Heat kernel estimates for random walks with degenerate weights

Sebastian Andres

joint work with J.-D. Deuschel, M. Slowik

University of Bonn / University of Cologne

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The underlying graph

Let $G = (V, E)$ be an infinite, connected, locally finite graph such that for some $d \geq 2$:

- **Volume regularity:** For all $x \in V$,

$$C_{\mathrm{reg}}^{-1}\, n^d \;\leq\; |B(x,n)| \;\leq\; C_{\mathrm{reg}}\, n^d \qquad \forall n \geq N_1(x),$$

  with $B(x,n) := \{y : \, d(x,y) \leq n\}$ ball w.r.t. the graph distance.

- **Local Sobolev inequality** (S_1^d)**:** For all $x \in V$,

$$\Bigg(\sum_{y \in B(x,n)} |u(y)|^{\frac{d}{d-1}} \Bigg)^{\frac{d-1}{d}} \leq C_{\mathrm{S}_1} \sum_{\substack{y \vee z \in B(x,n) \\ \{y,z\} \in E}} \big| u(y) - u(z) \big|, \qquad \forall n \geq N_2(x),$$

  for all $u \colon V \to \mathbb{R}$ with $\operatorname{supp} u \subset B(x,n)$.

Put **weights** (or **conductances**) $\omega_e \in (0, \infty)$ on the edges of G.

Random Walk

Choose a 'speed measure' $\pi_x(\omega)$, $x \in V$. (How? See next slide...)

For $\omega \in \Omega = (0,\infty)^E$ let P_x^ω be the probability law on $D([0,\infty), V)$ which makes the coordinate process X_t a Markov chain starting at x with generator

$$\mathcal{L}_\pi f(x) = \frac{1}{\pi_x} \sum_{y \sim x} \omega_{xy}(f(y) - f(x)).$$

Then X is reversible (symmetric) with respect to π. Write

$$\mu_x = \sum_{y\sim x} \omega_{xy}, \qquad \nu_x = \sum_{y \sim x} \frac{1}{\omega_{xy}}.$$

Example: **Random Conductance Model (RCM)**

- $G = (\mathbb{Z}^d, E_d)$
- $(\omega_e)_{e \in E_d}$ stationary ergodic random variables under some probability measure $\mathbb{P}$.

Choices for the 'speed measure' π

- $\pi_x = \mu_x = \sum_y \omega_{xy}$. This makes the times spent at each site x before a jump i.i.d. $\mathtt{exp}(1)$. Call this the **constant speed random walk (CSRW).**

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- $\pi_x = 1$ for all x. This makes the times spent at x i.i.d. $\mathtt{exp}(\mu_x)$. Call this the **variable speed random walk (VSRW).**

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- For either choice $\pi = \mu$ or $\pi \equiv 1$ define the heat kernel (transition density with respect to π) by

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- Problem: **Gaussian bounds (GB)** on $p_t^\omega(x,y)$, i.e. there exist $N_x(\omega)$ such that when $t \geq N_x$:

$$p_t^\omega(x,y) \leq c_1 t^{-d/2} \exp(-c_2 d(x,y)^2/t), \qquad \text{if } t \geq d(x,y),$$

and similar lower bounds.

Some Results on Gaussian bounds

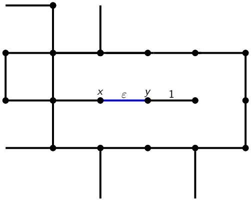
- "Elliptic": $0 < c_1 \leq \omega_e \leq c_2 < \infty$. Delmotte '99.
- SRW on percolation clusters: Barlow '04, Sapozhnikov '14.

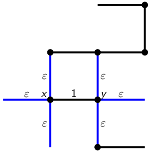
For the RCM with i.i.d. conductances:

- "Bounded below": $\omega_e \in [1, \infty)$. Barlow and Deuschel '10 (VSRW).
- "Bounded above": $\omega_e \in [0, 1]$ Berger, Biskup, Hoffmann, Kozma '08 showed that sub-Gaussian heat kernel decay can occur, so

  Gaussian bounds may fail!
- Boukhadra, Kumagai, Mathieu ('14): Sharp conditions on the tail of the conductances near 0.

Traps





Gaussian bounds and Harnack inequalities

Theorem (Delmotte '99)

If $0 < c_1 \leq \omega_e \leq c_2 < \infty$ *the following are equivalent:*

- *Gaussian upper and lower bounds on the heat kernel*
- *Volume doubling and local Poincaré inequality*
- *Parabolic Harnack inequality*

Similar results:

- Grigor'yan '92 and Saloff-Coste '92 on manifolds
- Sturm '96 on Dirichlet spaces
- Barlow, Chen '14: extension of Delmotte's result applicable to random graphs

Parabolic Harnack inequality (CSRW)

Theorem (A., Deuschel, Slowik (2013))

For any ball $B_n = B(x_0, n)$ and $t_0 \geq 0$ let $Q_n = [t_0, t_0 + n^2] \times B_n$. Suppose that $u > 0$ is caloric on Q_n, i.e. $\partial_t u - \mathcal{L}_\mu u = 0$ on Q_n. Then, for any $p, q \in (1, \infty)$ with

$$\frac{1}{p} + \frac{1}{q} \ < \ \frac{2}{d}$$

there exists $C_{\mathrm{H}} = C_{\mathrm{H}}(\|\mu\|_{p,B_n}, \|\nu\|_{q,B_n})$ such that

$$\max_{(t,x)\in Q_-} u(t,x) \ \leq \ C_{\mathrm{H}} \min_{(t,x)\in Q_+} u(t,x).$$

with $Q_- = \left[t_0 + \frac{1}{4}n^2, t_0 + \frac{1}{2}n^2\right] \times B_{n/2}$, $Q_+ = \left[t_0 + \frac{3}{4}n^2, t_0 + n^2\right] \times B_{n/2}$.

