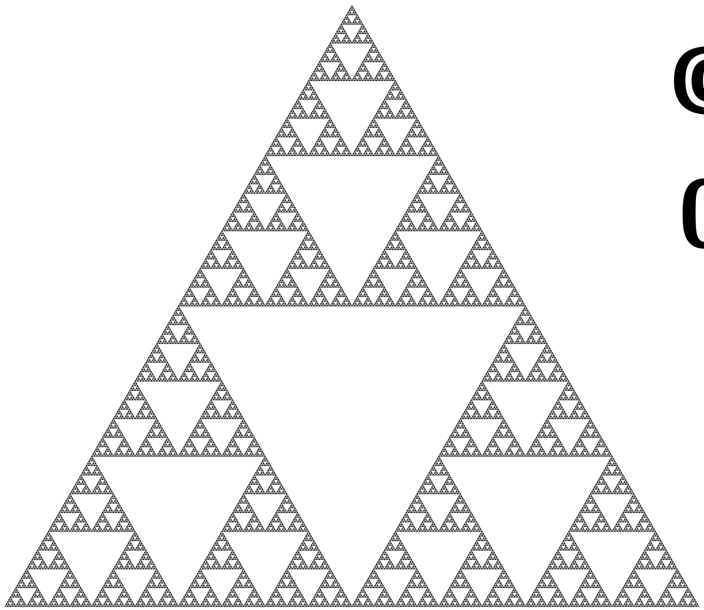
Localized upper bounds of heat kernels for diffusions via a multiple Dynkin-Hunt formula

Naotaka Kajino (Kobe University)

http://www.math.kobe-u.ac.jp/HOME/nkajino/

Joint work with Alexander Grigor'yan (U Bielefeld)

German-Japanese Stochast. Analysis & Applications



@ Tohoku University

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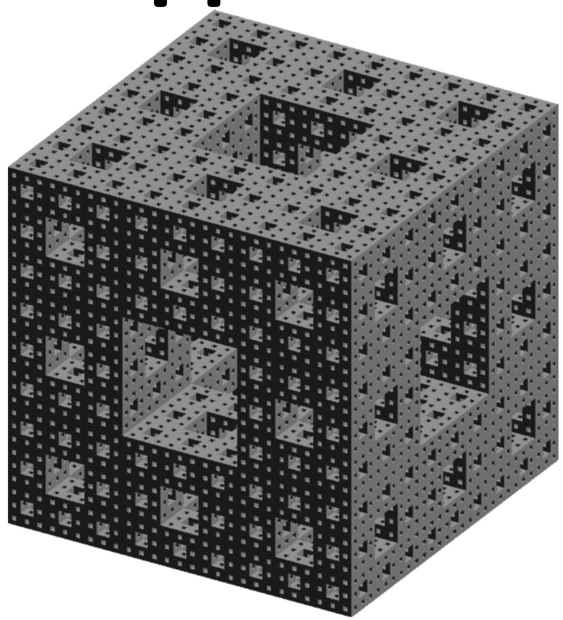

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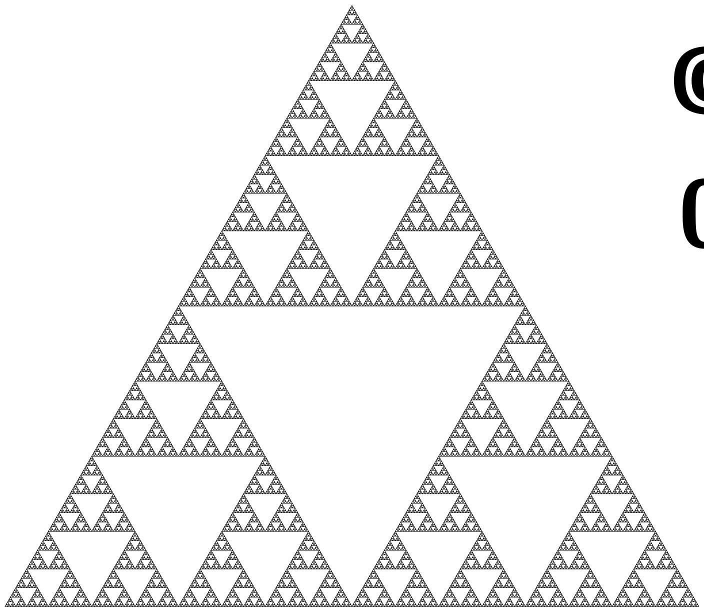

