Phase transition of random walk pinning model

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Pinning model

Homogeneous pinning model was introduced in physics literature to study the behavior of a polymer at an interface.
One of the simplest model is given as follows:

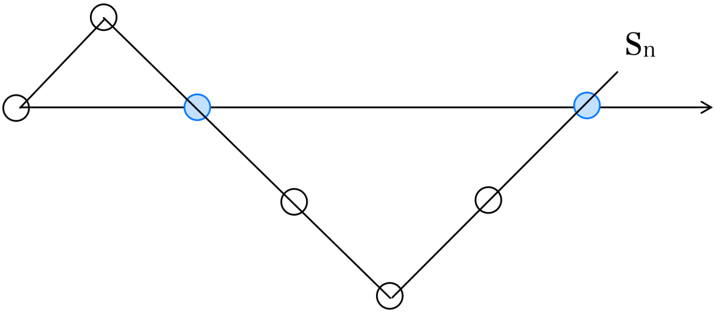
Setting

- *Polymers*:
  Let (S, P_S) be a simple random walk on $\mathbb{Z}$ starting from 0.
- *Polymer measure*: For $\beta \in \mathbb{R}$

$$P_n^\beta(dS) := \frac{1}{Z_n^\beta} \exp\left(\beta \sum_{k=1}^{n} \mathbf{1}\{S_k = 0\}\right) P_S(dS),$$

where

$$Z_n^\beta := P_S\left[\exp\left(\beta \sum_{k=1}^{n} \mathbf{1}\{S_k = 0\}\right)\right] \text{ (Partition function).}$$


S_n

Pinning model

Homogeneous pinning model is studied well. To study it, we often use the quantity, so-called the *free energy* which is defined by

$$F(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z_n^\beta \in [0, \infty).$$

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Phase transition

- If $\beta \leq 0$, then $F(\beta) = 0$.
- If $\beta > 0$, then $F(\beta) > 0$.

Pinning model

In each phase, the behavior of the path is definitely different:

Theorem A

$$P_n^\beta\left[\sum_{k=1}^n \mathbf{1}\{S_k=0\}\right]=\begin{cases}O(1), & \beta<0 \quad \text{(delocalized phase)}\\ O(\sqrt{n}), & \beta=0\\ O(n), & \beta>0 \quad \text{(localized phase).}\end{cases}$$

Remark: Generally, homogeneous pinning models are defined by using renewal processes like a return time of S.R.W. and we cannot consider the behavior of "S".

Random walk pinning model

Random walk pinning model is an inhomogeneous pinning model defined by using independent random walks, which was introduced by Birkner and Sun.

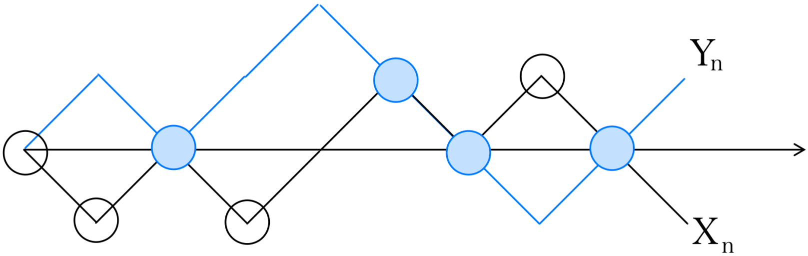
Setting

- Polymer: Let (X, P_X) be a S.R.W. on $\mathbb{Z}^d$ starting from 0.
- Environment: Let (Y, P_Y) be a S.R.W. on $\mathbb{Z}^d$ starting from 0.
- Polymer measure: For $\beta \geq 0$ and fixed Y,

$$\mu_{n,Y}^{\beta}(dX) := \frac{1}{Z_{n,Y}^{\beta}} P_X\left[\exp\left(\beta \sum_{k=1}^{n} \mathbf{1}\{X_k = Y_k\}\right) : dX\right],$$

where

$$Z_{n,Y}^{\beta} := P_X\left[\exp\left(\beta \sum_{k=1}^{n} \mathbf{1}\{X_k = Y_k\}\right)\right] \quad (\textit{quenched partition fn.}).$$


