A regularity theory for elliptic systems with random coefficients and application to homogenization

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Uniformly elliptic operator

$$\mathcal{L}_a u := -\nabla \cdot (a\nabla u), \qquad u : \mathbb{R}^d \to \mathbb{R}^N \qquad (N = 1 \text{ in this talk})$$

with $a(\cdot) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ uniformly elliptic, stationary and ergodic.

Homogenization: $\exists_1$ constant coefficient matrix $a_{hom} \in \mathbb{R}^{d\times d}$, s.t.

$\forall \mu > 0\ :\ (\mathcal{L}_a + \mu)^{-1} \approx (\mathcal{L}_{hom} + \mu)^{-1}$ on **large length scales**.

Goals of this talk:

I. Regularity (on large length scales)

$\mathcal{L}_a$ features the same regularity as $\mathcal{L}_{hom}$
above a critical length scale r_*

II. Application to quantitative homogenization

- Moment bounds on the corrector
- Quantitative two-scale expansion
- Starting point: stretched exp. moment bounds on r_*

III. Correlated coefficients

- II. requires mixing assumption (in terms of LSI)
- Results sensitive to strength of mixing

Plan of the talk

1. Stochastic Homogenization, notion of the corrector
2. Intrinsic $C^{1,\alpha}$-regularity, minimal radius r_*
3. Quantification of ergodicity, control of r_*
4. Application to quantitative stochastic homogenization

$$-\nabla \cdot a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon = f \qquad \text{in } H^1_0(U), \qquad U \subset\subset \mathbb{R}^d$$

with $a(\cdot) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ uniformly elliptic.

Assumption on the coefficients:

We suppose that $\langle\cdot\rangle$ denotes a stationary and ergodic ensemble on coeffcient fields

$$a : \mathbb{R}^d \to \mathbb{R}^{d\times d} \qquad \text{(resp. tensor valued)}$$

that are $\langle\cdot\rangle$-almost surely

(i) uniformly bounded

$$|a(x)\xi| \le |\xi| \qquad \text{for a.e. } x \in \mathbb{R}^d,$$

(ii) uniform elliptic in the sense of

$$\int \nabla\zeta \cdot a\nabla\zeta \ge \lambda \int |\nabla\zeta|^2 \qquad \text{for all } \zeta \in C_c^\infty(\mathbb{R}^d)$$

($\lambda > 0$ is a fixed, deterministic constant of ellipticity).

$$-\nabla \cdot a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon = f \qquad \text{in } H_0^1(U), \qquad U \subset\subset \mathbb{R}^d$$

with $a(\cdot) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ uniformly elliptic.

Classical theorem of elliptic homogenization:

$\exists a_{hom} \in \mathbb{R}^{d\times d}$ such that (for $\langle\cdot\rangle$-a.e. a and all r.h.s. f)

$$\begin{aligned} u_\varepsilon &\rightharpoonup u_{hom} && \text{weakly in } H_0^1(U) \\ a(\frac{\cdot}{\varepsilon})\nabla u_\varepsilon &\rightharpoonup a_{hom}\nabla u_{hom} && \text{weakly in } L^2(U) \end{aligned}$$

where $u_{hom} \in H_0^1(U)$ solves

$$-\nabla \cdot a_{hom}\nabla u_{hom} = f.$$

[Papanicolaou, Varadhan '79, Kozlov '79]

[60s, 70s: Spagnolo, Tartar, Bensoussan, Lions, Papanicolaou, ...]

Formula for the homogenized coefficients

$$a_{hom} e_i = \limsup_{r\uparrow\infty} \fint_{B_r} a(e_i + \nabla\phi_i)$$

Corrector equation

$$-\nabla \cdot a(e_i + \nabla\phi_i) = 0 \qquad \text{in } \mathbb{R}^d, \qquad \phi_i \in H^1_{loc}(\mathbb{R}^d)$$

with $\nabla\phi_i$ stationary, zero-expectation, finite second moment.

