Functional CLTs for non-symmetric random walks on crystal lattices

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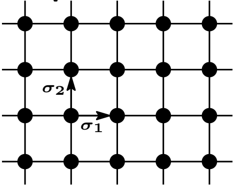
This talk is based on jointwork with
Satoshi Ishiwata (Yamagata) and Motoko Kotani (Tohoku).

♣ **Crystal Lattice X**

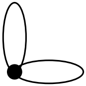
- $X = (V, E)$ **is a locally finite connected graph.**
- $\Gamma(\cong \mathbb{Z}^d) \curvearrowright X$, **freely**
- $X_0 = (V_0, E_0) := \Gamma \setminus X$ **is a finite graph.**

♠ **In other words,** X **is the abelian cover of a finite graph** X_0 **with the covering transformation group** Γ.

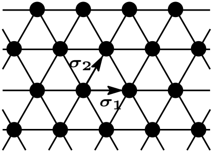
Square lattice



$\Gamma = \langle \sigma_1, \sigma_2 \rangle \simeq \mathbb{Z}^2$

$X_0 =$ 

Triangular lattice



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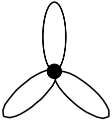
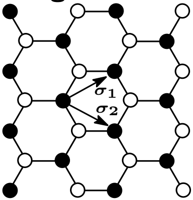
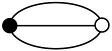
$X_0 =$ 

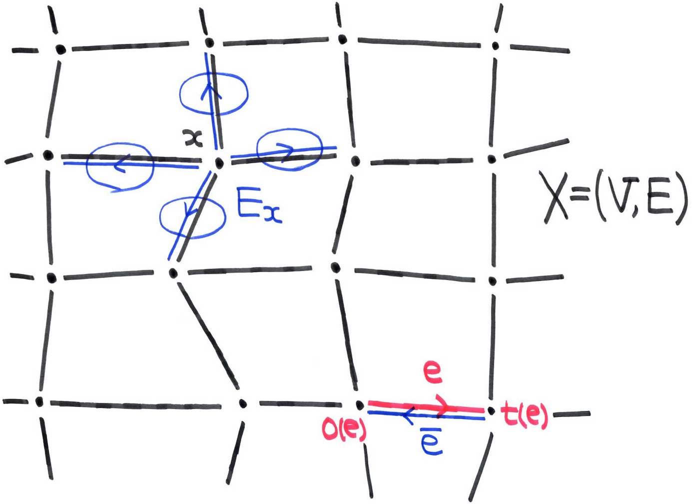
Hexagonal lattice



$\Gamma = \langle \sigma_1, \sigma_2 \rangle \simeq \mathbb{Z}^2$

$X_0 =$ 

Introduction



A random walk $\{x_n\}_{n=0}^{\infty}$ on X (resp. X_0) is characterized by a (Γ-invariant) transition probability $p = p(e) : E \to [0,1]$ with

$$\sum_{e \in E_x} p(e) = 1 \quad (x \in V), \qquad p(e) + p(\overline{e}) > 0 \quad (e \in E).$$

- A probability measure $\mathbb{P}_x$ $(x \in V)$ on

$$\Omega_x(X) := \{c = (e_1, e_2, \dots) \mid o(e_1) = x, \quad o(e_{n+1}) = t(e_n), n \in \mathbb{N}\}$$

is defined by

$$\mathbb{P}_x(\{c = (e_1, \dots, e_n, *, *, \dots\}) := p(e_1) \cdots p(e_n)$$

- $x_n(c) := o(e_{n+1})$ $(\in V)$, $c \in \Omega_x(X)$, $n = 0, 1, 2, \dots$

- $Lf(x) := \sum_{e \in E_x} p(e) f(t(e))$ **(transition operator)**
- $p(n, x, y) := (L^n \delta_y)(x)$ **(n-step transition probability)**

♣ **We assume the irreducibility on X_0:**

$$\forall x, y \in V_0, \exists n = n(x, y) \in \mathbb{N} \text{ s.t. } p(n, x, y) > 0.$$

