Invariance Principle for Variable Speed Random Walks on Trees

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Variable speed BMs and RWs on $\mathbb{R}$

1. **Donsker**: $\hat{X} = (\hat{X}_t)_{t\geq 0}$ (cont. time) RW on $\mathbb{Z}$, $X^n_t := \frac{1}{n}\hat{X}_{n^2 t}$
   - X^n_t is RW on $\frac{1}{n}\mathbb{Z}$ with (homogeneous) jump rates $n^2 = n \cdot n$

   $\rightsquigarrow$ $X^n \xrightarrow{\mathcal{L}} X$ in Skorohod pathspace, where X is Brownian motion

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$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t$$

   - "*variable speed*", *speed measure* $\nu = \frac{1}{\sigma^2} \cdot \lambda_{\mathbb{R}}$

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   - Need *variable rates* for RWs X^n on $\frac{1}{n}\mathbb{Z}$ to obtain $X^n \xrightarrow{\mathcal{L}} X$.
   - For measure ν_n on $\frac{1}{n}\mathbb{Z}$, let X^n jump from x at rate $\frac{n}{\nu_n(\{x\})}$

     $\rightsquigarrow$ We call X^n the **ν_n-random walk** (**ν_n-RW**)
   - **Example**: $\nu = \lambda_{\mathbb{R}}$, $\nu_n = n \cdot \#$ $\rightsquigarrow$ situation of 1
     Observe: $\nu_n \to \nu$ *vaguely*.

Speed-ν motions on $\mathbb{R}$

ν-BM: $\nu = \sigma^{-2}\lambda_{\mathbb{R}}$ solves $\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t$; ν-RW: jump rates $\frac{n}{\nu(\{x\})}$ on $\frac{1}{n}\mathbb{Z}$

What do ν-BMs and ν-RWs have in common?

- process on $\mathrm{supp}(\nu)$, does not jump over points in $\mathrm{supp}(\nu)$
- *natural scale*, i.e. $\mathbb{P}_x(\{\, \tau_a < \tau_b \,\}) = \frac{b-x}{b-a}$ if $x \in [a, b]$
- *speed* determined by ν $\rightsquigarrow$ **process characterised by ν**

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- *time transformation* of BM: Let $L_s(x)$ be local time of B at x

$$X_t = B_{s_t}, \qquad s_t = \inf\{ s \mid \int L_s \,\mathrm{d}\nu > t \} \tag{1}$$

- Alternative: characterise via *occupation time formula* $(x > a)$

$$\mathbb{E}_x\Big(\int_0^{\tau_a} f(X_t)\,\mathrm{d}t\Big) = \frac{1}{2}\int_a^{\infty} f(y)\big(|y-a| \wedge |x-a|\big)\,\nu(\mathrm{d}y) \tag{2}$$

$\rightsquigarrow$ For any *Radon measure ν* on $\mathbb{R}$, the **speed-ν motion** X on supp(ν) is defined by (1), or (2) + strong Markov property

Stone's invariance principle & Our goal

❷ SDE $\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t$, $\nu = \sigma^{-2}\lambda_{\mathbb{R}}$; ν_n-RW: rates $\frac{n}{\nu_n(\{x\})}$ on $\frac{1}{n}\mathbb{Z}$

Theorem ([Stone '63], continuity in ν of speed-ν motions)

$\nu_n \to \nu$ vaguely, $\mathrm{supp}(\nu_n) \to \mathrm{supp}(\nu)$ in local Hausdorff topology.
Then $X^n \xrightarrow{\mathcal{L}} X$ in pathspace. (X^n speed-ν_n motion, X speed-ν motion)

Example: can approximate SDE of ❷ by RWs on $\frac{1}{n}\mathbb{Z}$ with rates

$$n \cdot \nu\big([x, x + \tfrac{1}{n}]\big)^{-1} = n\Big(\int_x^{x+\frac{1}{n}} \frac{1}{\sigma^2(y)}\,\mathrm{d}y\Big)^{-1}$$