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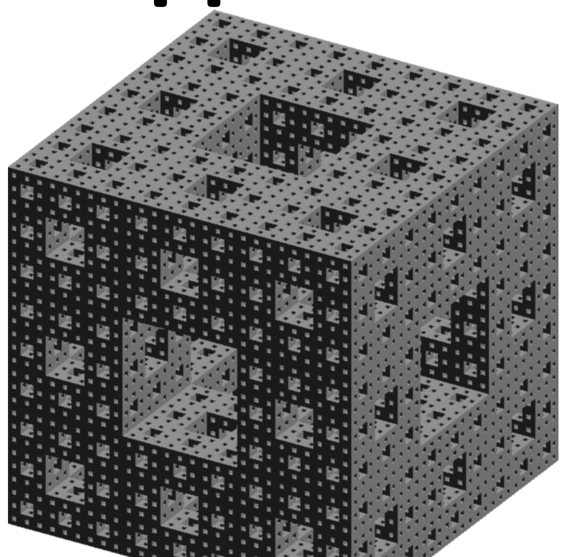

1 Theme: Ass. in fixed $U \Rightarrow$ heat kernel est. on U?

▷ (M,d): a loc. cpt separbl metric sp., $\Delta := \infty_M$

▷ $X = (\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M_\Delta})$: a diffusion on M

▷ ${}_{\emptyset\neq}U \subset M$: open, $\tau_U := \inf\{t \geq 0 \mid X_t \notin U\}$

Problem. Ass. on $X_t, t < \tau_U \Rightarrow \mathbb{P}_x[X_t \in dy]|_U \leq ?$

Aim. ${}^{\exists}p_t(x,y) := \mathbb{P}_x[X_t \in dy]|_U / d\mu \qquad (\beta > 1)$

(UHK)$_\beta$ $\leq F_t(x,y)\exp\left(-c\left(d(x,y)^\beta/t\right)^{\frac{1}{\beta-1}}\right)$

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Exmp. ● $\beta = 2$: Brownian motion on $\mathbb{R}^k$ & Riem. mfds

● $\beta > 2$: diffusions on fractals, Liouville B.M.(Andres-K.)

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▷ $F_t(x, y)$: $\dfrac{F_s(z, w)}{F_t(x, y)} \leq c_F \left(\dfrac{t \vee d(x, z)^\beta \vee d(y, w)^\beta}{s}\right)^{\alpha_F}$
$(s \leq t)$

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$$\leq F_t(x, y) \exp\left(-c(d(x, y)^\beta / t)^{\frac{1}{\beta-1}}\right)$$

Exmp. ● $F_t^{\mu,\beta}(x, y) := c/\sqrt{\mu(B(x, t^{1/\beta}))\mu(B(y, t^{1/\beta}))}$

μ is (VD): $\mu(B(x, 2r)) \leq c\mu(B(x, r))$

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● $F_t(x, y) = ct^{-\alpha}$ | μ is (VD): $\mu(B(x, 2r)) \leq c\mu(B(x, r))$

Known results 1: Gaussian estimates ($\beta = 2$)

(UHK)$_2$: ${}^{\exists}p_t(x,y) := \mathbb{P}_x[X_t \in dy]/d\mu$

$$\leq c_1 \mu(B(x,\sqrt{t}))^{-1/2} \mu(B(y,\sqrt{t}))^{-1/2} e^{-c_2 d(x,y)^2/t}$$

▷ M: a compl. Riem. mfds ($\Rightarrow {}^{\exists}p_t(x,y)$: smooth)

- $\mathrm{Ric}_M \geq 0 \Longrightarrow p_t(x,y) \asymp$ (RHS of (UHK)$_2$ above)

(BM: Li-Yau '86, Uniformly elliptic diffusions: Saloff-Coste '92)

- related to (VD), Poincaré, Sobolev, Faber-Krahn ineq.

analytic, localizable! (Grigor'yan '92, '94, Saloff-Coste '92)

▷ Generalize to loc. reg. Dirichlet sp.(Sturm '95, '96)

- Ass.: Intrinsic dist. is non-deg., compl.(exclude $\beta > 2$!)