Remark: **irreducibility on X $\Longrightarrow$ irreducibility on X_0**

Then by the Perron–Frobenius theorem,

$\exists!\ m = (m(x))_{x \in V_0}$ **(L-invariant measure) s.t.**

- $\sum_{x \in V_0} m(x) = 1,\ m(x) > 0\ (x \in V_0)$,
- $m(x) = {}^t Lm(x) \Big(:= \sum_{e \in (E_0)_x} p(\overline{e}) m(t(e)) \Big)\ (x \in V_0)$

- $\widetilde{m}(e) := p(e)m(o(e))$
- **We define the homological direction (of the RW) $\gamma_p \in H_1(X_0, \mathbb{R})$ by $\gamma_p := \sum_{e \in E_0} \widetilde{m}(e)e$.**
- **RW is (m-)symmetric $\underset{\text{def}}{\Longleftrightarrow}$ $\widetilde{m}(e) = \widetilde{m}(\overline{e})$ $\underset{\text{iff}}{\Longleftrightarrow}$ $\gamma_p = 0$**

♣ **Aim of this talk :**

Long time behavior of the (non-symmetric) RW

- **Generalizations of Donsker's invariance principle:**

$$\left(\frac{1}{\sqrt{n}} x_{[nt]}\right)_{t \geq 0} \Longrightarrow (B_t)_{t \geq 0} \quad \text{as } n \to \infty$$

- **(A refinement of) Local CLT:**

$$p(n,x,y)m(y)^{-1} \sim (2\pi n)^{-d/2} K \cdot \mathrm{vol}(\mathrm{Alb}^{\Gamma}) \times \exp\left(- \frac{\left|\Phi_0(x) - \Phi_0(y) - n\rho_{\mathbb{R}}(\gamma_p)\right|_{g_0}^2}{2n} \right)$$

$\Longrightarrow$ **Last week conference in Osaka**

♣ **(Usual) probabilist's viewpoint:**

Realize the crystal lattice into $\mathbb{R}^d$ with the canonical metric firstly, then study limit theorems.

◦ **Several text books of Spitzer, Woess, Lawler, ...**

♣ **(Some) geometer's viewpoint:**

Study the most "natural realization" of the crystal lattice through these limit theorems.

∘ Kotani–Shirai–Sunada ('98) ∘ Shirai ('03)

∘ Kotani–Sunada ('00∼'06) ... "standard realization" (harmonic realization & Albanese metric)

- **Berger–Biskup ('07), etc ... "harmonic coordinate"**

♣ **(modified) harmonic realization** $\Phi_0 : X \to \Gamma \otimes \mathbb{R}$,

$$L\Phi_0 - \Phi_0 = \rho_{\mathbb{R}}(\gamma_p)$$

(uniquely determined up to translation), where

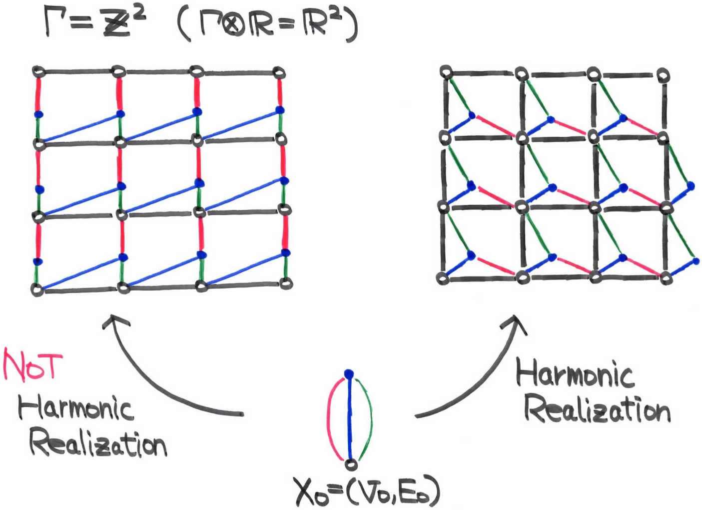
$\rho_{\mathbb{R}} : H_1(X_0, \mathbb{R}) \to \Gamma \otimes \mathbb{R}$ **is defined by**

$$\rho_{\mathbb{R}}([c]) \cdot o(\tilde{c}) = t(\tilde{c}) \text{ on } X \quad \text{for } [c] \in H_1(X_0, \mathbb{R}).$$

