Recurrence criteria for diffusion processes generated by divergence free perturbations of non-symmetric energy forms

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Contents

1. Analytic characterization of recurrence

- $E =$ Hausdorff topological space
- $\mathcal{B}(E) = \sigma(C(E))$
- μ is a σ-finite measure on $\mathcal{B}(E)$ with full support

(H1) $(T_t)_{t>0}$ is a submarkovian C_0-semigroup of contractions on $L^2(E, \mu)$

(H2) The adjoint semigroup $(\hat{T}_t)_{t>0}$ on $L^2(E, \mu)$ is also submarkovian

Let $(L, D(L))$ be the $L^2(E, \mu)$-**generator** of $(T_t)_{t>0}$, i.e.

$$D(L) = \{u \in L^2(E, \mu) \mid \exists Lu := \lim_{t \downarrow 0} \frac{T_t u - u}{t} \text{ in } L^2(E, \mu)\}$$

$(\hat{L}, D(\hat{L}))$ **adjoint operator** of $(L, D(L))$ in $L^2(E, \mu)$.

$(L, D(L))$ induces a **generalized Dirichlet form**

$$\mathcal{E}(u,v) := \begin{cases} -\int Lu \cdot v \, d\mu; & u \in D(L), \ v \in L^2(E,\mu) \\ -\int u \cdot \hat{L}v \, d\mu; & u \in L^2(E,\mu), \ v \in D(\hat{L}). \end{cases}$$

In general:

- $(L, D(L))$ **needs not be symmetric**, i.e. $(L, D(L)) = (\hat{L}, D(\hat{L}))$
- $(L, D(L))$ **needs not be sectorial**, e.g.

$$\left| \int Lu \cdot v \, d\mu \right| \leq const \cdot \sqrt{- \int Lu \cdot u \, d\mu} \cdot \sqrt{- \int Lv \cdot v \, d\mu}.$$

Remark: **If** $(\mathcal{E}, D(L))$ **is sectorial**, then it is closable in $L^2(E,\mu)$ and the **closure** $(\mathcal{E}, D(\mathcal{E}))$ **is a sectorial Dirichlet form on** $L^2(E,\mu)$.

Let $(G_\alpha)_{\alpha>0}$, $(\hat{G}_\alpha)_{\alpha>0}$, be the $L^2(E,\mu)$-resolvents of $(T_t)_{t>0}$, $(\hat{T}_t)_{t\geq 0}$.

$$(\hat{T}_t)_{t>0} \text{ is submarkovian } \Leftrightarrow (T_t)_{t>0} \text{ is an } L^1(E,\mu)\text{-contraction}$$

$\rightsquigarrow$ $(T_t)_{t>0}$ can be defined as a C_0-semigroup of contractions on $L^1(E,\mu)$.

Definition

For $f \in L^1(E,\mu)$, $f \geq 0$ μ-a.e.

$$Gf(x) := \lim_{N\to\infty} \int_0^N T_t f(x)dt = \lim_{\alpha\to 0} \underbrace{\int_0^\infty e^{-\alpha t} T_t f(x)dt}_{=G_\alpha f(x)} \ (\leq \infty)$$

is uniquely defined μ-a.e. G is called **potential operator** *associated with $(T_t)_{t>0}$.*

Since $(T_t)_{t>0}$ is sub-Markovian, $(T_t)_{t>0}$ and its potential operator G can also be defined on $L^\infty(E,\mu)$.

Definition

(i) $(T_t)_{t>0}$ *(or also* $\mathcal{E}$*) is* **recurrent**, *if for any* $f \in L^1(E,\mu)$ *with* $f \geq 0$ μ*-a.e., we have*

$$Gf = 0 \text{ or } \infty \ \mu\text{-a.e.}$$

(ii) $(T_t)_{t>0}$ *(or also* $\mathcal{E}$*) is* **transient**, *if there exists* $g \in L^1(E,\mu)$ *with* $g > 0$ μ*-a.e. such that*

$$Gg < \infty \ \mu\text{-a.e.}$$

(iii) $B \in \mathcal{B}(E)$ *is* **weakly invariant** *w.r.t.* $(T_t)_{t>0}$, *if for any* $t > 0$, $f \in L^2(E,\mu)$

$$T_t(f1_B)(x) = 0 \qquad \mu\text{-a.e. } x \in E \setminus B$$

(iv) $(T_t)_{t>0}$ *is* **strictly irreducible**, *if for any weakly invariant set* B *relative to* $(T_t)_{t>0}$, $\mu(B) = 0$ *or* $\mu(E \setminus B) = 0$.