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meaning: $1_{B(x,r)} \le {}^{\exists}\varphi \le 1_{B(x,2r)}$, "$|\nabla\varphi| \le r^{-1}$"

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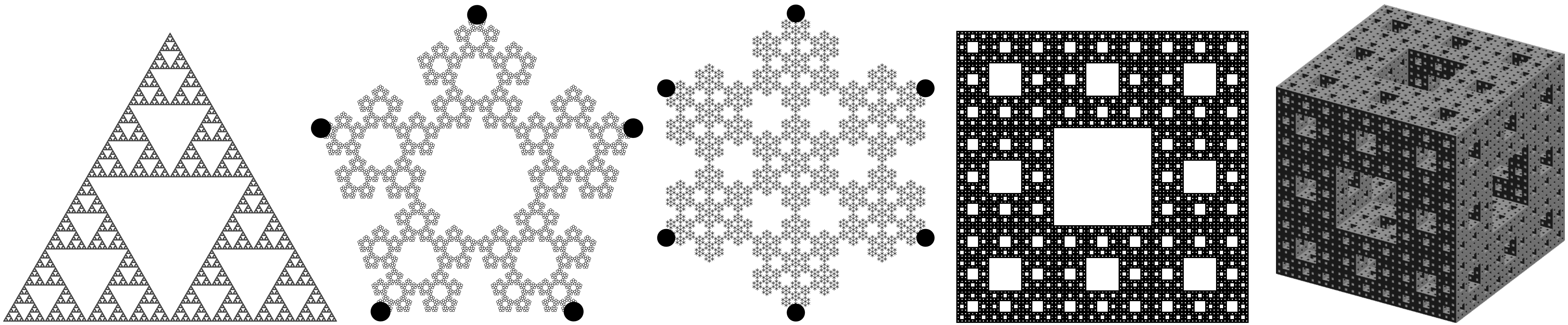

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- **Disadv.**: difficult to localize

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Rmk. Analytic esti.(BB-Kum.'06, Andres-Bar.'15): hard to verify!

2 Result: localized upper bounds of heat kernels

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▷ $X = (\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M_\Delta})$: a diffusion on M

▷ μ: a σ-finite Borel measure on M

▷ $N \subset M$: Borel, such that ${}^\forall x \in M \setminus N$, $\quad (\zeta := \tau_M)$

$\mathbb{P}_x[\tau_{M_\Delta \setminus N} = \infty] = 1 \qquad (M \setminus N$ is X-invariant)

$\mathbb{P}_x[\zeta < \infty, X_{\zeta-} \in M] = 0 \quad (X|_{M\setminus N}$: no killing inside)

● Assume any bdd closed subset of (M, d) is compact.

▷ $\beta \in (1, \infty)$, $R \in (0, \infty)$

▷ $\emptyset \neq U \subset M$: open with $\mathrm{diam}_d U \leq R$

▷ $F = F_t(x, y) : (0, R^\beta] \times U \times U \to (0, \infty)$: Borel

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▷ $N \subset M$: Borel, such that ${}^\forall x \in M \setminus N$, $\quad (\zeta := \tau_M)$

$\mathbb{P}_x[\tau_{M_\Delta \setminus N} = \infty] = 1 \qquad (M \setminus N$ is X-invariant$)$

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● Assume any bdd closed subset of (M,d) is compact.

▷ $\beta \in (1,\infty)$, $R \in (0,\infty)$

▷ $\emptyset \neq U \subset M$: open with $\mathrm{diam}_d U \leq R$

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2 Result: localized upper bounds of heat kernels

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- Let $c_F, \alpha_F, \delta \in (0,\infty)$, $\gamma \in (0,1)$ and assume:

(DB)$_{\beta}$: $\dfrac{F_s(z,w)}{F_t(x,y)} \leq c_F \left(\dfrac{t \vee d(x,z)^{\beta} \vee d(y,w)^{\beta}}{s} \right)^{\alpha_F}$ $({}^{\forall}s \leq {}^{\forall}t)$

(DU)$^{U,R}_{F}$: ${}^{\forall}(t,x) \in (0,R^{\beta}) \times (U \setminus N)$, ${}^{\forall}A \subset U$ Borel,

$$\mathbb{P}_x[X_t \in A,\, t < \tau_U] \leq \int_A F_t(x,y)\, d\mu(y).$$

(P)$^{U,R}_{\beta}$: ${}^{\forall}(x,r) \in (U \setminus N) \times (0,R)$ with $B(x,r) \subset U$,

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${}^\forall t\in(0,\infty)$, $\mathbb{P}_x[\tau_{B(x,r)}\le t]\le c_3\exp\bigl(-c_4(r^\beta/t)^{\frac{1}{\beta-1}}\bigr)$.

