An estimate of spectral gap for surface diffusion

Yukio NAGAHATA
Niigata University

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Plan

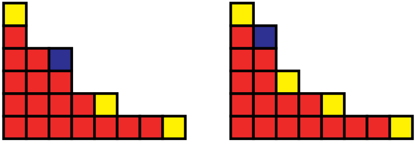
1. Introduction.
2. Results.
3. Idea of the proof.

Introduction

Model: Surface Diffusion

We consider evolutional model of a discrete surface:

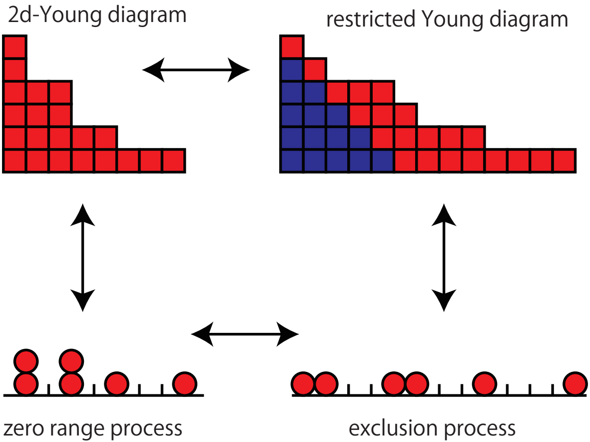
- surface is realized by 2-dim Young diagram
- number of cells in Young diagram is conserved quantity
- a cell at the edge can jump next corner with jump rate $1/(\mathrm{length})^2$



Introduction

Model: Surface Diffusion

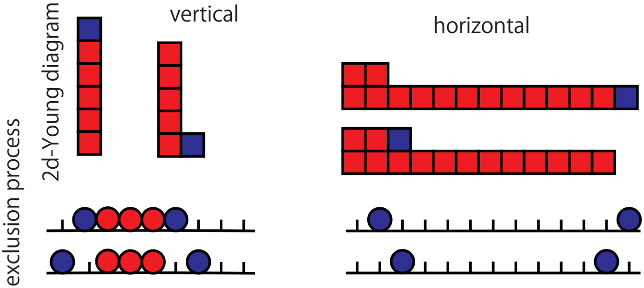
If we consider 2-dim Young diagram, restricted Young diagram, state space of zero range pr., and state space of exclusion pr., then each pair of them has bijection,



Introduction

Model: Surface Diffusion

In the exclusion model, we have two types of jumps.



Introduction

Notation

$\Lambda_n := \{1, 2, \ldots, n\}$
$\Sigma_n := \{0, 1\}^{\Lambda_n}$
$\Sigma_{n,K,M} := \{\eta \in \Sigma_n : \sum_{x \in \Lambda_n} \eta_x = K, \sum_{x \in \Lambda_n} x\eta_x = M\}$
$K = K_{\Lambda_n}(\eta) := \sum_{x \in \Lambda_n} \eta_x$: number of particles
$M = M_{\Lambda_n}(\eta) := \sum_{x \in \Lambda_n} x\eta_x$: (physical) moment
τ_x : shift operator

Introduction

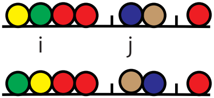
Notation

We set $\pi^{(i,j)}, \sigma^{(i,j)}$ for $i < j$ by

$$\pi^{(i,j)} f(\eta) := \sigma^{(i,j)} f(\eta) - f(\eta) = f(\sigma^{(i,j)}\eta) - f(\eta)$$

with

$$(\sigma^{(i,j)}\eta)_k = \begin{cases} \eta_i & \text{if } k = i-1 \\ \eta_{i-1} & \text{if } k = i \\ \eta_{j+1} & \text{if } k = j \\ \eta_j & \text{if } k = j+1 \\ \eta_k & \text{otherwise.} \end{cases}$$

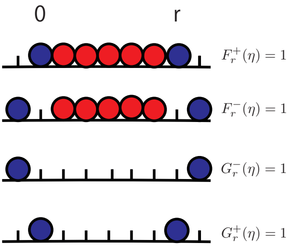


Introduction

<u>Notation</u>

We set $F_r^{\pm}, G^{\pm}$ as

$$F_r^+(\eta) := \mathbf{1}(\eta_{-1} = 0, \eta_0 = \ldots = \eta_r = 1, \eta_{r+1} = 0)(\eta)$$
$$F_r^-(\eta) := \mathbf{1}(\eta_{-1} = 1, \eta_0 = 0, \eta_1 = \ldots = \eta_{r-1} = 1, \eta_r = 0, \eta_{r+1} = 1)(\eta)$$
$$G_r^-(\eta) := \mathbf{1}(\eta_{-1} = 1, \eta_0 = \ldots = \eta_r = 0, \eta_{r+1} = 1)(\eta)$$
$$G_r^+(\eta) := \mathbf{1}(\eta_{-1} = 0, \eta_0 = 1, \eta_1 = \ldots = \eta_{r-1} = 0, \eta_r = 1, \eta_{r+1} = 0)(\eta).$$