Remark (cf. e.g. [Kuwae, 2010])

(a) $(T_t)_{t>0}$ *is transient, if and only if* $Gf < \infty$ μ*-a.e. for any* $f \in L^1(E,\mu)$ *with* $f \geq 0$ μ*-a.e.*

(b) *If* $g \in L^1(E,\mu)$ *with* $g > 0$ μ*-a.e., then* $\{x \in E : \ Gg(x) = \infty\}$ *is a weakly invariant set relative to* $(T_t)_{t>0}$*. Consequently, if* $(T_t)_{t>0}$ *is strictly irreducible, then it is either transient or recurrent.*

(c) *If there exists a strictly positive measurable function* $(p_t(x,y))_{x,y\in E}$ *with*

$$T_t f(x) = \int_E p_t(x,y) f(y) \mu(dy)$$

for any $x \in E$*,* $f \in L^2(E,\mu)$*, then* $(T_t)_{t>0}$ *is strictly irreducible.*

Theorem (Gim/Trutnau, 2015)

If there exists a sectorial Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ *with* $D(L)_b \subset D(\mathcal{E}^0)$ *and*

$$\mathcal{E}^0(u,u) \leq \mathcal{E}(u,u) \quad \text{for any} \quad u \in D(L)_b,$$

then the transience of $(\mathcal{E}^0, D(\mathcal{E}^0))$ *implies the transience of* $(T_t)_{t>0}$.

Remark

If $(\mathcal{E}^0, D(\mathcal{E}^0))$ *is a sectorial Dirichlet form on* $L^2(E,\mu)$*, then its symmetric part* $(\widetilde{\mathcal{E}}^0, D(\mathcal{E}^0))$ *is a symmetric Dirichlet form on* $L^2(E,\mu)$*. By the Theorem, we obtain: a sectorial Dirichlet form* $(\mathcal{E}^0, D(\mathcal{E}^0))$ *is transient, if and only if* $(\widetilde{\mathcal{E}}^0, D(\mathcal{E}^0))$ *is transient.*

Lemma (Gim/Tr, 2015)

If $(T_t)_{t>0}$ *is transient, then there exists a function* $g \in L^1(E,\mu)_b$ *with* $g > 0$ μ*-a.e. and* $\|Gg\|_{L^\infty(\mu)} < \infty$.

(H3) There exist a sectorial Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ with $D(L)_b \subset D(\mathcal{E}^0)$ and a linear operator $(N, D(N))$ on $L^2(E, \mu)$ such that

$$\mathcal{E}(u, v) = \mathcal{E}^0(u, v) + \int_E u \cdot Nv \, d\mu, \quad u \in D(L)_b, \quad v \in D(N) \cap D(\mathcal{E}^0)$$

and

$$\mathcal{E}^0(u, u) \leq \mathcal{E}(u, u), \ u \in D(L)_b.$$

The **extended Dirichlet** space of $D(\mathcal{E}^0)$ is defined as the set of all functions u for which there exists an $\mathcal{E}^0$-Cauchy sequence $(u_n)_{n\geq 1} \subset D(\mathcal{E}^0)$ such that

$$\lim_{n\to\infty} u_n = u \quad \mu\text{-a.e.}$$

(see [Oshima, Semi-Dirichlet forms, 2013]). Since the Dirichlet form $(\mathcal{E}^0, D(\mathcal{E}^0))$ is sectorial, for u in the extended Dirichlet space,

$$\mathcal{E}^0(u,u) := \lim_{n\to\infty} \mathcal{E}^0(u_n, u_n)$$

is independent of the choice of $(u_n)_{n\geq 1} \subset D(\mathcal{E}^0)$.

Theorem (Gim/Trutnau, 2015)

Suppose $(T_t)_{t>0}$ is transient and let $g \in L^1(E,\mu)_b$ with $g > 0$ μ-a.e. and $\|Gg\|_{L^\infty(\mu)} < \infty$. Then Gg is in the extended Dirichlet space of $D(\mathcal{E}^0)$ and

$$(g,u) = \underbrace{\mathcal{E}^0(Gg,u)}_{:=\lim_{\alpha\to 0}\mathcal{E}^0(G_\alpha g, u)} + \int_E Gg \cdot Nu \, d\mu$$

for any $u \in D := \{u \in D(N) \cap D(\mathcal{E}^0) : \; Nu \in L^1(E,\mu)\}$.