- $\rho_{\mathbb{R}}(\gamma_p)$ **is called the asymptotic direction.**

♠ **A discrete version of**

$$\partial_\alpha(A^{\alpha\beta}(x)\partial_\beta\Phi(x)^i) = \rho_{\mathbb{R}}(\gamma_p)^i \text{ with } A^{\alpha\beta} \neq A^{\beta\alpha}$$


Γ=ℤ² (Γ⊗ℝ=ℝ²)
NoT
Harmonic
Realization
Harmonic
Realization
X₀=(V₀,E₀)

♣ **Albanese metric g_0 on $\Gamma \otimes \mathbb{R}$: the dual metric of $\langle\langle \cdot, \cdot \rangle\rangle$ (restricted to $\mathrm{Hom}(\Gamma, \mathbb{R})$) through the maps $\rho_{\mathbb{R}}$ and ${}^t\rho_{\mathbb{R}} : \mathrm{Hom}(\Gamma, \mathbb{R}) = (\Gamma \otimes \mathbb{R})^* \hookrightarrow H^1(X_0, \mathbb{R})$.**

- **Due to the discrete Hodge–Kodaira theorem (Kotani–Sunada ('06)), we may identify $H^1(X_0, \mathbb{R})$ with**

$$\mathcal{H}^1(X_0) = \Big\{\omega : E_0 \to \mathbb{R} \,|\, \omega(\overline{e}) = -\omega(e), \quad (\delta_p \omega)(x) + \langle \gamma_p, \omega \rangle = 0, \quad x \in V_0 \Big\}, \quad \text{where}$$

$$(\delta_p \omega)(x) := -\textstyle\sum_{e \in (E_0)_x} p(e)\omega(e)$$

- **We equip $H^1(X_0, \mathbb{R}) \simeq \mathcal{H}^1(X_0)$ with the inner product**

$$\langle\langle \omega_1, \omega_2 \rangle\rangle := \sum_{e \in E_0} \omega_1(e)\omega_2(e)\widetilde{m}(e) - \langle \gamma_p, \omega_1 \rangle \langle \gamma_p, \omega_2 \rangle$$

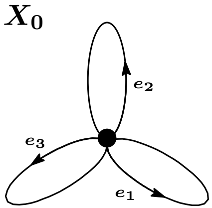
We can summarize as

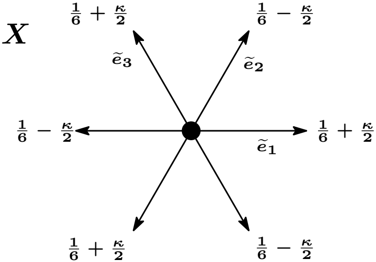
$$\begin{array}{ccc} (\Gamma \otimes \mathbb{R}, g_0) & \xleftarrow{\rho_{\mathbb{R}}} & H_1(X_0, \mathbb{R}) \\ \updownarrow \text{dual} & & \updownarrow \text{dual} \\ \mathrm{Hom}(\Gamma, \mathbb{R}) & \overset{{}^t\rho_{\mathbb{R}}}{\hookrightarrow} & H^1(X_0, \mathrm{R}) \simeq \big(\mathcal{H}^1(X_0), \langle\!\langle \cdot, \cdot \rangle\!\rangle\big) \end{array}$$

- $\mathrm{vol}(\mathrm{Alb}^{\Gamma}) := \mathrm{vol}(\Gamma \otimes \mathbb{R}/\Gamma, g_0)$

Remark: $\gamma_p = 0 \Longrightarrow \rho_{\mathbb{R}}(\gamma_p) = 0$.