Yn
Xn

Random walk pinning model

Also, we define the *annealed partition function* by

$$P_Y\left[Z^{\beta}_{n,Y}\right].$$

We set

$$L_n(X,Y) := \sum_{k=1}^{n} \mathbf{1}\{X_k = Y_k\}, \quad L(X,Y) := \sum_{n\geq 1} \mathbf{1}\{X_n = Y_n\}.$$

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Then, we have

$$Z^{\beta}_{Y} := \lim_{n\to\infty} Z^{\beta}_{n,Y} = P_X\left[\exp\left(\beta L(X,Y)\right)\right], \quad P_Y\text{-a.s.}$$
$$P_Y[Z^{\beta}_{Y}] = \lim_{n\to\infty} P_Y\left[Z^{\beta}_{n,Y}\right] = P_{X,Y}\left[\exp(\beta L(X,Y))\right].$$

Phase transitions I

Monotonicity implies the following phase transition:

Phase transition I

We set

$$\beta_1^q(d) := \sup\{\beta \geq 0 : Z_Y^\beta < \infty, \ P_Y\text{-a.s.}\}$$
$$\beta_1^a(d) := \sup\{\beta \geq 0 : P_Y[Z_Y^\beta] < \infty\}.$$

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It is trivial that

$$\beta_1^a(d) \leq \beta_1^q(d).$$

Free energies

We introduce the free energies of RWPM which are important quantities to analyze the RWPM.

Free energies

It is known that the following limits exist and they are non-random:

$$F^q(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z_{n,Y}^{\beta}$$

$$= \lim_{n\to\infty} \frac{1}{n} P_Y[\log Z_{n,Y}^{\beta}], \quad P_Y\text{-a.s.}$$

$$F^a(\beta) = \lim_{n\to\infty} \frac{1}{n} \log P_Y[Z_{n,Y}^{\beta}]$$

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\begin{aligned}
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&= \lim_{n\to\infty} \frac{1}{n} P_Y[\log Z_{n,Y}^{\beta}], \quad P_Y\text{-a.s.} && \textit{quenched free enegy} \\
F^a(\beta) &= \lim_{n\to\infty} \frac{1}{n} \log P_Y[Z_{n,Y}^{\beta}] && \textit{annealed free enegy}
\end{aligned}
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We have that

$$F^q(\beta) \le F^a(\beta), \quad \beta \ge 0$$

from Jensen's inequality.

Phase transitions II

Also, monotonicity of $F(\beta)$ yields the following phase transitions:

Phase transitions II

We set

$$\beta_2^q(d) := \sup\{\beta \geq 0 : F^q(\beta) = 0\}$$
$$\beta_2^a(d) := \sup\{\beta \geq 0 : F^a(\beta) = 0\}.$$

Phase transitions II

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Then,

$$\beta_2^a(d) \leq \beta_2^q(d).$$

Remark

We give a remark on the annealed model.
The annealed partition function $P_Y[Z_{n,Y}^{\beta}]$ can be rewritten by

$$P_{\tilde{X}}\left[\exp\left(\beta\sum_{k=1}^{n}\mathbf{1}\{\tilde{X}_k=0\}\right)\right],$$

where $\tilde{X}$ is a random walk on $\mathbb{Z}^d$ defined by $\tilde{X}_n = X_n - Y_n$.

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where $\tilde{X}$ is a random walk on $\mathbb{Z}^d$ defined by $\tilde{X}_n = X_n - Y_n$.
This representation is also an example of pinning model and there are many results on its partition function and free energy. Moreover, it is a discrete homopolymer model.