Upper Gaussian estimates for the CSRW

Theorem (A., Deuschel, Slowik (2014))

Let X be the CSRW and let $p, q \in (1, \infty)$ be such that

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{d}.$$

Assume that there exists $N_x(\omega)$ such that

$$\bar{\mu} := \sup_{x \in V} \sup_{n \geq N_x} \|\mu\|_{p, B(x,n)} < \infty, \qquad \bar{\nu} := \sup_{x \in V} \sup_{n \geq N_x} \|\nu\|_{q, B(x,n)} < \infty.$$

Then, there exist constants $c_i = c_i(d, p, q, \bar{\mu}, \bar{\nu}) > 0$ such that for any t and x with $\sqrt{t} \geq 2(N_x(\omega) \vee N_1(x) \vee N_2(x))$ and all $y \in V$,

$$p_t^\omega(x,y) \leq c_1\, t^{-d/2} \exp\big(-c_2 d(x,y)^2/t\big), \qquad \textit{if } t \geq d(x,y).$$

Idea of the proof – Davies' method

- For a suitable class of functions $\psi : V \to \mathbb{R}$ consider
  - semigroup $P_t^{\psi} f \;=\; e^{\psi}\left(P_t(e^{-\psi} f)\right)$
  - generator $\mathcal{L}^{\psi} f \;=\; e^{\psi}\big(\mathcal{L}_{\mu}(e^{-\psi} f)\big)$
  - Let $u(t,x)$ be the solution of the Cauchy problem

$$\begin{cases} \partial_t u - \mathcal{L}^{\psi} u &= 0, \\ u(t=0,\cdot) &= e^{\psi} f. \end{cases}$$

If $f = \mathbb{1}_{\{y\}}/\mu_y$ then $u(t,x) = e^{\psi(x)} p_t^{\omega}(x,y)$.

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- A-priori estimate

$$\|u(t,\,\cdot\,)\|_{\ell^2(V,\mu)} \;\leq\; e^{h(\psi)t}\,\|e^\psi f\|_{\ell^2(V,\mu)}$$

where

$$h(\psi) := 2\big(\cosh(\|\nabla\psi\|_\infty) - 1\big).$$

Maximal inequality via Moser iteration

- For $p, p_* \in [1, \infty]$ such that $1/p + 1/p_* = 1$, $\alpha \geq 1$, $\frac{1}{2} \leq \sigma' < \sigma \leq 1$,

$$\|u\|_{2\alpha\left(1+\frac{\rho-p_*}{\rho}\right), Q_{\sigma' n}, \mu} \leq c\left(1 \vee \|\mu\|_{p,B_n}\|\nu\|_{q,B_n}\right)^{\frac{1}{2\alpha}} \|u\|_{2\alpha, Q_{\sigma n}, \mu}$$

with a constant c (depending on $\|\nabla\psi\|_\infty$), provided that

$$\rho(q,d) - p_* > 0 \qquad \iff \qquad \frac{1}{p} + \frac{1}{q} < \frac{2}{d}.$$

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$$\max_{Q_{n/2}} u \leq c\left(1 \vee \|\mu\|_{p,B_n}\|\nu\|_{q,B_n}\right)^{\kappa} \|u\|_{2,Q_n,\mu}.$$

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- By the a-priori estimate

$$\max_{Q_{n/2}} u \leq c\left(1 \vee \|\mu\|_{p,B_n}\|\nu\|_{q,B_n}\right)^{\kappa} e^{h(\psi)n^2}\, n^{-d/2}\, \|e^{\psi} f\|_{\ell^2(V,\mu)}.$$

Finishing the sketch of the proof

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- For $t \asymp n^2$ large enough this can be written as

$$\|P_t^{\psi}(e^{\psi} f)\|_{\ell^{\infty}(V,\mu)} \leq c t^{-d/4} \, e^{h(\psi) t} \, \|e^{\psi} f\|_{\ell^2(V,\mu)}$$

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$$u(t,x) \leq c\, t^{-d/2} e^{2h(\psi)t} e^{\psi(y)}$$

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- Optimising over ψ yields upper Gaussian estimates.
- Advantage: Only balls with one fixed center point x are considered!

VSRW and chemical distance

- Now let X be the VSRW with generator

$$\mathcal{L}f(x) = \sum_{y \sim x} \omega_{xy}(f(y) - f(x)).$$

- The natural distance associated with X is the **chemical distance** defined by

$$d_\omega(x,y) \;:=\; \inf_\gamma \left\{ \sum_{i=0}^{l_\gamma - 1} 1 \wedge \omega(z_i, z_{i+1})^{-1/2} \right\},$$

  where the infimum is taken over all paths $\gamma = (z_0, \dots, z_{l_\gamma})$ connecting x and y.

- Let $\tilde{B}(x,r) := \{y \in V : d_\omega(x,y) \le r\}$.

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Then, there exist constants $c_i(d, p, q, \bar{\mu}, \bar{\nu}) > 0$ such that for any t and x with $\sqrt{t} \geq 2(\tilde{N}_x(\omega) \vee N_1(x) \vee N_2(x))$ and all $y \in V$,

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