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Then we define $c(x, y; \eta)$ by

$$c(x, y; \eta) := \frac{1}{|y-x|^2}\{\tau_x(F_{y-x}^+(\eta) + F_{y-x}^-(\eta)) + \tau_x(G_{y-x}^+(\eta) + G_{y-x}^-(\eta))\}$$

for $x < y$.

Introduction

Notation

Our generator is defined

$$Lf(\eta) := \sum_{x,y;x<y} c(x,y;\eta)\pi^{(x,y)}f(\eta).$$

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There are two conserved quantities;

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Hence uniform measure on $\Sigma_{n,K,M}$ for each fixed n, K, M becomes reversible measure.

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Hence uniform measure on $\Sigma_{n,K,M}$ for each fixed n, K, M becomes reversible measure.

Proposition [Funaki] (Equivalence of Ensembles)

Equivalence of ensembles holds true; i.e.,
there exists a function $\beta(x)$ such that the uniform measure on $\Sigma_{n,K,M}$ is approximated by inhomogeneous Bernoulli measure whose mean at site x is given by $\beta(x/N)$.

Results

Spectral gap

Set

$E_{n,K,M}$: the expectation w.r.t. uniform measure on $\Sigma_{n,K,M}$

$L_{n,K,M}$: the restriction of L on $\Sigma_{n,K,M}$

Definition of spectral gap

$$\lambda(n, K, M) := \inf\left\{ \frac{E_{n,K,M}[f(-L_{n,K,M})f]}{E_{n,K,M}[f^2]} \;\middle|\; E_{n,K,M}[f] = 0 \right\}$$

Results

Spectral gap

Proposition; An estimate of spectral gap [N]

There exists a constant C such that

$$\lambda(n, K, M) \geq \frac{C}{n^4}.$$

Results

Spectral gap; Upper bound estimate

We pick and fix "good" K, M such that β does not depend on the position. We take for $\alpha > 2$

$$f(\eta) := \sum_x x^\alpha \eta_x$$

Then we have

$$\begin{aligned} V[f] &= \beta(1-\beta)\frac{1}{2\alpha+1}n^{2\alpha+1} \\ D[f] &= \beta(\alpha-1)^2 n^{2\alpha-3} \end{aligned}$$

Hence we have

$$\lambda(n, K, M) \leq \frac{C}{n^4}.$$

Idea of the proof

Proof of the theorem

We consider mean field type process.

We set $T^{x,y;z}$ and $S^{x,y;z}$ for $x < y$ and $z < (y-x)/2$ by

$$T^{x,y;z}f(\eta) := S^{x,y;z}f(\eta) - f(\eta) = f(S^{x,y;z}\eta) - f(\eta)$$

with

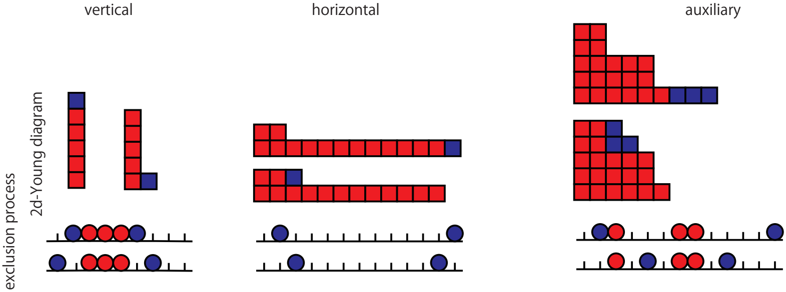
$$(S^{x,y;z}\eta)_k = \begin{cases} \eta_{x+z} & \text{if } k = x \\ \eta_x & \text{if } k = x+z \\ \eta_y & \text{if } k = y-z \\ \eta_{y-z} & \text{if } k = y \\ \eta_k & \text{otherwise.} \end{cases}$$

Idea of the proof

Proof of the theorem

We consider mean field type process.