But, the converse doesn't hold in general ! **(e.g. A class of non-symmetric RWs on the triangular lattice)**



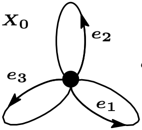


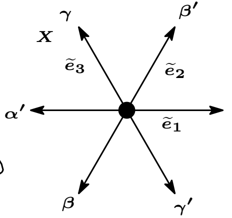
- $\gamma_p = \kappa(e_1 - e_2 + e_3) \in H_1(X_0, \mathbb{R})$
- $\rho_{\mathbb{R}}(\gamma_p) = 0 \in \Gamma \otimes \mathbb{R} \simeq \mathbb{R}^d$
- $\mathrm{vol}(\mathrm{Alb}^{\Gamma}) = \sqrt{3}$

◦ **Teruya ('12), Ishiwata-K-Teruya ('15, MJOU)**

$$p(n,x,y) \sim \sqrt{3}(2\pi n)^{-2/2} \exp\Big(-\frac{1}{2n}|\Phi_0(x)-\Phi_0(y)|^2_{g_0}\Big)$$
$$\times \Big\{1 + \big(-\frac{1}{2} - \frac{3\kappa^2}{2}\big)n^{-1}\Big\} \quad \text{as } n \to \infty.$$

Examples (Square lattice, Triangular lattice, Hexagonal lattice)





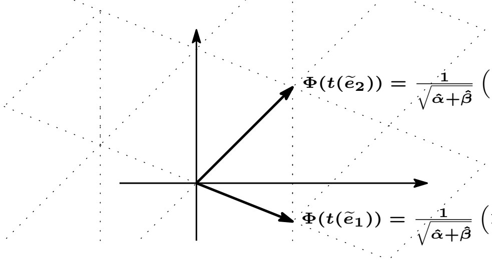
- $\hat{\alpha} := \alpha + \alpha'$ $\quad \hat{\beta} := \beta + \beta'$ $\quad \hat{\gamma} := \gamma + \gamma'$
- $m \equiv 1$

$$\gamma_p = (\alpha - \alpha')e_1 - (\beta - \beta')e_2 + (\gamma - \gamma')e_3$$

$$\rho_{\mathbb{R}}(\gamma_p) = \{(\alpha - \alpha') - (\gamma - \gamma')\}\,\sigma_1 + \{(\gamma - \gamma') - (\beta - \beta')\}\,\sigma_2$$

Assume $\kappa := \alpha - \alpha' = \beta - \beta' = \gamma - \gamma' \neq 0 \Longrightarrow \gamma_p \neq 0,\ \rho_{\mathbb{R}}(\gamma_p) = 0$

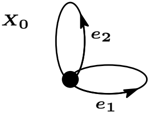
$$\mathrm{vol}(\mathrm{Alb}^{\Gamma})^{-2} = \hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\gamma} + \hat{\gamma}\hat{\alpha}$$

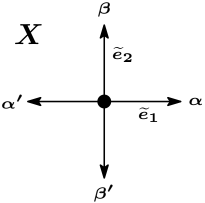


$$\Phi(t(\widetilde{e}_2)) = \frac{1}{\sqrt{\hat{\alpha}+\hat{\beta}}}\Big(1,\ \mathrm{vol}(\mathrm{Alb}^{\Gamma})\hat{\alpha}\Big)$$

$$\Phi(t(\widetilde{e}_1)) = \frac{1}{\sqrt{\hat{\alpha}+\hat{\beta}}}\Big(1,\ -\mathrm{vol}(\mathrm{Alb}^{\Gamma})\hat{\beta}\Big)$$

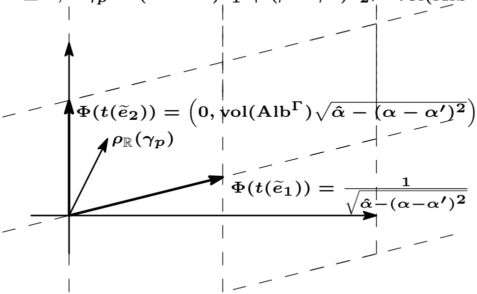
Examples (Square lattice, Triangular lattice, Hexagonal lattice)