Corollary (Gim/Tr, 2015)

(a) *If there exists a sequence of functions* $(\chi_n)_{n\geq 1} \subset D$ *with* $0 \leq \chi_n \leq 1$, $\lim_{n\to\infty} \chi_n = 1$ μ*-a.e. satisfying*

$$\lim_{n\to\infty} \left(\mathcal{E}^0(f, \chi_n) + \int_E f \cdot N\chi_n \, d\mu \right) = 0,$$

for any non-negative bounded f *(so in part. for* $f = Gg$*) in the extended Dirichlet space of* $D(\mathcal{E}^0)$*, then* $(T_t)_{t>0}$ *is not transient.*

(b) *If* $(T_t)_{t>0}$ *is strictly irreducible, then (a) represents a sufficient condition for recurrence of* $(T_t)_{t>0}$*, because if* $(T_t)_{t>0}$ *is strictly irreducible, then it is either transient or recurrent.*

2. Connection to recurrence in the classical probabilistic sense

Let $\mathbb{M} = (\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E_\triangle})$ with life time ζ be a strong Markov process with state space E, resolvent

$$R_\alpha f(x) := \mathbb{E}_x\Big[\int_0^\infty e^{-\alpha t} f(X_t)dt\Big], \quad x \in E, \ \alpha > 0, \ f \in B(E)_b$$

and transition semigroup

$$p_t f(x) := \mathbb{E}_x\Big[f(X_t)\Big], \quad x \in E, \ t > 0, \ f \in B(E)_b.$$

Suppose that the process $\mathbb{M}$ is associated with $\mathcal{E}$, i.e.

$$\left.\begin{aligned} R_\alpha f &= G_\alpha f \\ p_t f &= T_t f \end{aligned}\right\} \ \mu\text{-a.e. for any } \alpha > 0, \ t > 0, \ f \in B(E)_b \cup L^2(E,\mu)_b$$

In particular $f \in L^1(E,\mu)$, $f \geq 0$ μ-a.e.

$$Gf(x) = \mathbb{E}_x\Big[\int_0^\infty f(X_t)dt\Big].$$

$(T_t)_{t>0}$ is **recurrent**, if for any $f \in L^1(E,\mu)$ with $f > 0$ μ-a.e., we have

$$\mathbb{E}_x\left[\int_0^\infty f(X_t)dt\right] = \infty \quad \mu\text{-a.e. } x \in E.$$

$(T_t)_{t>0}$ is **transient**, if there exists $g \in L^1(E,\mu)$ with $g > 0$ μ-a.e., such that

$$\mathbb{E}_x\left[\int_0^\infty g(X_t)dt\right] < \infty \quad \mu\text{-a.e. } x \in E.$$

$B \in \mathcal{B}(E)$ is **weakly invariant** relative to $(T_t)_{t>0}$, if for any $t > 0$,

$$\mathbb{E}_x\left[1_B(X_t)\right] = 0 \quad \mu\text{-a.e. } x \in E \setminus B.$$

Define the **first hitting** time of $B \in \mathcal{B}(E)$ by

$$\sigma_B(\omega) := \inf\{t > 0 : \ X_t(\omega) \in B\}$$

and the **last exit time** from $B \in \mathcal{B}(E)$ by

$$L_B(\omega) := \sup\{t \geq 0 : \ X_t(\omega) \in B\} \in \mathcal{F}_\infty.$$

Proposition (following [Getoor, 1980])

(a) *$(T_t)_{t>0}$ is transient, if and only if there exists a sequence of Borel finely open sets $(B_n)_{n\geq 1}$ increasing to E up to some μ-negligible set N such that for any $x \in E \setminus N$, $n \geq 1$*

$$\mathbb{P}_x(L_{B_n} < \infty) = 1.$$

(b) *If $(T_t)_{t>0}$ is strictly irreducible recurrent, then $\mathbb{P}_x(\zeta = \infty) = 1$ for μ-a.e. $x \in E$ and for any non μ-polar[a] finely open set $B \in \mathcal{B}(E)$*

$$\mathbb{P}_x(L_B = \infty) = 1 \ \ \mu\text{-a.e. } x \in E.$$

[a] B is μ-polar if $\int_E P_x(\sigma_B < \infty)\mu(dx) = 0$

If $(T_t)_{t>0}$ is strictly irreducible and recurrent, then for any B open, $B \neq \emptyset$,

$$\mathbb{P}_x(\underbrace{\{L_B = \infty\}}_{=:\Lambda}) = 1 \quad \text{for} \quad \mu\text{-a.e. } x \in E$$

Assume that the **semigroup** p_t **of** $\mathbb{M}$ **is strong Feller** in the following sense: there exists a measurable function $(p_t(x,y))_{t>0,x,y\in E}$ with

$$p_t f(x) = \int_E p_t(x,y) f(y) \mu(dy) \quad \text{for any } x \in E, \ f \in B(E)_b$$

and

$$p_t f \text{ is continuous for any } f \in B(E)_b.$$

Since Λ **is shift invariant**, we get for $x \in E$

$$\mathbb{P}_x(\Lambda) = \mathbb{P}_x(\vartheta_t^{-1}(\Lambda)) = \mathbb{E}_x[\mathbb{E}_x[1_\Lambda \circ \vartheta_t \mid \mathcal{F}_t]] = \mathbb{E}_x[\mathbb{E}_{X_t}[1_\Lambda]] = p_t \mathbb{E}_\cdot[1_\Lambda](x)$$