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Goal

Generalise Stone's theorem to converging trees $T_n \to T$ replacing $\mathbb{R}$

Example: RW on Galton-Watson tree (conditioned on size n) $\xrightarrow{\mathcal{L}}$ BM on Aldous's *Brownian Continuum Random Tree (CRT)*

This particular case shown in [Croydon '08: *Convergence of simple...*]

Stone's invariance principle & Our goal

2. SDE $\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t$, $\nu = \sigma^{-2}\lambda_{\mathbb{R}}$; ν_n-RW: rates $\frac{n}{\nu_n(\{x\})}$ on $\frac{1}{n}\mathbb{Z}$

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Goal

Generalise Stone's theorem to converging trees $T_n \to T$ replacing $\mathbb{R}$

Result (informal): *continuity* of $(T, \nu) \mapsto X$ for the speed-ν motion X on T

ToDo for precise formulation:

1. Define what we mean by *tree*: metric measure tree (T, ν)
2. Define *ν-motions* on metric measure trees
3. Define *convergence* of metric measure trees and of processes *living on different spaces* T_n, T

Metric trees for us, trees are metric spaces

A *metric space* (T, r) is **0-hyperbolic** if the *4-point condition* holds:

$$r(x_1,x_2)+r(x_3,x_4) \leq \max\{r(x_1,x_3)+r(x_2,x_4),\ r(x_1,x_4)+r(x_2,x_3)\}$$

0-hyperbolicity implies for $x, y, z \in T$:

- $[x, y] := \{ m \in T \mid r(x, y) = r(x, m) + r(m, y) \} \subseteq T$ is isometric to a subset of the interval $\big[0, r(x, y)\big] \subseteq \mathbb{R}$,
- $\#\big([x, y] \cap [y, z] \cap [z, x]\big) \leq 1$
- If (T, r) is connected, it is an *$\mathbb{R}$-tree*

Note: $\mathbb{R}$-trees (usually) have curvature $-\infty$ $\rightsquigarrow$ *no CD-condition*

(T, r) is **Heine-Borel** if *closed, bounded sets are compact*.

A measure ν is **boundedly-finite** if $\nu(A) < \infty$ for *bounded* $A \subseteq T$

Definition (Metric measure tree)

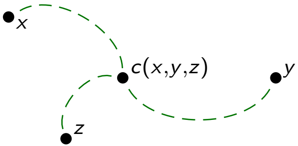
A (rooted) **metric measure tree** (**mm-tree**) is (T, r, ρ, ν), where

- (T, r) is a *0-hyperbolic Heine-Borel* space with

$$[x, y] \cap [y, z] \cap [z, x] = c(x, y, z) \in T$$

- $\rho \in T$, called the **root**
- ν is a *boundedly-finite* measure on T with full support, $\mathrm{supp}(\nu) = T$

$c(x, y, z)$ is the *branch point* corresponding to x, y, z



From now on, $\mathcal{X} = (T, r, \rho, \nu)$, $\mathcal{X}_n = (T_n, r_n, \rho_n, \nu_n)$ are mm-trees

Particular cases of metric trees

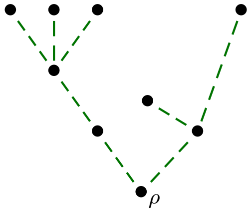
An **edge** in (T, r) is a pair (x, y), $x \neq y \in T$, with $[x, y] = \{x, y\}$

finite trees

- (T, E) finite (undirected) *graph tree* with *edge weights* $r_{x,y} > 0$, $(x, y) \in E$
- Metric r on T: maximal extension of $r(x, y) := r_{x,y}$, $(x, y) \in E$

$\rightsquigarrow$ (T, r) topologically discrete, 0-hyperbolic

$\rightsquigarrow$ *edges* are precisely the elements of E



finite tree: edges (dashed) are not in the space (T, r)

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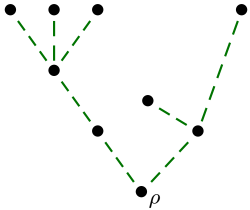