Existence: e.g. via regularization (massive term)

Sublinear growth property [cf. Sidovaricius & Sznitman '04]

$$\lim_{r\uparrow\infty} r^{-2} \fint_{B_r} (\phi_i - \fint_{B_r} \phi_i)^2 = 0 \qquad \langle\cdot\rangle\text{-a.s.}$$

follows from $\langle\nabla\phi_i\rangle = 0$ & ergodicity.

Role played by the corrector (I)

Harmonic coordinates $\Psi_i(x) = x_i + \phi_i(x)$

- (a-harmonic): $-\nabla \cdot a\nabla\Psi_i = 0$
- (coordinate map): $\lim_{r\uparrow\infty} r^{-2} \fint_{B_r} |\Psi_i(x) - x_i|^2 = 0$
- (macroscopic gradient): $e_i = \lim_{r\uparrow\infty} \fint_{B_r} \nabla\Psi_i$
- (macroscopic flux): $a_{hom}e_i = \lim_{r\uparrow\infty} \fint_{B_r} a\nabla\Psi_i$

Flux corrector $a(e_i + \nabla\phi_i) - a_{hom}e_i = q_i = \nabla \cdot \sigma_i$

$\sigma_i \in \mathbb{R}^{d\times d}_{skew}$ is defined by

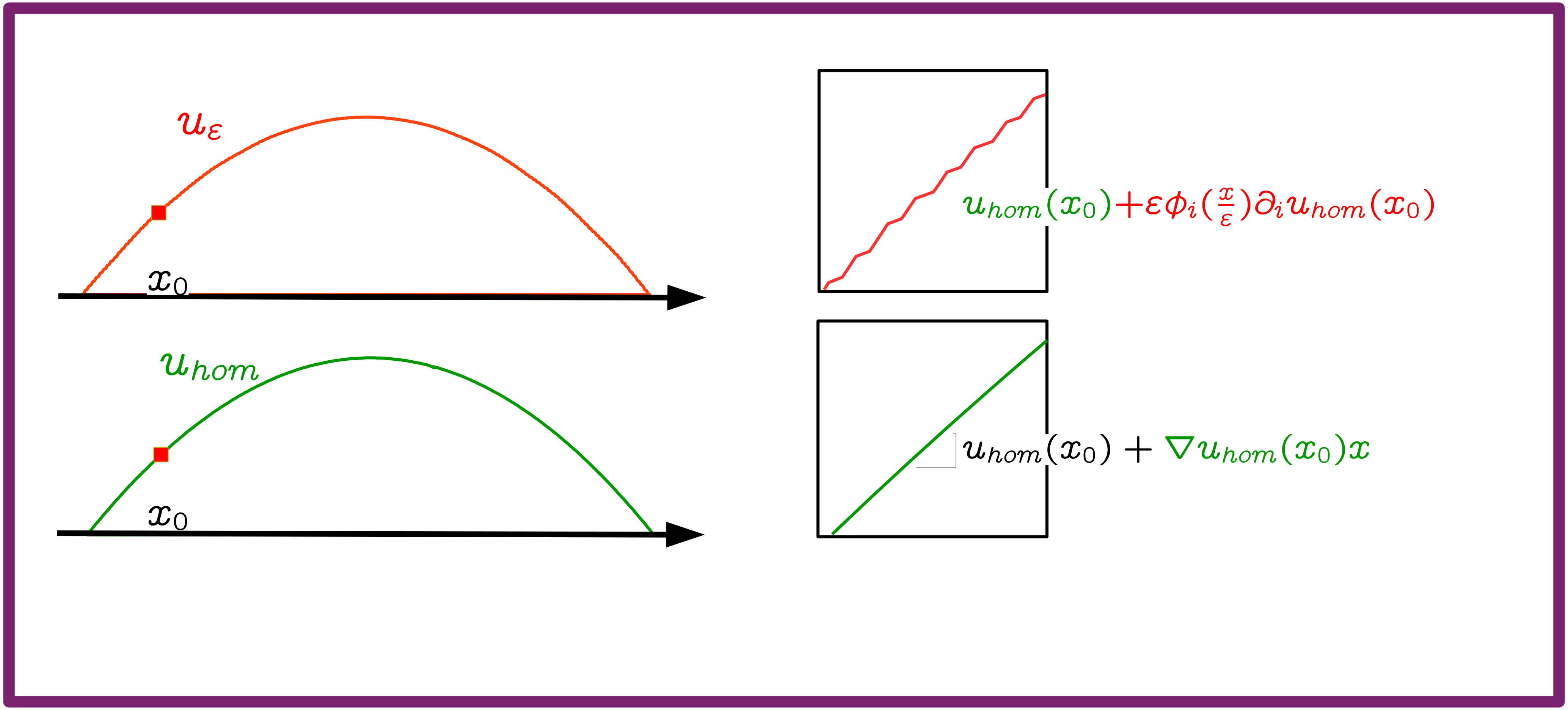
$$\begin{aligned} -\triangle\sigma_{ijk} &= \partial_j q_{ik} - \partial_k q_{ij} \\ \partial_k\sigma_{ijk} &= q_{ij} \end{aligned}$$