hence since μ has full support

$$\mathbb{P}_x(\Lambda) = 1 \text{ for any } x \in E.$$

3. Application to a class of diffusions on Euclidean space

- $E \subset \mathbb{R}^d$ open or closed with $dx(\partial E) = 0$
- $d\mu := \rho dx$, where $\rho \in L^1_{loc}(E, dx)$ with $\rho > 0$ dx-a.e.
- $A = (a_{ij}) \in L^1_{loc}(E, \mu)$ $1 \le i, j \le d$ and for each relatively compact open set $V \subset E$, there exists $\nu_V > 0$ such that

$$\nu_V^{-1}|\xi|^2 \le \sum_{i,j=1}^d \widetilde{a}_{ij}(x)\xi_i\xi_j \le \nu_V|\xi|^2 \qquad \text{(locally elliptic)}$$

  for all $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$, $x \in V$.

Suppose

$$\mathcal{E}^0(f, g) = \sum_{i,j=1}^{d} \int_E a_{ij}(x)\partial_i f(x)\partial_j g(x)\mu(dx) \quad f, g \in C_0^\infty(E).$$

is closable on $L^2(E, \mu)$ and that $(\mathcal{E}^0, C_0^\infty(E))$ satisfies the **strong sector condition**, i.e. there is a constant $K > 0$ such that

$$|\mathcal{E}^0(f, g)| \le K\sqrt{\mathcal{E}^0(f, f)}\sqrt{\mathcal{E}^0(g, g)} \qquad \forall f, g \in C_0^\infty(E).$$

Denote the closure of $(\mathcal{E}^0, C_0^\infty(E))$ on $L^2(E,\mu)$ by $(\mathcal{E}^0, D(\mathcal{E}^0))$.

$\rightsquigarrow$ $(\mathcal{E}^0, D(\mathcal{E}^0))$ non-symmetric regular sectorial Dirichlet form on $L^2(E,\mu)$.

Let $(L^0, D(L^0))$ be the linear operator corresponding to $(\mathcal{E}^0, D(\mathcal{E}^0))$ on $L^2(E,\mu)$.

For any $V \subset\subset E$, $(\mathcal{E}^0, C_0^\infty(V))$ is closable on $L^2(V,\mu)$.

Denote its closure by $(\mathcal{E}^{0,V}, D(\mathcal{E}^{0,V}))$, then $D(\mathcal{E}^{0,V}) \subset D(\mathcal{E}^0)$ and

$$D(\mathcal{E}^{0,E}) := \bigcup_{V\subset\subset E} D(\mathcal{E}^{0,V}) \subset D(\mathcal{E}^0).$$

Let $B(x) := (B_1(x), \dots, B_d(x)) \in L^2_{loc}(E, \mathbb{R}^d, \mu)$ satisfy

$$\int_E \langle B(x), \nabla f(x)\rangle \mu(dx) = 0 \qquad \text{for any} \quad f \in C_0^\infty(E),$$

hence for any $f \in D(\mathcal{E}^{0,E})$.

Using a similar technique as in [Stannat, 1999], we get:

Lemma (Gim/Tr, 2015)

There exists a closed operator $(\overline{L}, D(\overline{L}))$ on $L^1(E,\mu)$ which is the generator of sub-Markovian C_0-semigroup of contractions $(\overline{T}_t)_{t>0}$ satisfying the following properties:

(a) $(\overline{L}, D(\overline{L}))$ *is a closed extension of* $Lu = L^0u + \langle B, \nabla u\rangle$, $u \in D(L^0)_{0,b}$ *on* $L^1(E,\mu)$.

(b) $D(\overline{L})_b \subset D(\mathcal{E}^0)$ *and for* $u \in D(\overline{L})_b$, $v \in D(\mathcal{E}^{0,E})_b$, *we have*

$$\mathcal{E}^0(u,v) - \int_E \langle B, \nabla u\rangle v d\mu = -\int_E \overline{L}u \cdot v\, d\mu$$

and

$$\mathcal{E}^0(u,u) \leq -\int_E \overline{L}u \cdot u\, d\mu.$$

Denote the C_0-resolvent of $(\overline{L}, D(\overline{L}))$ by $(\overline{G}_\alpha)_{\alpha>0}$.

Since $(\overline{T}_t)_{t>0}$ is a sub-Markovian C_0-semigroup of contractions on $L^1(E,\mu)$ and $L^1(E,\mu)_b \subset L^2(E,\mu)$ densely, we can construct uniquely a sub-Markovian C_0-semigroup of contractions $(T_t)_{t>0}$ on $L^2(E,\mu)$ such that $T_t \equiv \overline{T}_t$ for $t>0$ on $L^1(E,\mu)\cap L^2(E,\mu)$ (cf. the Riesz-Thorin interpolation Theorem).