$\textbf{(DU)}_F^{U,R}$: ${}^{\exists}p^U = p_t^U(x,y) \le F_t(x,y)$ on $(0,R^\beta) \times (U \setminus N) \times U$

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3 Key: a *multiple Dynkin-Hunt formula*

- $\triangleright$ M: a loc. cpt separbl metrizable sp., $\Delta := \infty_M$
- $\triangleright$ $X = (\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M_\Delta})$: **Hunt** proc. on M
- $\triangleright$ ${}_{\emptyset\neq}U \subset M$: open, $\tau_U := \inf\{t \geq 0 \mid X_t \notin U\}$
- $\triangleright$ $B \subset M$: Borel, $\dot{\sigma}_B := \inf\{t \geq 0 \mid X_t \in B\}$
- $\triangleright$ $\mathcal{P}_t u(x) := \mathbb{E}_x[u(X_t)]$, $\mathcal{P}_t^U u(x) := \mathbb{E}_x[u(X_t)1_{\{t<\tau_U\}}]$

Thm 3 (multiple D-H formula). $\overline{B}^M \subset U$, $\tau_1 := \tau_U$,

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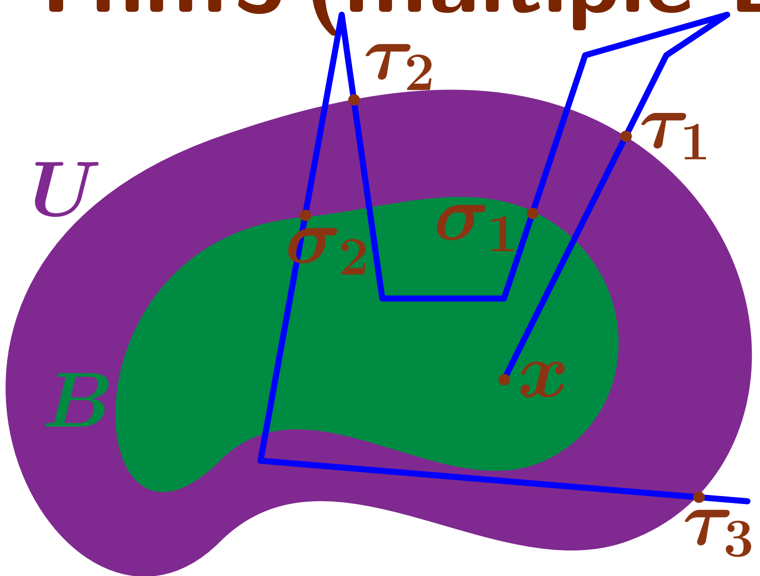

$$\sigma_n := \tau_n + \dot{\sigma}_B \circ \theta_{\tau_n} \text{ and}$$

$$\tau_{n+1} := \sigma_n + \tau_U \circ \theta_{\sigma_n}.$$

$\triangleright\ \mathcal{P}_t u(x) := \mathbb{E}_x[u(X_t)],\ \mathcal{P}_t^U u(x) := \mathbb{E}_x[u(X_t)\mathbf{1}_{\{t<\tau_U\}}]$