$\nabla\sigma_{ijk}$ is stationary, zero expectation and finite second moment.

Role played by the corrector (II)

Two-scale Expansion

$$u_\varepsilon(x) \approx u_{hom}(x) + \varepsilon \phi_i(\tfrac{x}{\varepsilon}) \partial_i u_{hom}(x) \qquad \text{away from } \partial U$$



Role played by the corrector (II)

Two-scale Expansion

$$u_\varepsilon(x) \approx u_{hom}(x) + \varepsilon\phi_i(\tfrac{x}{\varepsilon})\partial_i u_{hom}(x) \qquad \text{away from } \partial U$$

Classical estimate (for periodic coefficients)

Suppose that a, f and U are smooth. Consider

$$v_\varepsilon := u_\varepsilon - (u_{hom} + \varepsilon\phi_i(\tfrac{\cdot}{\varepsilon})\partial_i u_{hom})$$

Then

$$\forall U' \subset\subset U \ : \qquad \|v_\varepsilon\|_{H^1(U')} \lesssim \varepsilon.$$

[cf. Avellaneda-Lin '87, Allaire-Amar '99, Gerard-Varet – Masmoudi '12]

Goal:
Similar result in the stochastic case, yet optimal scaling is different...

A cartoon of the periodic case

Suppose $U = \mathbb{R}^d$ and $u_{hom} \in \mathcal{S}(\mathbb{R}^d)$.
Then:

$$-\nabla \cdot a(\tfrac{\cdot}{\varepsilon})\nabla v_\varepsilon = -\nabla \cdot (\varepsilon\sigma_i(\tfrac{\cdot}{\varepsilon}) + \varepsilon\phi_i(\tfrac{\cdot}{\varepsilon})a(\tfrac{\cdot}{\varepsilon}))\nabla\partial_i u_{hom}$$

$$\nabla\partial_i u_{hom} \cdot (a(e_i + \nabla\phi_i) - a_{hom}e_i) = -\nabla \cdot \sigma_i \nabla\partial_i u_{hom}$$

and thus

$$\int |\nabla v_\varepsilon|^2 \leq C(d,\lambda,\Lambda)\varepsilon^2 \int |D^2 u_{hom}|^2 (|\sigma_i(\tfrac{\cdot}{\varepsilon})| + |\phi_i(\tfrac{\cdot}{\varepsilon})|)^2$$

$$\to (\int |D^2 u_{hom}|^2)(\int_{\mathbb{T}} (|\sigma_i| + |\phi_i|)^2)$$

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Upcoming result in a nutshell:

$-\nabla \cdot a\nabla$ is as good as $-\nabla \cdot a_{hom}\nabla$ on length scales $\geq r_*(a)$

- [Avellaneda, Lin '87]: periodic case, compactness method
- [Armstrong, Smart '14]: convex energies, finite range condition
- [Armstrong, Mourrat '15]: monotone operators, mixing condition

Intrinsic regularity

Regularity of harmonic functions: $-\triangle u = 0$ in B_R

$$\forall r \in (0, R] : \qquad \min_{e\in\mathbb{R}^d} \fint_{B_r} |\nabla u - e|^2 \leq C(d)\left(\frac{r}{R}\right)^2 \fint_{B_R} |\nabla u|^2$$

Lemma (Regularity of a-harmonic functions). [Gloria, N. & Otto '15]

Suppose $-\nabla \cdot a\nabla u = 0$ in B_R, then

$$\mathrm{Exc}(r) := \min_{e\in\mathbb{R}^d} \fint_{B_r} |\nabla u - (e + \nabla\phi_e)|^2$$

$$\leq C(d,\lambda)\Big((1+\delta)\left(\frac{r}{R}\right)^2 + \left(\delta^{\frac{1}{(d+2)^2}} + \delta\right)\left(\frac{R}{r}\right)^d\Big) \fint_{B_R} |\nabla u|^2$$

$$\text{with } \delta := \sup_{\rho\in(2r,R)} \frac{1}{\rho^2} \fint_{B_\rho} \left|(\phi,\sigma) - \fint_{B_\rho} (\phi,\sigma)\right|^2$$

Cartoon of the proof

$$\begin{aligned} -\nabla\cdot a_{hom}\nabla u_{hom} &= 0 \qquad B_R \\ u_{hom} &= u \qquad \partial B_R \end{aligned}$$

$$v = u - (u_{hom} + \phi_i \partial_i u_{hom})$$

$$\min_{e\in\mathbb{R}^d} \fint_{B_r} |\nabla u - (\mathbb{I} + \nabla\phi)e|^2$$

$$\leq \fint_{B_r} |\nabla u - (\mathbb{I} + \nabla\phi)(\fint_{B_r} \nabla u_{hom})|^2$$

$$\lesssim \fint_{B_r} |\nabla u - (\mathbb{I} + \nabla\phi)\nabla u_{hom}|^2 + \fint_{B_r} |\mathbb{I} + \nabla\phi|^2 |\nabla u_{hom} - \fint_{B_r} \nabla u_{hom}|^2$$

$$\lesssim \fint_{B_r} |\nabla v|^2 + \left(1 + r^{-2} \fint_{B_{2r}} (\phi - \fint_{B_{2r}} \phi)^2\right) r^2 \sup_{B_r} |D^2 u_{hom}|^2$$

$$\begin{aligned} -\nabla \cdot a_{hom} \nabla u_{hom} &= 0 \qquad B_R \\ u_{hom} &= u \qquad \partial B_R \end{aligned}$$

$$v = u - (u_{hom} + \phi_i \partial_i u_{hom})$$

$$\min_{e \in \mathbb{R}^d} \fint_{B_r} |\nabla u - (\mathbb{I} + \nabla \phi) e|^2$$

$$\lesssim \fint_{B_r} |\nabla v|^2 + \left(1 + r^{-2} \fint_{B_{2r}} (\phi - \fint_{B_{2r}} \phi)^2\right) r^2 \sup_{B_r} |D^2 u_{hom}|^2$$