Let $(L, D(L))$ be the generator of $(T_t)_{t>0}$ and $(G_\alpha)_{\alpha>0}$ be the corresponding C_0-resolvent. Clearly, $G_\alpha \equiv \overline{G}_\alpha$ for $\alpha>0$ on $L^1(E,\mu)\cap L^2(E,\mu)$.

Let $(\widehat{L}, D(\widehat{L}))$ be the adjoint operator of $(L, D(L))$ in $L^2(E,\mu)$. Then

$$\mathcal{E}(f,g) := \begin{cases} (-Lf, g) & f\in D(L),\ g\in L^2(E,\mu), \\ (-\widehat{L}g, f) & g\in D(\widehat{L}),\ f\in L^2(E,\mu), \end{cases}$$

satisfies (H1). Clearly (H2) holds since $(L, D(L))$ satisfies the same assumptions as $(\widehat{L}, D(\widehat{L}))$.

In particular, the bilinear form $\mathcal{E}$ is an extension of

$$\int_E \langle A\nabla f, \nabla g\rangle d\mu - \int_E \langle B, \nabla f\rangle g d\mu, \qquad f, g \in C_0^\infty(E) \cap D(L^0).$$

Put

$$Nv := \langle B, \nabla v\rangle, \; v \in D(N) := D(\mathcal{E}^{0,E})_b.$$

Then

$$D = D(N) \cap D(\mathcal{E}^0) = D(\mathcal{E}^{0,E})_b$$

and $\mathcal{E}$ satisfies assumption (H3), i.e.

$$\mathcal{E}^0(u,u) \leq \mathcal{E}(u,u), \; u \in D(L)_b$$

and

$$(-Lu, v) = \mathcal{E}^0(u,v) + \int_E \langle B, \nabla v\rangle u d\mu, \; u \in D(L)_b, \; v \in D.$$

By the results of 1. we get:

Corollary (Gim/Tr, 2015)

(a) *If $(\mathcal{E}^0, D(\mathcal{E}^0))$ is transient, then $(T_t)_{t>0}$ is also transient.*

(b) *If there exists a sequence of functions $(\chi_n)_{n\geq 1} \subset D$ with $0 \leq \chi_n \leq 1$, $\lim_{n\to\infty} \chi_n = 1$ μ-a.e. satisfying*

$$\lim_{n\to\infty} \left(\mathcal{E}^0(g, \chi_n) + \int_E \langle B, \nabla \chi_n \rangle g d\mu \right) = 0 \qquad (\star)$$

for any non-negative bounded g in the extended Dirichlet space of $D(\mathcal{E}^0)$, then $(T_t)_{t>0}$ is not transient.

Remark

$(\star)$ holds, if

$$\lim_{n\to\infty} \left(\mathcal{E}^0(\chi_n, \chi_n) + \int_E |\langle B, \nabla \chi_n \rangle| d\mu \right) = 0,$$

Furthermore, since $-B$ satisfies the same assumptions as B, the co-form is then also not transient.

Theorem (Gim/Tr, 2015)

If $(\overline{T}_t)_{t>0}$ is recurrent, then there exists a sequence of functions $(\chi_n)_{n\geq 1}$ in $D(\overline{L})_b$ with $0 \leq \chi_n \leq 1$ and $\lim_{n\to\infty} \chi_n = 1$ μ-a.e. satisfying

$$\lim_{n\to\infty} (-\overline{L}\chi_n, \chi_n) = 0.$$

Furthermore, $\lim_{n\to\infty} -\overline{L}\chi_n = 0$ μ-a.e. and in $L^1(E,\mu)$. In particular, $(\widetilde{\mathcal{E}}^0, D(\mathcal{E}^0))$ is recurrent, i.e. $\lim_{n\to\infty} \widetilde{\mathcal{E}}^0(\chi_n, \chi_n) = 0$.

Corollary

If $(\overline{T}_t)_{t>0}$ is recurrent, then it is conservative, i.e. for all $t > 0$, $\overline{T}_t 1 = 1$ μ-a.e..

Explicit conditions for recurrence

Assume that there exists a non-negative continuous function $\phi(x)$ on E with

$$\nabla\phi \in L^{\infty}_{loc}(E, \mathbb{R}^d, \mu)$$

such that for $r > 0$

$$E_r := \{x \in E : \ \phi(x) < r\}$$

is a relatively compact open set in E and $\cup_{r>0} E_r = E$. (For instance, if E is closed or $E = \mathbb{R}^d$, we may choose $\phi(x) = |x|$.)