Note that $T^{x,y;1} = \pi^{(x-1,y-1)}$.



Idea of the proof

Proof of the theorem

We consider mean field type process.

We define $c^m(x, y, z; \eta)$ for $x < y$ and $z < (y - x)/2$ by

$$c^m(x, y, z; \eta) := \frac{1}{(y-x)^2}\{\mathbf{1}(\eta_x = \eta_y = 1, \eta_{z+z} = \eta_{y-z} = 0) + \mathbf{1}(\eta_x = \eta_y = 0, \eta_{z+z} = \eta_{y-z} = 1)\}$$

Then we define a generator of the mean field type process by

$$L^m f(\eta) := \sum_{x,y,z;x<y,z<(y-x)/2} c^m(x, y, z; \eta) T^{x,y;z} f(\eta).$$

Idea of the proof

<u>Proof of the theorem</u>

We consider mean field type process.

$L^m_{n,K,M}$: the restriction of L^m on $\Sigma_{n,K,M}$

Definition of spectral gap for mean field type

$$\lambda^m(n,K,M) := \inf\left\{ \frac{E_{n,K,M}[f(-L^m_{n,K,M})f]}{E_{n,K,M}[f^2]} \;\middle|\; E_{n,K,M}[f] = 0 \right\}$$

Idea of the proof

Proof of the theorem

We consider mean field type process.

An estimate of spectral gap for mean field type

There exists a constant C such that

$$\lambda^m(n, K, M) \geq C.$$

A comparison estimate

There exists a constant C such that

$$\begin{aligned} E_{n,K,M}[f(-L^m_{n,K,M})f] &\leq Cn^3K^2E_{n,K,M}[f(-L_{n,K,M})f] \\ &\leq Cn^5E_{n,K,M}[f(-L_{n,K,M})f]. \end{aligned}$$

Idea of the proof

Proof of the theorem

If we separate these two propositions, our spectral gap estimate becomes $O(1/n^5)$.

However if we marge the proof of these two propositions, our spectral gap estimate becomes $O(1/n^4)$.

The main idea is

A comparison estimate with restriction

There exists a set A and a constant C such that

$$E_{n,K,M}[f(-L^m_{n,K,M})f\mathbf{1}(A)] \leq Cn^4 E_{n,K,M}[f(-L_{n,K,M})f\mathbf{1}(A)].$$

This estimate do not make sense if we separate our proof.

Idea of the proof

Proof of the theorem: Lu-Yau's martingale method

The proof of the spectral gap estimate for mean field type is due to Lu-Yau's martingale (induction) method.

In general, we have

$$V[f] = E[(f - E[f])^2] = E[E[(f - E[f|\eta_n])^2|\eta_n]] + E[(E[f|\eta_n] - E[f])^2]$$

We assume that the first part is already estimated by using $n-1$-th Dirichlet form as an induction assumption, i.e., there is $W(n-1)$ such that

$$\begin{aligned} &E[(f - E[f|\eta_n])^2|\eta_n] \\ &\leq W(n-1)E[\sum_{x,y,z;x<y,z<(y-x)/2} c(x,z;\eta)(\pi^{(x,z)}f(\eta))^2|\eta_n] \end{aligned}$$

The second part is rewritten by

$$E[(E[f|\eta_n] - E[f])^2] = P(\eta_n = 1)P(\eta_n = 0)(E[f|\eta_n = 1] - E[f|\eta_n = 0])^2.$$

Idea of the proof

Proof of the theorem: Lu-Yau's martingale method

separate case

By direct computation, we have

$$
\begin{aligned}
&P(\eta_n = 1)P(\eta_n = 0)(E[f|\eta_n = 1] - E[f|\eta_n = 0])^2 \\
&\leq \frac{C_1}{n^2} E[\sum_{x,z;z<(n-x)/2} (T^{x,n;z} f(\eta))^2] + \frac{C_2}{n^2} V[f] \\
&\leq C_1 E[\sum_{x,z;z<(n-x)/2} c^m(x,n,z;\eta)(T^{x,n;z} f(\eta))^2] + \frac{C_2}{n^2} V[f]
\end{aligned}
$$

for some constant C_1, C_2 which are independent of n, K, M, f.
(This is very hard computation.)