Assume (for simplicity) $(\star)$: $v = 0$ in ∂B_r, so that

$$\fint_{B_r} |\nabla v|^2 \lesssim \left(r^{-2} \fint_{B_r} (|\sigma| + |\phi|)^2\right) r^2 \sup_{B_r} |D^2 u_{hom}|^2$$

combine with

$$\begin{aligned} r^2 \sup_{B_r} |D^2 u_{hom}|^2 \lesssim \min_e \fint_{B_{2r}} |\nabla u_{hom} - e|^2 &\lesssim (\frac{r}{R})^2 \fint_{B_R} |\nabla u_{hom}|^2 \\ &\lesssim (\frac{r}{R})^2 \fint_{B_R} |\nabla u|^2. \end{aligned}$$

$$\Rightarrow \qquad Exc(r) \lesssim (\frac{r}{R})^2 \left(1 + r^{-2} \fint_{B_{2r}} |(\phi, \sigma) - \fint_{B_{2r}} (\phi, \sigma)|^2\right) \fint_{B_R} |\nabla u|^2$$

$$\begin{aligned} -\nabla \cdot a_{hom} \nabla u_{hom} &= 0 \qquad B_R \\ u_{hom} &= u \qquad \partial B_R \end{aligned}$$

$$v = u - (u_{hom} + \phi_i \partial_i u_{hom})$$

$$\min_{e \in \mathbb{R}^d} \fint_{B_r} |\nabla u - (\mathbb{I} + \nabla \phi) e|^2$$

$$\lesssim \fint_{B_r} |\nabla v|^2 + \left(1 + r^{-2} \fint_{B_{2r}} (\phi - \fint_{B_{2r}} \phi)^2\right) r^2 \sup_{B_r} |D^2 u_{hom}|^2$$

Assume (for simplicity) $(\star)$: $v = 0$ in ∂B_r, so that

Instead of $(\star)$ the rigorous proof uses a cut-off function

$$v = u - (u_{hom} + \eta \phi_i \partial_i u_{hom})$$

and requires to control a boundary layer.
This is the source of the additional term

$$\left((\delta^{\frac{1}{(d+2)^2}} + \delta)(\frac{R}{r})^d\right) \fint_{B_R} |\nabla u|^2$$

Excess decay & minimal radius

$$-\nabla \cdot a\nabla u = 0 \qquad \text{in } B_R$$

$$Exc(r) := \inf_{e\in\mathbb{R}^d} \fint_{B_r} |\nabla u - (e + \nabla\phi_e)|^2$$

Theorem (Excess decay). [Gloria, N. & Otto '15]

For all $\alpha \in (0,1)$ there exists $C(d,\lambda,\alpha) > 0$ s.t.

$$\forall r \in (r_*, R) \ : \qquad Exc(r) \le C(d,\lambda,\alpha)\left(\frac{r}{R}\right)^{2\alpha} Exc(R).$$

$$r_* := r_*(a) := \inf\left\{ \bar\rho \ : \ \frac{1}{\rho^2}\fint_{B_\rho} |(\phi,\sigma) - \fint_{B_\rho}(\phi,\sigma)|^2 \le \frac{1}{C(d,\lambda,\alpha)} \ \forall \rho \ge \bar\rho \right\}$$

Note: Since $\langle\cdot\rangle$ is ergodic, have $r_*(a) < \infty$ for $\langle\cdot\rangle$-a.e. a.

Consequences of excess decay

Lipschitz estimate

$$\forall r \in (r_*, R) \ : \qquad \fint_{B_r} |\nabla u|^2 \leq C(d,\lambda) \fint_{B_R} |\nabla u|^2$$

Non-degeneracy of the corrector

$$\forall r \geq r_* \ : \qquad \frac{1}{2}|e|^2 \leq \fint_{B_r} |e + \nabla \phi_e|^2 \leq C(d,\lambda)|e|^2$$

Corollary (Liouville property).

Every a-harmonic function u that grows sub-quadratically in the sense of

$$\lim_{R\uparrow\infty} R^{-2(1+\alpha)} \fint_{B_R} u^2 = 0 \qquad (\text{for some } \alpha < 1)$$

is a-linear, i.e. $u(x) = \text{const} + \xi \cdot x + \phi_\xi(x)$.