Define for $r > 0$,

$$v(r) := \underbrace{\int_{E_r} \langle A(x)\nabla\phi(x), \nabla\phi(x)\rangle \mu(dx)}_{=:v_1(r)} + \underbrace{\int_{E_r} \phi(x) \cdot |\langle B(x), \nabla\phi(x)\rangle| \mu(dx)}_{=:v_2(r)}$$

Theorem (Gim/Tr, 2015)

Assume that $(T_t)_{t>0}$ is strictly irreducible. If the sequence a_n defined by

$$a_n := \int_1^n \frac{r}{v(r)} dr$$

satisfies $\lim_{n\to\infty} a_n = \infty$ *and* $\lim_{n\to\infty} \frac{\log(v_2(n)\vee 1)}{a_n} = 0$, *then $(T_t)_{t>0}$ is recurrent.*

Proof.

Define $\chi_n(x) := \psi_n(\phi(x))$, where for $r > 0$

$$\psi_n(r) := \begin{cases} 1 & 0 \le r \le 1, \\ 1 - \frac{1}{a_n}\int_1^r \frac{t}{v(t)} dt & 1 \le r \le n, \\ 0 & r \ge n. \end{cases}$$

Then $\chi_n \in D(\mathcal{E}^{0,E})_b$, $0 \le \chi_n \le 1$, $\lim_{n\to\infty} \chi_n = 1$ μ-a.e. and

$$\mathcal{E}^0(\chi_n, \chi_n) + \int_E |\langle B, \nabla \chi_n \rangle| d\mu \le \frac{2}{a_n} + \frac{1}{a_n^2 v(1)} + \frac{\log(v_2(n) \vee 1)}{a_n}.$$

□

Corollary

Assume that $(T_t)_{t>0}$ is strictly irreducible. The conditions of the Theorem are satisfied if one of the following conditions is fulfilled for sufficiently large r and some constant $C > 0$:

(a) $v_1(r) \leq Cr^2$ *and* $v_2(r) \leq C \log r$

(b) $v(r) \leq Cr^\alpha$ *for some constant* $\alpha < 2$*.*

4. Examples and Counterexamples

Counterexamples to the symmetric case

- $E = \mathbb{R}$
- $d\mu := \varphi(x)dx, \varphi : \mathbb{R} \to \mathbb{R}^+$ is locally bounded above and below by strictly positive constants, $\varphi' \in L^2_{loc}(\mathbb{R}, dx)$
- $(\mathcal{E}^0, D(\mathcal{E}^0))$ is given as the closure of

$$\mathcal{E}^0(f,g) := \frac{1}{2}\int_{\mathbb{R}} f'(x)g'(x)\mu(dx), \;\; f,g \in C_0^\infty(\mathbb{R})$$

  on $L^2(\mathbb{R}, \mu)$.
- $B(x) := \frac{const.}{\varphi} \in L^2_{loc}(\mathbb{R}, d\mu)$ (is μ-divergence free)

Then, using the construction method of 3. one can see that (H1)-(H3) are satisfied with $D := D(\mathcal{E}^0)_{0,b}$. The constructed $\mathcal{E}$ is an extension of

$$\mathcal{E}(f,g) = \frac{1}{2}\int_{\mathbb{R}} f'(x)g'(x)\mu(dx) - \int_{\mathbb{R}} B(x)f'(x)g(x)\,\mu(dx), \;\; f,g \in C_0^\infty(\mathbb{R}).$$

Moreover

$$C_0^\infty(\mathbb{R}) \subset D(L^0)_{0,b} \subset D(L)_{0,b}$$

and

$$Lu = \frac{1}{2}u'' + \Big(\frac{\varphi'}{2\varphi} + B\Big)u', \qquad u \in C_0^\infty(\mathbb{R}).$$

We consider three explicit cases:

(a) $\varphi(x) = e^{-|x|},\ B = \frac{1}{2}e^{|x|}$ (b) $\varphi(x) = \inf(1, \frac{1}{|x|}),\ B = \frac{b}{\varphi},$

(c) $\varphi(x) \equiv 1,\ B \equiv b.$

In all explicit cases, choose $(\chi_n)_{n\geq 1} \subset C_0^\infty(\mathbb{R})$ such that $1_{B_n(0)} \leq \chi_n \leq 1_{B_{2n}(0)}$, $\lim_{n\to\infty} \chi_n = 1$, $0 \leq \chi_n \nearrow 1$ μ-a.e. and $\|\chi_n'\|_{L^\infty(\mu)} \leq 2/n$. Then

$$\lim_{n\to\infty}(-L\chi_n, \chi_n) = \lim_{n\to\infty} \mathcal{E}(\chi_n, \chi_n) = \lim_{n\to\infty} \underbrace{\mathcal{E}^0(\chi_n, \chi_n)}_{\leq \frac{const.}{n}} = 0.$$

and so $(\mathcal{E}^0, D(\mathcal{E}^0))$ **is recurrent**, but we will see $\mathcal{E}$ **is not recurrent**.