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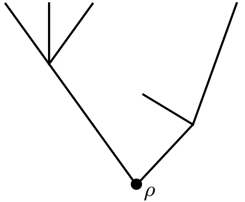
$\mathbb{R}$-trees

- *Connected* 0-hyperbolic $\rightsquigarrow$ *no edges*
- infinite degrees, dense leaves possible
- Hausdorff dimension arbitrary, topological dimension one $\rightsquigarrow$ *fractal*



corresponding **$\mathbb{R}$-tree**: no edges

Speed-ν motions on trees

$\mathcal{X} = (T, r, \rho, \nu)$ is an mm-tree. Recall $T = \mathrm{supp}(\nu)$, $c(x, y, z)$ is branch point

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Proposition ([Athreya, L., Winter '14$^+$: *Invariance...*], compact case)

If (T, r) is compact, then there exists a unique strong Markov process $X = (X_t)_{t\geq 0}$ on T satisfying the occupation time formula

$$\mathbb{E}_x\Big[\int_0^{\tau_y} f(X_s)\,\mathrm{d}s\Big] = 2\int_T r\big(y, c(x,y,z)\big) f(z)\,\nu(\mathrm{d}z)$$

for all $x, y \in T$, f bounded measurable, τ_y first hitting time of y.
ν is reversible for X. We call X the **speed-ν motion** *on (T, r).*

Proofidea.

Uniqueness: Resolvent calculation
Existence: Construct via *Dirichlet form* (see following slides) □

Speed-ν motions on trees

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Remark (general case)

- If (T, r) is not compact, we still get a unique strong Markov process X, also called **speed-ν motion**, by approximating with balls $B_R(\rho)$ $(R \to \infty)$. Or using the Dirichlet form
- If X is recurrent, we still obtain the *occupation time formula*

The Dirichlet form

Unsurprisingly, the *Dirichlet form* of the *speed-ν motion* in defined on $L^2(\nu)$ as the closure of $\big(\mathcal{E}, \mathcal{D}(\mathcal{E})\big)$ with

$$\mathcal{E}(f,g) = \tfrac{1}{2}\int_T \nabla f \cdot \nabla g \,\mathrm{d}\lambda_T, \qquad f,g \in \mathcal{D}(\mathcal{E}),$$
$$\mathcal{D}(\mathcal{E}) = \big\{\, f \in L^2(\nu) \cap \mathcal{C}_\infty \,\big|\, \nabla f \in L^2(\lambda_T) \,\big\},$$

but we have to define the *length measure* λ_T and the *gradient* ∇ properly.

The Dirichlet form: Length measure on metric trees

$\mathcal{E}(f,g) = \frac{1}{2}\int_T \nabla f \cdot \nabla g \,\mathrm{d}\lambda_T$, $\mathcal{D}(\mathcal{E}) = \{\, f \in L^2(\nu) \cap \mathcal{C}_\infty \mid \nabla f \in L^2(\lambda_T) \,\}$

Lemma (length measure λ_T)

*There is a unique measure $\lambda_T = \lambda_{(T,r,\rho)}$ (**length measure**) with*

$$\lambda_T\big([\rho,x]\big) = r(\rho,x) \ \forall x \in T \qquad \text{and} \qquad \lambda_T(L) = 0,$$

where L consists of ρ and all leaves that are not isolated in (T,r).

- For $T = \mathbb{R}$, $\lambda_\mathbb{R}$ is the Lebesgue-measure
- λ_T need *not* be *locally finite*, but is always *σ-finite*
- For **$\mathbb{R}$-trees** (i.e. (T,r) connected), λ_T is the usual length measure (1-dim. Hausdorff measure on $T \setminus L$)
  $\rightsquigarrow$ *non-atomic* and *independent of ρ*
- If (T,r) has an **edge**, λ_T has an *atom* at the end further away from ρ. Its mass equals the length of the edge.
  In particular, λ_T *depends* in this case *on ρ*