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Upcoming result in a nutshell:

Up to now:

$$\text{Ergodicity implies } r_* < \infty \text{ a.s.}$$

Next:

$$\text{Quantification of ergodicity via LSI implies } \langle \exp(\tfrac{1}{C} r_*^{d(1-\beta)}) \rangle < \infty.$$

Quantification of ergodicity via LSI and SG

Discrete case

$\{a(x)\}_{x\in\mathbb{Z}^d}$ i.i.d. i. e. $\langle\cdot\rangle = \mathbb{Z}^d$-fold product measure

$$\Rightarrow \ \forall F(a) \ : \ \mathrm{Ent}(F^2) = \langle F^2 \log \frac{F^2}{\langle F^2\rangle}\rangle \leq \frac{1}{2}\langle \sum_{x\in\mathbb{Z}^d} |\frac{\partial F}{\partial a(x)}|^2\rangle$$

$$\Rightarrow \ \forall F(a) \ : \ \langle (F - \langle F\rangle)^2\rangle \leq \langle \sum_{x\in\mathbb{Z}^d} |\frac{\partial F}{\partial a(x)}|^2\rangle$$

Morally speaking, (SG), (LSI) corresponds to integrable correlations.

- (SG) introduced to homogenization in the context of statistical mechanics by [Naddaf & Spencer '97]
- Systematic use in quantitative stochastic homogenization

  (SG): [Gloria, Otto '11], [Gloria, N., Otto '15]

  (LSI): [Otto, Marahrens '14], [Ben-Artzi, Marahrens, N. '14]

Coarsened LSI

Given a partition $\{D\}$ of $\mathbb{R}^d$ we define the Dirichlet Energy

$$\|\frac{\partial F}{\partial a}\|^2 := \sum_D \Big(\int_D |\frac{\partial F}{\partial a}| \Big)^2 \qquad \text{(for } F(a) \in \mathbb{R}\text{)}$$

Definition: c-LSI

We say $\langle \cdot \rangle$ satisfies c-LSI(ρ, β) ($\rho > 0$ and $0 \le \beta < 1$), if there exists a partition $\{D\}$ of $\mathbb{R}^d$ with

$$\mathrm{diam}(D) \sim (\mathrm{dist}(D) + 1)^\beta$$

such that for all random variables F:

$$\mathrm{Ent}(F^2) \le \frac{1}{\rho} \langle \|\frac{\partial F}{\partial a}\|^2 \rangle.$$

$\beta = 0 \quad \Rightarrow \quad \{D\}$ equipartition(*standard LSI*)
$\beta = 1 \quad \Rightarrow \quad \{D\}$ dyadic decomposition(*excluded*)

“vertical derivative”

Consider

- a function $F(a) \in \mathbb{R}$, $a : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ coefficient field.
- an infinitesimal perturbation $\delta a(x)$ supported in $D \subset\subset \mathbb{R}^d$

Functional derivative

$$\lim_{t\downarrow 0} \frac{F(a+t\delta a) - F(a)}{t} = \int_{\mathbb{R}^d} \frac{\partial F(a,x)}{\partial a} \delta a(x)\, dx.$$

A dual expression for the $L^1(D)$-norm of $\frac{\partial F}{\partial a}$ is

$$\int_D \left|\frac{\partial F(a,\cdot)}{\partial a}\right| \quad := \quad \sup\{\limsup_{t\downarrow 0} \frac{F(a+t\delta a) - F(a)}{t} \;: \\ \delta a \text{ supported in } D, \sup_{x\in D} |\delta a| \le 1\,\}$$

Example: Correlated Gaussian random field

Example [Gloria, N. & Otto '15]

Let $\omega(x)$ be a stationary, centered scalar Gaussian random field with covariance

$$c(x) := \langle \omega(x)\omega(0)\rangle .$$

Suppose that the covariance decays mildly in the sense of:

$$\int_{\mathbb{R}^d} |c(|x|)|(|x|+1)^{-\beta d}\, dx < \infty$$

for some $\beta \in [0,1)$.