Idea : **Symmetrize** $\mathcal{E}$

We have

$$\mathcal{E}(u,v) = -\int_{\mathbb{R}} Lu \cdot v \underbrace{\varphi\, dx}_{=\mu}.$$

We can find (a symmetrizing measure) $\widetilde{\mu}$, such that

$$\widetilde{\mathcal{E}}(u,v) := -\int_{\mathbb{R}} Lu \cdot v\, d\widetilde{\mu} = -\int_{\mathbb{R}} u \cdot Lv\, d\widetilde{\mu}, \ \forall u, v \in C_0^{\infty}(\mathbb{R})$$

and such that $\widetilde{\mathcal{E}}$ coincides with a sym. Dirichlet form $(\widetilde{\mathcal{E}}, D(\widetilde{\mathcal{E}}))$ on $C_0^{\infty}(\mathbb{R})$.

Moreover, if

$$(G_\alpha)_{\alpha>0} \leftrightarrow \mathcal{E} \ \text{ and } \ (\widetilde{G}_\alpha)_{\alpha>0} \leftrightarrow (\widetilde{\mathcal{E}}, D(\widetilde{\mathcal{E}})),$$

then

$$G_\alpha f = \widetilde{G}_\alpha f, \ \forall f \in L^2(\mathbb{R}, \mu) \cap L^2(\mathbb{R}, \widetilde{\mu}).$$

Conclusion : $\mathcal{E}$ is recurrent (transient) $\Longleftrightarrow$ $\widetilde{\mathcal{E}}$ is recurrent (transient).

This idea works with

$$\widetilde{\varphi}(x) := \exp\left(\int_0^x \frac{\varphi'(s)+2b}{\varphi(s)}ds\right) = \varphi(x)\exp\left(\int_0^x \frac{2b}{\varphi(s)}ds\right).$$

Note that our form $\mathcal{E}$ is

$$\mathcal{E}(f,g) := \underbrace{\frac{1}{2}\int_{\mathbb{R}} f'g'\varphi dx}_{=\mathcal{E}^0(f,g)} - \int_{\mathbb{R}} Bf'g\varphi dx,\ \ f,g \in C_0^\infty(\mathbb{R}).$$

In case of:

(a) $\varphi(x) = e^{-|x|},\ \ B(x) = \frac{1}{2}e^{|x|}$, we have

$$\int_0^\infty \frac{1}{\widetilde{\varphi}(x)}dx = \int_0^\infty \frac{e^x}{\exp(e^x-1)}dx = 1.$$

$\Longrightarrow$ $\widetilde{\mathcal{E}}$ is not recurrent by scale function arguments (cf. e.g book Mandl).

$\Longrightarrow$ $\mathcal{E}$ is **not recurrent**.

Moreover, $\mathcal{E}$ is **not conservative** and **does not satisfy the weak sector condition**,

$$\sup_{u,v\in C_0^\infty(\mathbb{R})} \frac{|\mathcal{E}(u,v)|}{\|u\|_{D(\mathcal{E}^0)}\|v\|_{D(\mathcal{E}^0)}} = \infty.$$

(b) $\varphi(x) = \inf(1, \frac{1}{|x|}),\ B(x) = \frac{1}{2}\varphi^{-1}(x),$

$$\int_0^\infty \frac{1}{\widetilde{\varphi}(x)}dx < 1 + \int_1^\infty \frac{1}{\exp(\frac{x^2+1}{2})}dx < \infty.$$

$\Longrightarrow$ $\widetilde{\mathcal{E}}$ is not recurrent $\Longrightarrow$ $\mathcal{E}$ is **not recurrent**.

Here $\mathcal{E}$ is **conservative** and **does not satisfy the strong sector condition**, i.e.

$$\sup_{u,v\in C_0^\infty(\mathbb{R})} \frac{|\mathcal{E}(u,v)|}{\mathcal{E}^0(u,u)^{1/2}\mathcal{E}^0(v,v)^{1/2}} = \infty.$$

(but we do not know whether the weak sector condition is satisfied).

(c) $\varphi(x) \equiv 1,\ B(x) \equiv b > 0$ then

$$\int_0^\infty \frac{1}{\widetilde{\varphi}(x)}dx = \int_0^\infty \frac{1}{\exp(2bx)}dx = \frac{1}{2b}.$$

$\Longrightarrow$ $\widetilde{\mathcal{E}}$ is not recurrent $\Longrightarrow$ $\mathcal{E}$ is **not recurrent**.

Here $\mathcal{E}$ is **conservative** and **satisfies the weak sector condition** but **not the strong sector condition** (otherwise $\mathcal{E}^0$ is transient, since $\mathcal{E}$ strictly irreducible).