- $\hat{\alpha} := \alpha + \alpha'$
- $\hat{\beta} := \beta + \beta'$

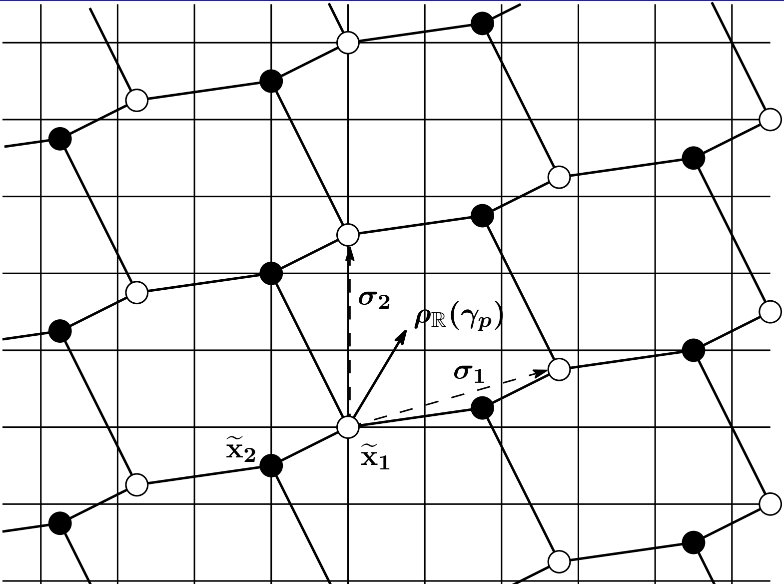
$$\Longrightarrow m \equiv 1, \quad \gamma_p = (\alpha - \alpha')e_1 + (\beta - \beta')e_2, \quad \mathrm{vol}(\mathrm{Alb}^\Gamma)^{-2} = \hat{\alpha}\hat{\beta} - \hat{\alpha}(\beta - \beta')^2 - (\alpha - \alpha')^2\hat{\beta}$$



$$\Phi(t(\widetilde{e}_2)) = \left(0, \mathrm{vol}(\mathrm{Alb}^\Gamma)\sqrt{\hat{\alpha} - (\alpha - \alpha')^2}\right)$$

$$\Phi(t(\widetilde{e}_1)) = \frac{1}{\sqrt{\hat{\alpha} - (\alpha - \alpha')^2}} \left(1, \mathrm{vol}(\mathrm{Alb}^\Gamma)(\alpha - \alpha')(\beta - \beta')\right)$$

Examples (Square lattice, Triangular lattice, Hexagonal lattice)



- **We define a RW $\{\xi_n\}_{n=0}^{\infty}$ (starting from 0) on $\Gamma \otimes \mathbb{R}$ by**

$$\xi_n(c) := \Phi_0(x_n(c)), \quad c \in \Omega_{x_0}(X),$$

where $x_0 \in V$ is a fixed basepoint such that $\Phi_0(x_0) = 0$.

♣ **(LLN, Kotani–Sunada ('06))**

$$\lim_{n\to\infty} \frac{1}{n}\xi_n(c) = \rho_{\mathbb{R}}(\gamma_p), \quad \mathbb{P}_{x_0}\text{-a.s. } c \in \Omega_{x_0}(X).$$

Define $\mathbb{X}^{(n)} : \Omega_{x_0}(X) \to \mathcal{W} := C([0,\infty), \Gamma \otimes \mathbb{R})$ by the piecewise linear interpolation of

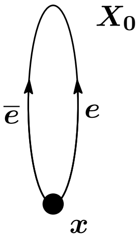
$$\mathcal{X}_t^{(n)}(c) := \frac{1}{\sqrt{n}}\Big(\xi_{[nt]}(c) - [nt]\rho_{\mathbb{R}}(\gamma_p)\Big), \quad t \geq 0.$$

Theorem 1 : (1st Functional CLT)

$$\mathbb{X}^{(n)} \Longrightarrow (B_t)_{t\geq 0} \quad \text{as } n \to \infty, \quad \text{where}$$

$(B_t)_{t\geq 0}$ is a $\Gamma \otimes \mathbb{R}$–valued standard BM with $B_0 = 0$.