Known results

1. $d = 1, 2$ ([3])

$$\beta_1^a(d) = \beta_1^q(d) = \beta_2^a(d) = \beta_2^q(d) = 0$$

2. $d \geq 3$ (annealed [4] et.al.)

$$0 < \beta_1^a(d) = \beta_2^a(d).$$

3. $d \geq 3$ (quenched [1, 3] et.al.)

$$0 < \beta_i^a(d) < \beta_i^q(d), \quad i = 1, 2.$$

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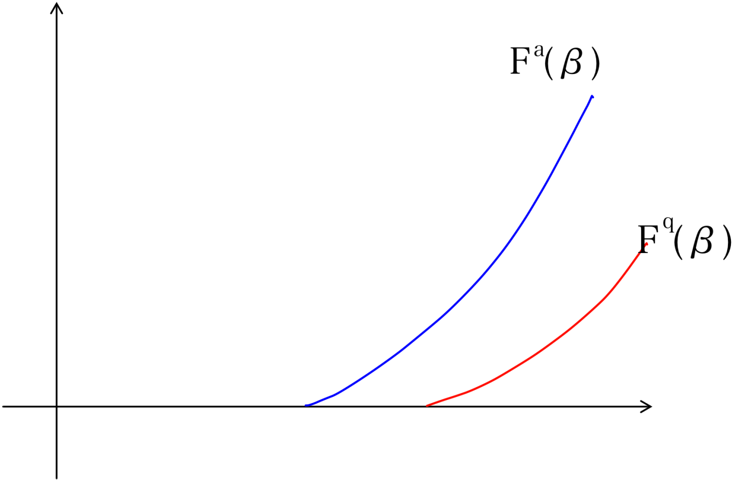
Thus, we have that for $d \geq 3$

$$0 < \beta_1^a(d) = \beta_2^a(d) < \beta_1^q(d) \leq \beta_2^q(d).$$


$F^a(\beta)$
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Main result 1

When we look at the critical points, we have the following result.

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$$\beta_1^q(d) = \beta_2^q(d)$$

for $d \geq 1$. Moreover, the quenched free energy $F^q(\beta)$ is given by

$$F^q(\beta) = s^{-1}\left(-\log\left(e^{\beta} - 1\right)\right), \ \beta \geq \beta_1^q(d),$$

where s is a continuous, convex, and strictly decreasing function which has a certain variational representation.

Corollary

1. *If* $\beta < \beta_1^q(d)$, *then*

$$\limsup_{n\to\infty} \mu_{n,Y}^{\beta}[L_n(X,Y)] < \infty$$

P_Y*-a.s.*

2. *If* $\beta > \beta_1^q(d)$, *then*

$$\liminf_{n\to\infty} \mu_{n,Y}^{\beta}\left[\frac{1}{n}L_n(X,Y)\right] > 0,$$

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Corollary

1. *If* $\beta < \beta_1^q(d)$, *then*

$$\limsup_{n\to\infty} \mu_{n,Y}^{\beta}[L_n(X,Y)] < \infty$$

P_Y-a.s. delocalized phase

2. *If* $\beta > \beta_1^q(d)$, *then*

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P_Y-a.s. localized phase

Investigating the variational representation of s, we have the asymptotics of $F^q(\beta)$ for the case $d = 1, 2$.

Corollary

1. *($d = 1$)*

$$F^q(\beta) \asymp \beta^2, \quad \beta \searrow 0.$$

2. *($d = 2$)*

$$\log F^q(\beta) \asymp -\beta^{-1}, \quad \beta \searrow 0.$$

Main results 2

So far, we investigated the path property of $\mu^{\beta}_{n,Y}$ from a view point of the collision local time. In the next theorem, we will see the path property of $\mu^{\beta}_{N,Y}$ in the distribution of X in $\mathbb{R}^d$.

Theorem 2

When $\beta < \beta_1^q(d)$, we have that

$$\mu^{\beta}_{n,Y}\left(\frac{X_n}{\sqrt{n}} \in \cdot\right) \Rightarrow \mu(\cdot), \quad P_Y\text{-a.s.},$$

where μ is a Gaussian measure on $\mathbb{R}^d$ with mean 0 and covariance matrix $\frac{1}{d}I$.

Remark

1. We don't know whether $Z_Y^\beta < \infty$ or not at critical point $\beta = \beta_1^q(d)$ $(d \geq 3)$.
2. For continuous homopolymer model, Cranston and Molchanov gave some path properties for delocalized phase, localized phase, and the critical case.