Muckenhoupt weights

Let

- $E = \mathbb{R}^d$, $d \geq 2$
- $d\mu := \rho(x)dx$, $\rho > 0$ dx-a.e. such that

$$\mathcal{E}^0(f,g) := \frac{1}{2}\int_{\mathbb{R}^d} \langle \nabla f, \nabla g \rangle \mu(dx), \;\; f,g \in C_0^\infty(\mathbb{R}^d)$$

  is closable on $L^2(\mathbb{R}^d, \mu)$ with closure $(\mathcal{E}^0, D(\mathcal{E}^0))$.
- $B \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d, \mu)$ is μ-divergence free and there exist $M > 0$ and $\alpha \in \mathbb{R}$ such that

$$|\langle B(x), x \rangle| \leq M(1 + |x|)^\alpha$$

  for sufficiently large $|x|$.

Then as in 3. we can construct a generalized Dirichlet form $\mathcal{E}$ satisfying (H1)-(H3) and which is an extension of

$$\mathcal{E}(f,g) = \frac{1}{2}\int_{\mathbb{R}^d} \langle \nabla f, \nabla g \rangle \mu(dx) - \int_{\mathbb{R}^d} \langle B(x), \nabla f(x) \rangle g(x)\mu(dx), \;\; f,g \in C_0^\infty(\mathbb{R}^d).$$

Definition

Let $\psi \in \mathcal{B}(\mathbb{R}^d)$ with $\psi > 0$ dx-a.e. and A a positive constant

(i) *ψ is called a Muckenhoupt $\mathcal{A}_\beta$-weight (in notation $\psi \in \mathcal{A}_\beta$), $\beta \in (1,2]$, if for every ball $B \subset \mathbb{R}^d$,*

$$\left(\int_B \psi dx\right)\left(\int_B \psi^{-\frac{1}{\beta-1}} dx\right) \leq A \left(\int_B 1\, dx\right)^2.$$

(ii) *ψ is called a Muckenhoupt $\mathcal{A}_1$-weight (in notation $\psi \in \mathcal{A}_1$), if for every ball $B \subset \mathbb{R}^d$*

$$\left(\int_B \psi dx\right) \operatorname{ess.sup}_{x \in B} \frac{1}{\psi(x)} \leq A \int_B 1\, dx.$$

For $\varphi \in \mathcal{A}_\beta$, $\beta \in [1,2]$, the closability follows since

$$\mathcal{A}_\beta \subset \mathcal{A}_2 \quad \text{and} \quad \frac{1}{\varphi} \in L^1_{loc}(\mathbb{R}^d, dx) \quad \forall \varphi \in \mathcal{A}_2.$$

(a): Let $d = 2$ and ρ be a Muckenhoupt $\mathcal{A}_1$-weight.

Then for sufficiently large $r > 1$

$$v_1(r) = \mu(B_r) \leq Ar^2$$

and

$$v_2(r) = \int_{B_r} |\langle B(x), x\rangle|\rho(x)dx \leq C(1+r)^{\alpha+2}$$

By the last Corollary of 3. **if $\alpha \leq -2$, then $\mathcal{E}$ is not transient**.

(b): Let φ be a Muckenhoupt $\mathcal{A}_\beta$-weight with $1 \leq \beta \leq 2$. Then for $r > 1$ and some constant A, we have that

$$v_1(r) \sim Ar^{\beta d}$$

and by resuts of [Sturm 96, AOLDS III.] $(\mathcal{E}^0, D(\mathcal{E}^0))$ admits a strictly positive heat kernel $(p_t^0(x,y))_{t>0,\ x,y\in\mathbb{R}^d}$. Hence $(\mathcal{E}^0, D(\mathcal{E}^0))$ is strictly irreducible. By the Corollary again,

$$\beta d > 2 \Rightarrow \mathcal{E}^0 \text{ is transient} \Rightarrow \mathcal{E} \text{ is transient} \quad \forall \alpha \in \mathbb{R}.$$

(c): Let $\rho(x) := |x|^\eta$ with $\eta > -d$. Then

$$v_1(r) = \int_{B_r} |x|^\eta dx = Cr^{d+\eta}$$

and for sufficiently large $r > 1$,

$$v_2(r) = \int_{B_r} |\langle B(x), x\rangle| \cdot |x|^\eta dx \leq \begin{cases} C(1+r)^{\alpha+d+\eta} & \alpha+d+\eta \neq 0, \\ C\log(1+r) & \alpha+d+\eta = 0. \end{cases}$$

Then $\mathcal{E}$ is **not transient, if one of the following conditions is satisfied**

(b1) $d+\eta = 2$ and $\alpha \leq -2$.

(b2) $d+\eta \in (0,2)$ and $\alpha+d+\eta < 2$.

Similarly to [Stannat, 1999], [Tr 2003], $\exists$ **diffusion process associated with** $\mathcal{E}$ **and if** $d+\eta \in (0,1]$**, it is not a semimartingale**. Thus (b2) asserts that one can determine non-transience or recurrence even in the non semimartingale case.