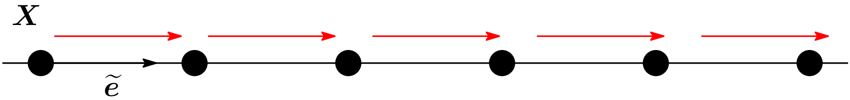
Remark:



$$p(e) = 1,\ p(\overline{e}) = 0$$

$$\Longrightarrow m(x) \equiv 1,\ \gamma_p = e$$

$$\Longrightarrow \langle\!\langle \omega_1, \omega_2 \rangle\!\rangle \equiv 0$$



$$\|\mathbf{x}\|_{\Gamma\otimes\mathbb{R}} = \begin{cases} 0 & (\mathbf{x} = 0) \\ \infty & (\mathbf{x} \neq 0) \end{cases}$$

We introduce a family of transition probabilities $\{p_\varepsilon\}_{0\le\varepsilon\le1}$ by $p_\varepsilon(e) := p_0(e) + \varepsilon q(e)$, where

$$\begin{aligned} p_0(e) &:= \frac{1}{2}\Big(p(e) + \frac{m(t(e))}{m(o(e))}p(\overline{e})\Big), \\ q(e) &:= \frac{1}{2}\Big(p(e) - \frac{m(t(e))}{m(o(e))}p(\overline{e})\Big). \end{aligned}$$

Lemma 1: (1) $\gamma_{p_\varepsilon} = \varepsilon\gamma_p$ for $0 \le \varepsilon \le 1$.

(2) $p_\varepsilon(e) > 0,\ e \in E_0$ for $0 \le \varepsilon < 1$

(3) p_0: (m-)symmetric, q: (m-)anti-symmetric

♠ $\{p_\varepsilon\}$ interpolates between the original (non-symmetric) RW and a symmetric RW.

- $\mathcal{H}^1_{(\varepsilon)}(X_0)$ denotes the set of the modified harmonic 1-forms associated with p_ε equipped with the inner product

$$\langle\langle \omega_1, \omega_2 \rangle\rangle_{(\varepsilon)} := \sum_{e \in E_0} \omega_1(e)\omega_2(e)p_\varepsilon(e)m(o(e)) - \varepsilon^2 \langle \gamma_p, \omega_1 \rangle \langle \gamma_p, \omega_2 \rangle$$

♠ Since the identification $H^1(X_0, \mathbb{R}) \simeq \mathcal{H}^1_{(\varepsilon)}(X_0)$ depends on ε, we write ${}^t\rho_{\mathbb{R}}(\omega)$ as $\omega^{(\varepsilon)}$ for $\omega \in \mathrm{Hom}(\Gamma, \mathbb{R})$.

- **Albanese metric** $g_0^{(\varepsilon)}$ on $\Gamma \otimes \mathbb{R}$: the dual metric of $\langle\langle \cdot, \cdot \rangle\rangle_{(\varepsilon)}$
- $(\Gamma \otimes \mathbb{R})_{(\varepsilon)} := (\Gamma \otimes \mathbb{R}, g_0^{(\varepsilon)})$

Lemma 2 : For $\omega_1, \omega_2 \in \mathrm{Hom}(\Gamma, \mathbb{R})$ and $\mathrm{x}, \mathrm{y} \in \Gamma \otimes \mathbb{R}$,

(1) $\langle\!\langle \omega_1^{(\varepsilon)}, \omega_2^{(\varepsilon)} \rangle\!\rangle_{(\varepsilon)} \to \langle\!\langle \omega_1^{(0)}, \omega_2^{(0)} \rangle\!\rangle_{(0)}$ as $\varepsilon \searrow 0$,

(2) $\langle \mathrm{x}, \mathrm{y} \rangle_{g_0^{(\varepsilon)}} \to \langle \mathrm{x}, \mathrm{y} \rangle_{g_0^{(0)}}$ as $\varepsilon \searrow 0$.