The Dirichlet form: Gradient on metric trees

$\mathcal{E}(f,g) = \frac{1}{2}\int_T \nabla f \cdot \nabla g \, \mathrm{d}\lambda_T, \;\; \mathcal{D}(\mathcal{E}) = \{ f \in L^2(\nu) \cap \mathcal{C}_\infty \mid \nabla f \in L^2(\lambda_T) \}$

Lemma (∇, [Athreya, L., Winter '14$^+$: *Invariance...*])

If $f: T \to \mathbb{R}$ is absolutely continuous, then there exists a unique (up to λ_T-zero sets) **gradient** $\nabla f \in L^1_{\mathrm{loc}}(\lambda_T)$ *with*

$$f(x) - f(\rho) = \int_{[\rho,x]} \nabla f \, \mathrm{d}\lambda_T \qquad \forall x \in T$$

- ∇ *depends on* ρ, even for $\mathbb{R}$-trees, where λ_T does not
- $T = \mathbb{R}_+$, $\rho = 0$: ∇ is the usual gradient on $\mathbb{R}$
- $T = \mathbb{R}$, $\rho = 0$: $\nabla f(x) = \mathrm{sgn}(x) f'(x) = \pm f'(x)$
- **finite tree**: $\nabla f(x) = f(x) - f(y)$ where (x,y) is the unique edge towards ρ, i.e. with $y \in [\rho, x]$

The Dirichlet form: speed-ν motion

$\mathcal{E}(f,g) = \frac{1}{2}\int_T \nabla f \cdot \nabla g \,\mathrm{d}\lambda_T$, $\mathcal{D}(\mathcal{E}) = \{\, f \in L^2(\nu) \cap \mathcal{C}_\infty \mid \nabla f \in L^2(\lambda_T) \,\}$

Fact [Fukushima, Oshima, Takeda '94]: To a regular Dirichlet form on $L^2(\nu)$ corresponds a ν-symmetric Markov process with generator G characterised by $\mathcal{E}(f,g) = \langle Gf, g\rangle_\nu$

Proposition ([Athreya, L., Winter '14$^+$: *Invariance...*])

$(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is closable and its closure is a regular Dirichlet form. The corresponding Markov process $X = (X_t)_{t\geq 0}$ is the speed-ν motion.

- Although λ_T and ∇ depend on ρ, the form $\mathcal{E}$ does not
- $\mathcal{E}$ need not be *conservative*, i.e. X may hit ∞ in finite time
- X has *continuous paths* iff (T, r) is an $\mathbb{R}$*-tree*, i.e. connected
- *Jumps* occur precisely over *edges*
- $T = \mathbb{R}$, $\nu = \lambda_{\mathbb{R}}$: X is standard Brownian motion
- $T = \mathbb{R}$, ν **arbitrary**: X is process considered in [Stone '63]

The Dirichlet form: speed-ν motion

$$\mathcal{E}(f,g) = \frac{1}{2}\int_T \nabla f \cdot \nabla g \,\mathrm{d}\lambda_T, \quad \mathcal{D}(\mathcal{E}) = \{\, f \in L^2(\nu) \cap \mathcal{C}_\infty \mid \nabla f \in L^2(\lambda_T) \,\}$$

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- (T, r) **$\mathbb{R}$-tree**: X is the *ν-BM* [Athreya, Eckhoff, Winter '13]
- (T, r) **finite tree**: X jumps from x to $y \sim x$ with rate

$$\gamma_{x,y} = \big(2\nu(\{x\})r(x,y)\big)^{-1}$$

- (T, r) **finite** tree with **unit edge-lengths**: $r_{x,y} = 1$ for $x \sim y$.
  ν **counting measure** $\rightsquigarrow$ *degree-dependent* total jump rate

$$\gamma_x = \sum_{y\sim x} \gamma_{x,y} = \frac{1}{2}\deg(x)$$

  We get the *constant speed* (simple) RW with

$$\nu(\{x\}) = \frac{1}{2}\deg(x)$$

Convergence of mm-trees

$\mathcal{X}_n = (T_n, r_n, \rho_n, \nu_n)$, $\mathcal{X} = (T, r, \rho, \nu)$ are mm-trees. Recall $T_n = \mathrm{supp}(\nu_n)$