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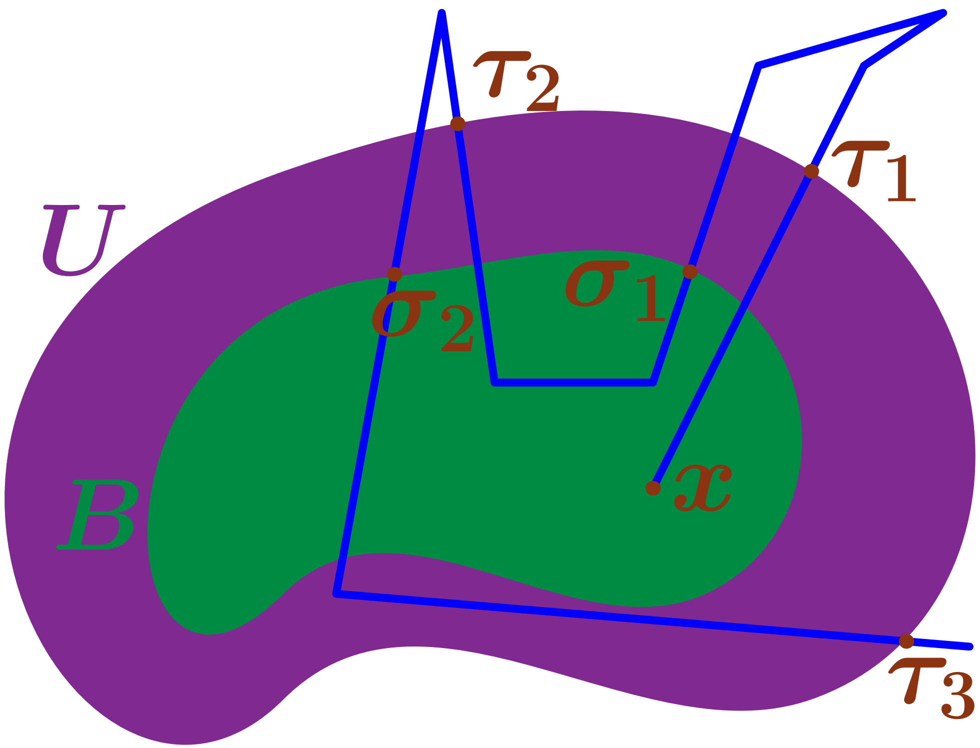
$$\mathcal{P}_t u(x) = \mathcal{P}_t^U u(x) + \sum_{n\in\mathbb{N}} \mathbb{E}_x\left[\mathbf{1}_{\{\sigma_n \leq t\}} \mathcal{P}_{t-\sigma_n}^U u(X_{\sigma_n})\right].$$



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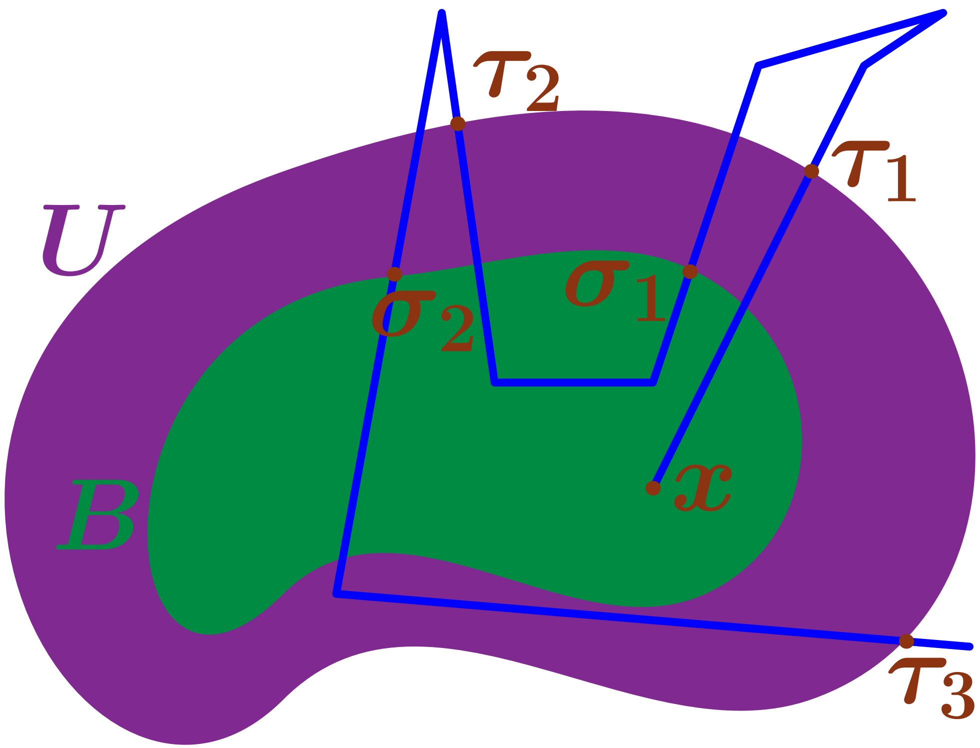