- $L_\varepsilon = L_0 + \varepsilon Q$: the transition operator associated with p_ε
- (modified) harmonic realization $\Phi_0^{(\varepsilon)} : X \to \Gamma \otimes \mathbb{R}$:

$$L_\varepsilon \Phi_0^{(\varepsilon)} - \Phi_0^{(\varepsilon)} = \rho_{\mathbb{R}}(\gamma_{p_\varepsilon}) \Big(= \varepsilon \rho_{\mathbb{R}}(\gamma_p) \Big)$$

(uniquely determined up to translation)

- A RW $\{\xi_n^{(\varepsilon)}\}_{n=0}^{\infty}$ (starting from 0) on $(\Gamma \otimes \mathbb{R})_{(0)}$ is defined by

$$\xi_n^{(\varepsilon)}(c) := \Phi_0^{(\varepsilon)}(x_n(c)), \quad c \in \Omega_{x_0}(X),$$

where $x_0 \in V$ is a fixed basepoint such that $\Phi_0^{(\varepsilon)}(x_0) = 0$.

Define $\mathbb{Y}^{(\varepsilon,n)} : \Omega_{x_0}(X) \to \mathcal{W}_{(0)} := C([0,\infty),(\Gamma \otimes \mathbb{R})_{(0)})$ by the piecewise linear interpolation of

$$\mathcal{Y}_t^{(\varepsilon,n)}(c) := \frac{1}{\sqrt{n}} \xi_{[nt]}^{(\varepsilon)}(c), \quad t \geq 0.$$

<u>Theorem 2 :</u> (2nd Functional CLT)

$$\mathbb{Y}^{(n^{-1/2},n)} \Longrightarrow \Big(B_t + \rho_{\mathbb{R}}(\gamma_p)t\Big)_{t\geq 0} \quad \text{as } n \to \infty, \text{ where}$$

$(B_t)_{t\geq 0}$ is a $(\Gamma \otimes \mathbb{R})_{(0)}$-valued standard BM with $B_0 = 0$.

- **transition-shift operator** L_{γ_p} **on** $C(X \times H_1(X_0, \mathbb{R}))$ **:**

$$L_{\gamma_p} f(x, \mathrm{z}) := \sum_{e \in E_x} p(e) f(t(e), \mathrm{z}+\gamma_p), \ \ x \in V, \ \mathrm{z} \in H_1(X_0, \mathbb{R})$$

- **scaling operator** P_ε : $C_\infty(\Gamma \otimes \mathbb{R}) \to C_\infty(X \times \mathrm{H}_1(X_0, \mathbb{R}))$

$$P_\varepsilon f(x, \mathrm{z}) := f\big(\varepsilon(\Phi_0(x) - \rho_{\mathbb{R}}(\mathrm{z}))\big)$$

♣ **Ergodic theorem:**

$$\frac{1}{N}\sum_{j=0}^{N-1} L^j h(x) = \sum_{y \in V_0} h(y) m(y) + O(1/N), \ \ \ x \in V_0$$

$$\Longrightarrow \ \| \tfrac{1}{N\varepsilon^2}(I - L_{\gamma_p}^N) P_\varepsilon f - P_\varepsilon(\tfrac{1}{2}\Delta_{g_0}) f \|_\infty \to 0 \ \text{as} \ N\varepsilon^2 \searrow 0$$

By Trotter's approximation theorem,

$\|L_{\gamma_p}^{[nt]} P_{n^{-1/2}} f - P_{n^{-1/2}} e^{-t\frac{\Delta_{g_0}}{2}} f\|_\infty \to 0$ as $n \to \infty$. Thus

$$
\begin{aligned}
& L_{\gamma_p}^{[nt]} P_{1/\sqrt{n}} f(x_0, 0) \\
&\quad = \sum_{c \in \Omega_{x_0,[nt]}(X)} p(c) f\left(\frac{1}{\sqrt{n}} (\Phi(t(c)) - [nt]\rho_{\mathbb{R}}(\gamma_p)) \right) \\
&\quad \to \int_{\Gamma \otimes \mathbb{R}} f(\mathrm{y}) \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|\mathrm{y}|_{g_0}^2}{2t}} d\mathrm{y}.
\end{aligned}
$$

♣ Tightness: 4th moment estimate & harmonicity

♣ In the proof of Thm 2, we need to make use of the purturbation theory (cf. Parry–Pollicott's monograph).