Then the associated ensemble satisfies LSI with parameter β.

Two-fold usage of LSI and SG

- As a Poincaré Inequality

$$\langle \fint_B (\phi - \fint_B \phi)^2 \rangle \lesssim \fint_B \langle \| \frac{\partial \phi}{\partial a} \|^2 \rangle \lesssim \text{Expression of } (\nabla \phi)$$

- In the spirit of concentration of measure

If F is Lipschitz, i.e. $\|\frac{\partial F}{\partial a}\| \leq 1$,
then $F - \langle F \rangle$ has exponential moments, i.e. $\langle \exp(\rho(F - \langle F \rangle)) \rangle < \infty$.

[Herbst' argument]

Optimal control of r_*

Theorem [Gloria, N. & Otto '15]
Suppose that $\langle\cdot\rangle$ satisfies LSI(ρ,β) with $\rho>0$ and $\beta\in[0,1)$. Then

$$\langle \exp(\frac{1}{C(d,\lambda,\rho,\beta)} r_*^{d(1-\beta)})\rangle < 2$$

$$r_* := r_*(a) := \inf\Big\{\,\bar\rho\ :\ \frac{1}{\rho^2}\fint_{B_\rho}|(\phi,\sigma)-\fint_{B_\rho}(\phi,\sigma)|^2 \le \frac{1}{C(d,\lambda,\alpha)}\ \forall\rho\ge\bar\rho\,\Big\}$$

Proof invokes massive term approximation (ϕ_T,σ_T), sensitivity estimate, concentration of measure.

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Assumption: $\langle\cdot\rangle$ satisfies LSI(ρ,β) with $\rho > 0$ and $\beta \in [0,1)$.

- $d \geq 2$ – dimension
- $\beta \in [0,1)$ – decay of correlation ($\beta \uparrow \Leftrightarrow$ cor. $\uparrow$)
- $\beta = 1 - \frac{2}{d}$ – critical exponent
- $\mathcal{C}(x)$ – stationary random field with stretched exponential moments
$$\langle \exp(\frac{1}{C}\mathcal{C}^{\frac{2(1-\beta)}{2-\varepsilon}})\rangle < 2$$
- $\epsilon = \epsilon(d,\lambda)$ – hole filling exponent

Theorem (decay of spatial averages of the corrector's gradient)

$$|\fint_{B_r(x)} (\nabla\phi, \nabla\sigma)| \le \mathcal{C}(x) r^{-\frac{d}{2}(1-\beta)}.$$

[Gloria, N. & Otto '15]

Sensitivity Estimate

Theorem [Gloria, N. & Otto '15]

Let $\{D\}$ denote a β-partition with $\beta \in [0,1)$, and

$$Fh := \int_{\mathbb{R}^d} h \cdot g \qquad \text{with } \operatorname{supp}(g) \subset B_r, \ \Big(\fint_{B_r} |g|^{2p}\Big)^{\frac{1}{2p}} \leq r^{-d}$$

Then for $0 < p-1 \ll 1$ and suitable $\kappa > 0$ have

$$\|\frac{\partial F \nabla(\phi,\sigma)}{\partial a}\|^2 \lesssim \Big(\frac{(r+r_*)^\beta}{r}\Big)^d r^{-d\frac{p-1}{p}}$$

$$\times \Big(\int \Big(\big(\frac{|x|}{r+r_*}+1\big)^{-\kappa+d\frac{\beta}{p}}\big(\frac{r_*(x)}{(|x|+1)^\beta}+1\big)^{d(1-\varepsilon)}\Big)^{\frac{p}{p-1}} dx\Big)$$

In particular:

$$\langle(\fint_{B_r} \nabla(\phi,\sigma))^2\rangle \lesssim \langle\|\frac{\partial F \nabla(\phi,\sigma)}{\partial a}\|^2\rangle \lesssim r^{-d(1-\beta)}\langle r_*^\gamma\rangle$$

for some $\gamma > 0$.