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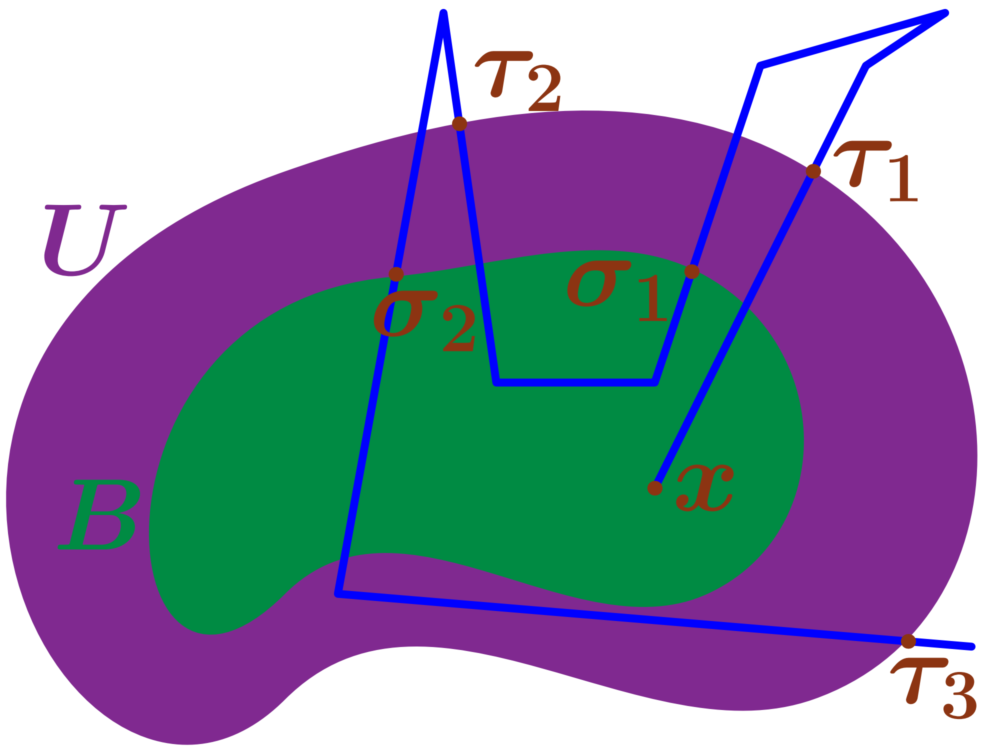

Prf. **With** $\tau_0 := 0 =: \sigma_0$,

$u(X_t) = u(X_t)\mathbf{1}_{\{X_t \in B\}}$

$\triangleright\ \mathcal{P}_t u(x) := \mathbb{E}_x[u(X_t)],\ \mathcal{P}_t^U u(x) := \mathbb{E}_x[u(X_t)\mathbf{1}_{\{t<\tau_U\}}]$