Definition (Gromov-vague & Gromov-Hausdorff-vague topology)

$\mathcal{X}_n \xrightarrow{\mathrm{Gv}} \mathcal{X}$ (**Gromov-vague** convergence) if $(T_n, r_n), (T, r)$ can be *isometrically embedded* in a *metric space* (E, d) such that, in this embedding, $\rho_n \to \rho$ and for balls $B = B_R(\rho)$ of radius R in (E, d)

$$\nu_n\restriction_B \xrightarrow[n\to\infty]{\mathrm{w}} \nu\restriction_B \qquad \text{for almost all } R > 0$$

- *Gv-topology* is a simple modification of *Gromov-weak topology* (finite measures) [Greven, Pfaffelhuber, Winter '09]
- *Gromov's $\underline{\square}_1$-metric* [Gromov '99: *Metric structures...*] induces the Gw-topology, shown in [L. '13: *Equivalence of Gromov-Prohorov...*]

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$$\nu_n\restriction_B \xrightarrow[n\to\infty]{\mathrm{w}} \nu\restriction_B \qquad \text{for almost all } R > 0$$

$\mathcal{X}_n \xrightarrow{\mathrm{GHv}} \mathcal{X}$ (**Gromov-Hausdorff-vague** convergence) if additionally

$$T_n \cap B \xrightarrow{\mathrm{Hausdorff}} T \cap B \qquad \text{for almost all } R > 0$$

- *GHv-topology* closely related to *Gromov-Hausdorff-Prohorov* metric but subtle difference: full-support assumption / equivalence classes

$\rightsquigarrow$ The GHP-metric is *not complete* on spaces of spaces with full support

The gap between Gv and GHv topologies

Proposition ([Athreya, L., Winter '14^{+}: *The gap between...*])

- *The GHv-topology is Polish, the Gv-topology is Lusin*

- The Gv-topology *not* Polish. It becomes Polish if we drop the *Heine-Borel* assumption

The gap between Gv and GHv topologies

Proposition ([Athreya, L., Winter '14$^+$: *The gap between...*])

- *The GHv-topology is Polish, the Gv-topology is Lusin*
- $\mathcal{X}_n \xrightarrow{\text{GHv}} \mathcal{X}$ *if and only if* $\mathcal{X}_n \xrightarrow{\text{Gv}} \mathcal{X}$ *and the lower mass-bound*

$$\liminf_{n\to\infty} \inf_{x\in B_R(\rho_n)} \nu_n\big(B_\delta(x)\big) > 0 \qquad \forall R > 0 \tag{3}$$

In this case, (E, d) *can be chosen as Heine-Borel space.*

- Gv-topology is induced by the algebra of functions of the form

$$\Phi(\mathcal{X}) = \int_{T^m} \varphi\big(r(x_i, x_j)_{i,j=0,\dots,m}\big)\, \nu^{\otimes m}(\mathrm{d}x), \quad x_0 := \rho,$$

for $m \in \mathbb{N}$, $\varphi \in \mathcal{C}_c\big(\mathbb{R}^{(m+1)\times(m+1)}\big)$. Can use Le Cam:

$\rightsquigarrow$ For random variables: $\mathcal{X}_n \underset{\text{Gv}}{\xrightarrow{\mathcal{L}}} \mathcal{X} \Leftrightarrow \mathbb{E}[\Phi(\mathcal{X}_n)] \to \mathbb{E}[\Phi(\mathcal{X})]\ \forall \Phi$

- (3) acts as a *tightness condition*

Convergence of processes

We use the same *embedding approach* as for the definition of Gv- and GHV-topology to define convergence of *processes living on different spaces*:

Definition (convergence of processes on different spaces)

Let $X^n = (X^n_t)_{t\geq 0}$, $n \in \mathbb{N} \cup \{\infty\}$, be stochastic processes with values in (T_n, r_n). We say that X^n **converges** to $X = X^\infty$ **in pathspace** or **f.d.d.** if (T_n, r_n) can be *isometrically embedded* in a *metric space* (E, d) such that, in this embedding, X^n converges to X in pathspace or f.d.d., respectively, as E-valued processes.

