 \in \mathfrak{m}_l$ where $l \neq 0$.

If $\dim(\mathfrak{m}_l) \neq 2, 4, 7$,

$$\bar{\mathcal{L}} = \frac{|Y_0|^2}{\lambda(Y_0)\dim(\mathfrak{m}_l)}\Delta_{\mathfrak{m}_l}.$$

The number $\lambda(Y_0)$ can be computed from eigenvalues of $\mathcal{L}_0$.

Are they Brownian motions? $\sum_i L_{Y_i}L_{Y_i}$, λ_l.

Thank You!