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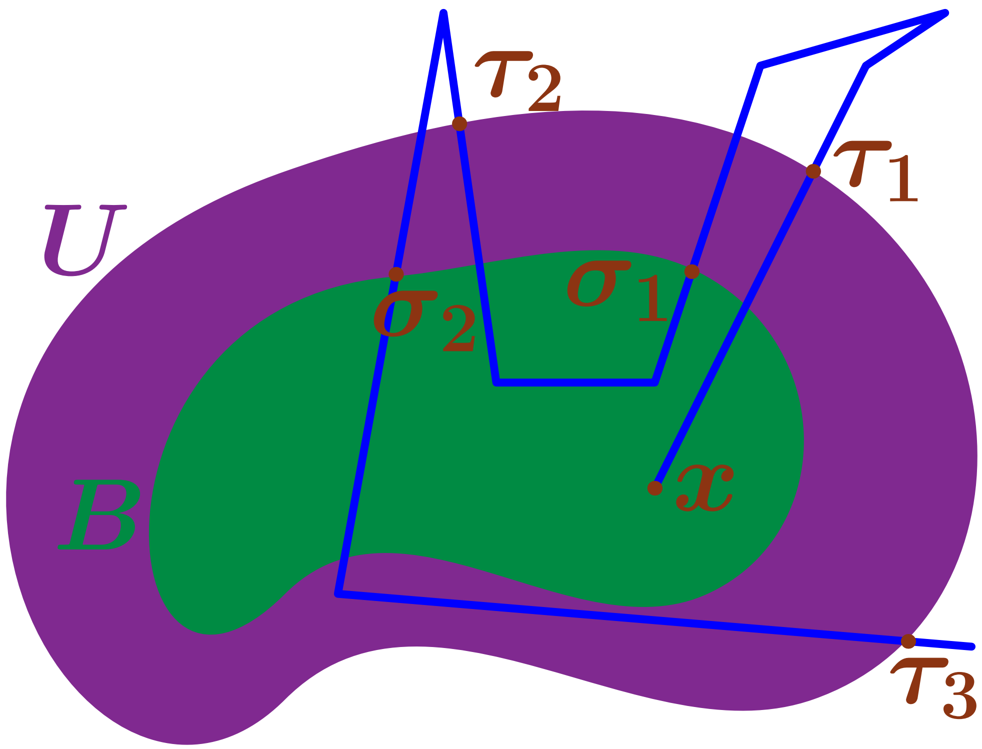

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$$u(X_t) = u(X_t)\mathbf{1}_{\{X_t\in B\}} = \sum_{n\geq 0} u(X_t)\mathbf{1}_{\{X_t\in B,\ \tau_n \leq t < \tau_{n+1}\}}$$

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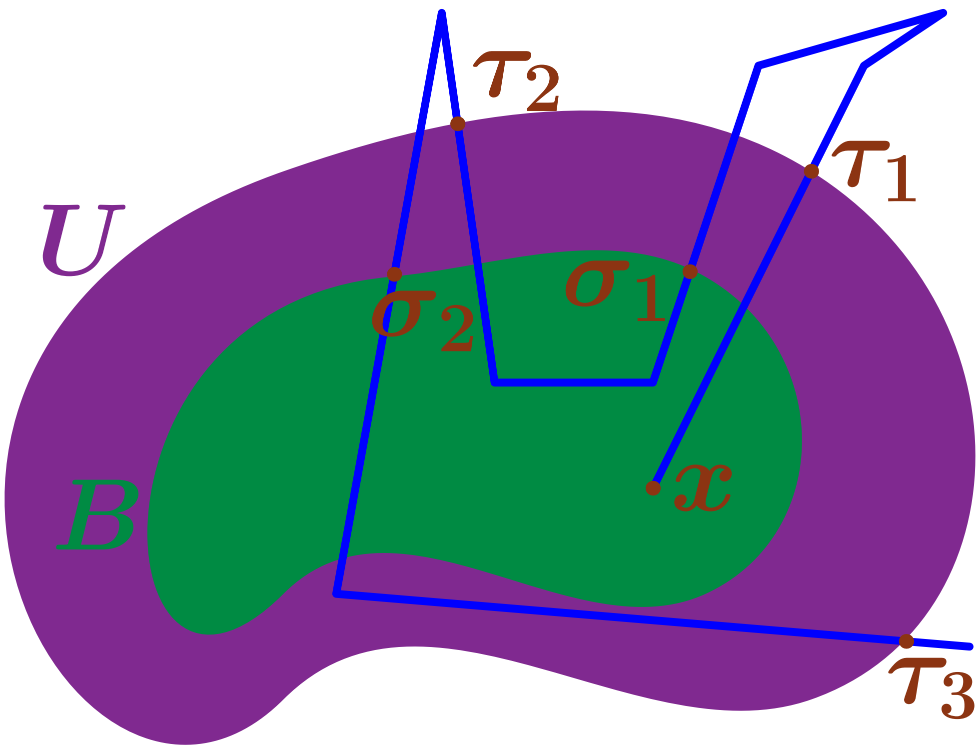