Sensitivity estimate – Ingredients of the proof (ρ=1)

$$\|\frac{\partial F\nabla\phi_i}{\partial a}\|^2 = \sum_D \left(\int_D |\frac{F\nabla\phi_i}{\partial a}|\right)^2$$

Identify vertical derivative

$$\zeta = \tfrac{d}{dt}\phi_i(a+t\delta a) \text{ solves } -\nabla\cdot a\nabla\zeta = -\nabla\cdot\delta a(\nabla\phi_i + e_i)$$

Duality

$$\int_D |\frac{\partial F\nabla\phi_i}{\partial a}| \leq \int_D |\nabla v||\nabla\phi_i + e_i| \qquad \text{with} \qquad -\nabla\cdot a^*\nabla v = -\nabla\cdot g.$$

$$\leq (\int_D |\nabla v|^2\omega)^{\frac{1}{2}}(\int_D |\nabla\phi_i + e_i|^2\omega^{-1})^{\frac{1}{2}}$$

Weighted L2 regularity for dual equation

$$\left(\int_{\mathbb{R}^d} |\nabla v|^2\omega\right)^{\frac{1}{2}} \lesssim (\int_{\mathbb{R}^d} |g|^2\omega)^{\frac{1}{2}} \qquad \text{for } \omega(x) = (\frac{|x|}{r+r_*}+1)^{\gamma}$$

Hole filling and Non-degeneracy

$$\fint_D |\nabla\phi_i + e_i|^2 \lesssim 1 \ \wedge\ (\frac{\mathrm{diam}(D)}{r_* + \mathrm{dist}(D)})^{\epsilon d}$$

Theorem (moment bounds on the corrector)

$$\left(\fint_{B(x)} |(\phi,\sigma)|^2\right)^{\frac{1}{2}} \lesssim |\fint_B (\phi,\sigma)| + \mathcal{C}(x) \begin{cases} 1 & \text{for } 0 \le \beta < 1-\frac{2}{d} \\ \log^{\frac{1}{2}}(2+|x|) & \text{for } 0 \le \beta = 1-\frac{2}{d} \\ 1+|x|^{\frac{d}{2}(\beta-(1-\frac{2}{d}))} & \text{for } 1-\frac{2}{d} < \beta. \end{cases}$$

[Gloria, N. & Otto '15]

Suppose that $\langle\cdot\rangle$ satisfies LSI(ρ,β) with $\rho > 0$ and $\beta \in [0,1)$.
Let $u_\varepsilon, u_{hom} \in H^1(\mathbb{R}^d)$ satisfy

$$-\nabla \cdot a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon = -\nabla \cdot a_{hom}\nabla u_{hom}.$$

Theorem (Quantitative 2scale expansion).

$$\|\nabla u_\varepsilon - (\nabla u_{hom} + \partial_j u_{hom}\nabla\phi_j(\tfrac{\cdot}{\varepsilon}))\|_{L^2(\mathbb{R}^d)} \leq C\left(\int |D^2 u_{hom}(x)|^2 p_{d,\beta}(x)\,dx\right)^{\frac12} \varepsilon\, p_{d,\beta}(\tfrac{1}{\varepsilon})$$

with

$$p_{d,\beta}(r) := \begin{cases} 1 & \text{for } 0 \leq \beta < 1-\frac{2}{d} \\ \log(1+r)^{\frac12} & \text{for } 0 \leq \beta = 1-\frac{2}{d} \\ 1 + r^{\frac{d}{2}(\beta-(1-\frac{2}{d}))} & \text{for } \beta > 1-\frac{2}{d} \end{cases}$$

[Gloria, N. & Otto '15]

Thank you for your attention!

For details see:

arXiv:1409.2678

A regularity theory for random elliptic operators
Antoine Gloria, Stefan Neukamm, Felix Otto