Idea of Proof

It is known that

$$F^q(\beta) = \lim_{n\to\infty} \frac{1}{n} \log P_Y \left[\exp(\beta L_n(X,Y)) : X_n = Y_n\right]$$
$$= \lim_{n\to\infty} \frac{1}{n} \log Z_{n,Y}^{\beta,\mathrm{pin}}, \quad P_Y\text{-a.s.} \tag{1}$$

We introduce

$$K(\beta, r) = \sum_{n=1}^{\infty} e^{-rn} Z_{n,Y}^{\beta,\mathrm{pin}}$$

for $r \geq 0$.

It follows from (1) that

$$r < F^q(\beta) \Rightarrow K(\beta, r) = \infty,$$
$$r > F^q(\beta) \Rightarrow K(\beta, r) < \infty.$$

Thus, $F^q(\beta)$ is determined by looking at $K(\beta, r)$.

Expanding

$$\exp(\beta L_n(X,Y)) = \prod_{j=1}^{n} \exp\left(\beta \mathbf{1}\{X_j = Y_j\}\right) = \prod_{j=1}^{n} \left(1 + \left(e^{\beta} - 1\right) \mathbf{1}\{X_j = Y_j\}\right),$$

we have that

$$Z_{\mathrm{Y}}^{\beta} = 1 + K(\beta, 0)$$

$$K(\beta, r) = \sum_{k \geq 1} \left(e^{\beta} - 1\right)^{k} \sum_{1 \leq j_1 < \cdots < j_k < \infty} e^{-r j_k} P_{\mathrm{X}}\left(X_{j_i} = Y_{j_i}; i = 1, \cdots, k\right). \tag{2}$$

Lemma

We have that

$$s(r) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{1\le j_1<\cdots<j_k<\infty} e^{-rj_k} P_X\left(X_{j_i} = Y_{j_i}; i = 1, \cdots, k\right)$$

exists P_Y-a.s. and s is continuous and strictly decreasing in $r \geq 0$.

Combining it with (2),

$$\log(e^{\beta} - 1) + s(F^q(\beta)) = 0$$

and also

$$\log(e^{\beta_1^q} - 1) + s(0) = 0.$$

To prove Lemma, we have used the quenched LDP for word sequences which was proved by Birkner, Greven, and den Hollander [1].

Word sequences

1. $\rho(n) = \dfrac{p_{2n}(0)}{\sum_{n\geq 1} p_{2n}(0)}$, $n \geq 1$.
2. $\{\tau_i : i \geq 1\}$: i.i.d. r.v.'s with law $\rho(n)$.
3. $\xi_i = Y_i - Y_{i-1}$, $i \geq 1$ is increments of Y. (letter)

Then, we define new r.v.'s $\{\zeta_i : i \geq 1\}$ by

$$\zeta_i = (\xi_{T_{i-1}+1}, \cdots, \xi_{T_i}), \quad \text{(word)}$$

where $T_0 = 0$ and $T_i = T_{i-1} + \tau_i$.

Roughly, we can rewrite as

$$\sum_{1\le j_1<\cdots<j_k<\infty} e^{-rj_k} P_X\left(X_{j_i}=Y_{j_i}; i=1,\cdots,k\right)$$
$$=\mathbb{P}\left[\exp\left(k\int f_r(dy)R_k(dy)\right)\middle|\,\xi\right],$$

where f_r is a bounded function on word set and R_k is an empirical measure of k-tuples of words.

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where f_r is a bounded function on word set and R_k is an empirical measure of k-tuples of words.
Since Birkner et. al. proved the quenched LDP for R_k, we can apply the Varadhan's lemma in the right hand side. So, we obtain Lemma.

In the proof of CLT, we also used the quenched LDP for words. Especially, we saw the continuity of the limit

$$\tilde{s}(\alpha) = \lim_{k\to\infty} \frac{1}{k} \log P_X \left(X_{j_i} = Y_{j_i} : i = 1, \cdots, k\right)^{\alpha}, \quad \alpha \in (\frac{3}{4}, \infty)$$

at $\alpha = 1$.

Originally, Birkner and Sun introduced random walk pinning model to give an lower bound of the weak-strong disorder critical point of directed polymers in random environment and parabolic Anderson model with Brownian noise.
Also, the coincidence of the critical points may be related to the conjecture of the coincidence of the weak-strong-very strong disorder critical points of directed polymers in random environment.

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in preparation.

Thank you for your attention!