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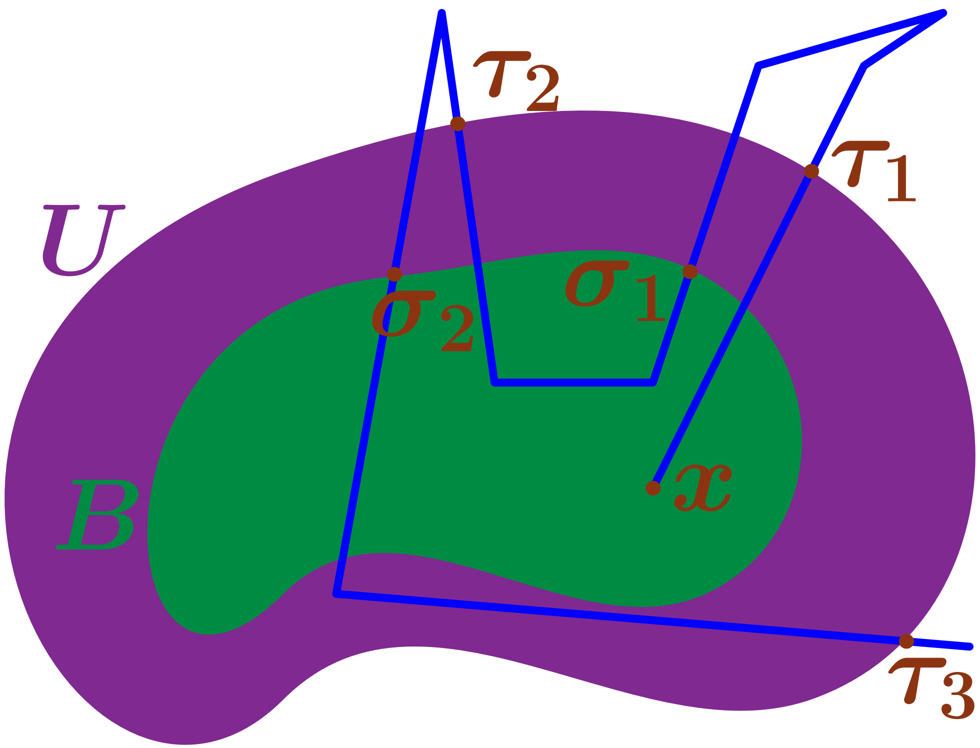

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$\rightsquigarrow$ strong Markov at time σ_n. $\square$

4 Verifying (DU): μ-a.e. HK est. $\Leftrightarrow$ $\mathcal{E}$-q.e. HK est.

▷ M: a loc. cpt separbl metrizable sp., $\Delta := \infty_M$

▷ $X = (\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M_\Delta})$: **Hunt** proc. on M

▷ μ: a Borel meas. on M, $\mu(\mathrm{cpt}) < \infty$, $\mu(\mathrm{open}_{\neq\emptyset}) > 0$

● Ass. X is μ-symm. and its Dirich. form is regular

▷ $I \subset (0,\infty)$: open interv., $J \subset I$: countbl, dense

▷ $U, V, W \subset M$: $\mathrm{open}_{\neq\emptyset}$

$(\mathrm{DU})_F^{U,R}$: $\forall (t,x) \in (0, R^\beta) \times (U \setminus N)$, $\forall A \subset U$ Borel,

$$\mathbb{P}_x[X_t \in A,\ t < \tau_U] \leq \int_A F_t(x,y)\, d\mu(y).$$

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Thm2 (*cf.* Barlow-Bass-Chen-Kassmann '09). $\boxed{(1) \Longleftrightarrow (2)}$!

(1) ${}^\forall t \in J$, ${}^\forall w \in L^2(M,\mu)$ with $w \geq 0$ μ-a.e., $\mathcal{P}^U_t(w\mathbf{1}_W)(x) \leq \int_W H_t(x,y)w(y)\,d\mu(y)$ μ-a.e. $x \in V$.

$H \geq {}^\exists p^U : I \times V \times W \to [0,\infty)$ Borel, ${}^\forall t \in I$, ${}^\forall x \in V \setminus N$, ${}^\forall A \subset W$ Borel, $\mathbb{P}_x[X_t \in A,\ t < \tau_U] = \int_A p^U_t(x,y)\,d\mu(y)$,

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(2) ${}^{\exists}N \subset M$: Borel, properly exceptional for X,

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