The Result

$\mathcal{X}_n = (T_n, r_n, \rho_n, \nu_n)$; X^n is the speed-ν_n motion on (T_n, r_n) started in ρ_n

Theorem ([Athreya, L., Winter '14$^+$: *Invariance principle...*])

Assume X is conservative, and consider the conditions $(R > 0)$

1. *Edge-length bound:*
   $\limsup_{n\to\infty} \sup\{ r_n(x,y) \mid r_n(\rho_n, x) < R,\ (x,y) \text{ edge} \} < \infty$
2. *Gromov-vague convergence:* $\mathcal{X}_n \xrightarrow{\text{Gv}} \mathcal{X}$
3. *Lower mass-bound:* $\liminf_{n\to\infty} \inf_{x \in B_R(\rho_n)} \nu_n\big(B_\delta(x)\big) > 0$
4. *Diameter bound:* $\sup_n \operatorname{diam}(T_n, r_n) < \infty$

If 1, 2 and 3 hold, then X^n converges in pathspace to X.
If 1, 2 and 4 hold, then X^n converges f.d.d. to X.

- 1 is a weak condition; trivially satisfied for $\mathbb{R}$-trees (no edges)
- 4 can be weakened, but some condition is needed
- 2 + 3 is equivalent to *Gromov-Hausdorff-vague* convergence

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Theorem ([Athreya, L., Winter '14+: *Invariance principle...*])

Consider the conditions $(R > 0)$

1. *Edge-length bound:*
   $\limsup_{n\to\infty} \sup\{ r_n(x,y) \mid r_n(\rho_n, x) < R,\ (x,y) \text{ edge} \} < \infty$
2. *Gromov-vague convergence:* $\mathcal{X}_n \xrightarrow{\text{Gv}} \mathcal{X}$
3. *Lower mass-bound:* $\liminf_{n\to\infty} \inf_{x\in B_R(\rho_n)} \nu_n\big(B_\delta(x)\big) > 0$

If 1, 2 and 3 hold, then X^n converges in pathspace to a process Y on the one-point compactification of T, and Y killed at infinity coincides with X

- Speed ν-motions are always killed at infinity
- The limit process Y may hit ∞ and not stay there
  - $\leadsto$ Y looses the Markov property at ∞ and defines *entrance laws* for X from ∞

Proof strategy

For pathspace convergence (under the lower mass-bound):

1. **Tightness** using the *Aldous criterion*
2. **Strong Markov property** of all limit processes (compact case)
   - show *equicontinuity* of the maps $P_n \colon (t, x) \mapsto \mathcal{L}_x(X_t^n)$, $n \in \mathbb{N}$, where $\mathcal{L}_x$ is the law of a process started in x; use *Arzelà-Ascoli*
3. Show **occupation time formula** for all limit processes

   $\rightsquigarrow$ All *limit processes coincide* with X in the compact case
4. approximate non-compact trees with compact trees

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For the f.d.d.-case (without the lower mass-bound):

5. $\hat{T}_n \subseteq T_n$: $\nu_n(T_n \setminus \hat{T}_n) < \varepsilon$, $(\hat{T}_n, r_n, \nu_n)$ *satisfies mass-bound*
6. Show **closeness of marginals** of X^n and $\hat{X}^n$
   - Use a simple *heat-kernel bound* satisfied for speed-ν motions X:

$$\left\| q_t(x,\cdot) \right\|_2^2 \leq \nu(T)^{-1} + \mathrm{diam}(T) \cdot t^{-1} \qquad \forall x \in T,\ t > 0$$

7. Use *pathspace convergence* of $\hat{X}^n$ and Markov property

Thank you for your attention!

Arigatō gozaimasu!