Idea of the proof

Proof of the theorem: Lu-Yau's martingale method

marge case

By direct computation, we have

$$
\begin{aligned}
&P(\eta_n = 1)P(\eta_n = 0)(E[f|\eta_n = 1] - E[f|\eta_n = 0])^2 \\
&\le \frac{C_1}{n^2} E[\sum_{x,z;z<(n-x)/2} (T^{x,n;z} f(\eta))^2 \mathbf{1}(A)] + \frac{C_2}{n^2} V[f] + P(A^c)V[f] \\
&\le C_1 E[\sum_{x,z;z<(n-x)/2} c^m(x,n,z;\eta)(T^{x,n;z} f(\eta))^2 \mathbf{1}(A)] + \frac{C_3}{n} V[f]
\end{aligned}
$$

for some constant C_1, C_2, C_3 which are independent of n, K, M, f and C_3 is small enough.
(This is very hard computation.)

Idea of the proof

Proof of the theorem: Lu-Yau's martingale method

We apply

A comparison estimate with restriction

There exists a set A and a constant C such that

$$E[\sum_{x,z;z<(n-x)/2} c^m(x,n,z;\eta)(T^{x,n;z}f(\eta))^2\mathbf{1}(A)]$$
$$\leq Cn^3E[\sum_{x,z} c(x,z;\eta)(\pi^{(x,z)}f(\eta))^2\mathbf{1}(A)]$$

Note that this inequality implies

$$E_{n,K,M}[f(-L^m_{n,K,M})f\mathbf{1}(A)] \leq Cn^4E_{n,K,M}[f(-L_{n,K,M})f\mathbf{1}(A)].$$

Idea of the proof

Proof of the theorem: Lu-Yau's martingale method

We have

$$
\begin{aligned}
V[f] &\le W(n-1)E[\sum_{x,z} c(x,z;\eta)(\pi^{(x,z)}f(\eta))^2|\eta_n] \\
&\quad + C_1' n^3 E[\sum_{x,z} c(x,z;\eta)(\pi^{(x,z)}f(\eta))^2] + \frac{C_3}{n}V[f] \\
&\le (1+\frac{C_3'}{n})(W(n-1)+C_1' n^3)[\sum_{x,z} c(x,z;\eta)(\pi^{(x,z)}f(\eta))^2 \mathbf{1}(A)]
\end{aligned}
$$

This inequality says that

$$
W(n) \le (1+\frac{C_3'}{n})(W(n-1)+C_1' n^3).
$$

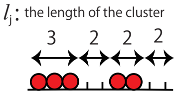
Hence we have

$$
W(n) \le Cn^4 \quad \text{for some constant } C.
$$

Idea of the proof

Why we need A and how to select A

Given a configuration η we set $\{l_i\}$ the length of the cluster;



Then we should have to use following inequality;

$$\begin{aligned}
E[(f(\eta)-f(\eta'))^2] &= E[\{\sum_j (f(\eta^{j-1})-f(\eta^j))\}^2] \\
&= E[\{\sum_j l_j \times \frac{1}{l_j}(f(\eta^{j-1})-f(\eta^j))\}^2] \\
&\leq E[\sum_j l_j^2 \sum_j \frac{1}{l_j^2}(f(\eta^{j-1})-f(\eta^j))^2] \\
&\leq \sup_\eta \sum_j l_j^2 E[\sum_j \frac{1}{l_j^2}(f(\eta^{j-1})-f(\eta^j))^2]
\end{aligned}$$

Idea of the proof

Why we need A and how to select A

It is not difficult to see that

$$\sup_{\eta} \sum_j l_j^2 = Cn^2, \quad \text{for some constant } C,$$

but

$$E[\sum_j l_j^2] = C'n, \quad \text{for some constant } C'.$$

Idea of the proof

Why we need A and how to select A

Hence if we set

$$A := \{\eta; \sum_j l_j^2 < C''n\}, \quad \text{for some constant } C'' \gg C'$$

Then $P(A^c) < O(1/n)$ and

$$E[(f(\eta) - f(\eta'))^2 \mathbf{1}(A)] \leq E[\sum_j l_j^2 \sum_j \frac{1}{l_j^2}(f(\eta^{j-1}) - f(\eta^j))^2 \mathbf{1}(A)]$$

$$\leq \sup_{\eta \in A} \sum_j l_j^2 E[\sum_j \frac{1}{l_j^2}(f(\eta^{j-1}) - f(\eta^j))^2 \mathbf{1}(A)]$$

$$\leq C''nE[\sum_j \frac{1}{l_j^2}(f(\eta^{j-1}) - f(\eta^j))^2 \mathbf{1}(A)]$$

as we required

Thank you for your attention!