♣ **symmetric case (i.e., $\gamma_p = 0$)**

- $\Phi : X \to \Gamma \otimes \mathbb{R}$, **periodic, i.e.,** $\Phi(\sigma x) = \Phi(x) + \sigma$

$$d\Phi(\tilde{e}) := \Phi(t(\tilde{e})) - \Phi(o(\tilde{e})), \quad e \in E_0$$

- g: **flat metric on** $\Gamma \otimes \mathbb{R}$
- $\mathcal{E}(\Phi, g) = \frac{1}{2} \sum_{e \in E_0} \left| d\Phi(\tilde{e}) \right|_g^2 \widetilde{m}(e)$: **Energy functional**
- **A variational characterization: (Kotani–Sunada ('00))**

$\mathcal{E}(\Phi_0, g_0) \le \mathcal{E}(\Phi, g)$ **holds for all** (Φ, g) **with**

$$\mathrm{vol}(\Gamma \otimes \mathbb{R}/\Gamma, g) = \mathrm{vol}(\Gamma \otimes \mathbb{R}/\Gamma, g_0) =: \mathrm{vol}(\mathrm{Alb}^{\Gamma})$$

- $\varphi = \varphi(\tau) : \mathbb{R} \to \mathbb{R}$: a smooth function bounded from above and $\varphi(\tau) = \tau$ around $\tau = 0$
- g: a fixed flat metric on $\Gamma \otimes \mathbb{R}$
- $\Phi_N : X \to (\Gamma \otimes \mathbb{R}, g)$, $N = 2, 3, \dots$ is the unique minimizer of a functional

$$\begin{aligned}\mathcal{E}_g^N(\Phi) &= \frac{1}{2}\sum_{e \in E_0} \left|d\Phi(\tilde{e})\right|_g^2 \widetilde{m}(e) \\ &\quad -\varphi\Big(\sum_{x \in \mathcal{F}} \big\langle Q\Phi_{N-1}(x), \Phi(x)\big\rangle_g m(x)\Big) \\ &\quad +\sum_{x \in \mathcal{F}} \big\langle \rho_{\mathbb{R}}(\gamma_p), \Phi(x)\big\rangle_g m(x)\end{aligned}$$

of periodic realizations Φ with $\sum_{x \in \mathcal{F}} \Phi(x)m(x) = 0$.

♣ $\sum_{x\in\mathcal{F}} \langle Q\Phi(x), \Phi(x)\rangle_g m(x) = 0$

Theorem 3 : **(A variational characterization of Φ_0)**

For any periodic realization $\Phi_1 : X \to (\Gamma \otimes \mathbb{R}, g)$, there exists a subsequence $\{\Phi_{N(j)}\}$ such that $\Phi_{N(j)} \to \Phi_0$ as $j \to \infty$.

♣ **Takeyuki Nagasawa ('99): A minimizing movement approach to the (non-stationary) Navier-Stokes equation**

- $b(u,u,u) := \int \langle (u\cdot\nabla)u, u\rangle dx = 0$

$$\Longrightarrow \quad b(u_{N-1}, u_{N-1}, u)$$

Theorem 4 : (A variational characterization of g_0)

Let $\Phi_0 : X \to \Gamma \otimes \mathbb{R}$ be the (modified) harmonic realization.

Then the Albanese metric g_0 is the (unique) minimizer of a functional

$$\mathcal{E}_{\Phi_0}(g) := \frac{1}{2} \sum_{e \in E_0} \left\langle d\Phi(\tilde{e}), d\Phi(\tilde{e}) - \rho_{\mathbb{R}}(\gamma_p) \right\rangle_g \widetilde{m}(e)$$

$$\left(= \frac{1}{2} \sum_{e \in E_0} \left| d\Phi(\tilde{e}) \right|_g^2 \widetilde{m}(e) - \left| \rho_{\mathbb{R}}(\gamma_p) \right|_g^2 \right)$$

of flat metrics g on $\Gamma \otimes \mathbb{R}$ with

$$\mathrm{Vol}(\Gamma \otimes \mathbb{R}/\Gamma, g) = \mathrm{Vol}(\mathrm{Alb}^{\Gamma}).$$

The End

Many thanks for your kind attention !