Large time asymptotics of Feynman-Kac functionals for symmetric α-stable processes

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Basic materials and problem

- $\{X_t\}$: transient symmetric α-stable process on $\mathbb{R}^d$ $(0 < \alpha < 2)$
- $(\mathcal{E}, \mathcal{F})$: Dirichlet form associated with $\{X_t\}$ on $L^2(\mathbb{R}^d)$

$$\mathcal{E}(u,u) = \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(y) - u(x))^2 \frac{A_{d,\alpha}}{|x-y|^{d+\alpha}} dxdy, \quad A_{d,\alpha} = \frac{\alpha 2^{\alpha-2} \Gamma(\frac{d+\alpha}{2})}{\pi^{d/2} \Gamma(1 - \frac{\alpha}{2})}$$

- $p(t,x,y)$: transition density function $\mathbb{E}_x[f(X_t)] = \int_{\mathbb{R}^d} p(t,x,y) f(y) dy$
- $G(x,y)$: Green kernel $G(x,y) = \int_0^\infty p(t,x,y) dt$
- μ: positive Radon smooth measure on $\mathbb{R}^d$ in a certain class
- A_t^μ: PCAF in the Revuz correspondence with μ
- Schrödinger form: $\mathcal{E}^\mu(u,u) = \mathcal{E}(u,u) - \int_{\mathbb{R}^d} u^2 d\mu$
- Feynman-Kac functional: $\mathbb{E}_x[\exp(A_t^\mu)] = \int_{\mathbb{R}^d} p^\mu(t,x,y) dy$

Problem

What is the large time asymptotic behavior of the Feynman-Kac functional like?

Green tightness and comparison with a Dirichlet form

Definition

A positive Radon smooth measure μ is Green-tight if it satisfies

$$\lim_{a\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|\le a} G(x,y)\mu(dy)=0, \quad \lim_{R\to\infty}\sup_{x\in\mathbb{R}^d}\int_{|y|>R} G(x,y)\mu(dy)=0$$

Definition

Spectral bottom of the time-changed process by μ

$$\lambda(\mu):=\inf\left\{\mathcal{E}(u,u)\ \middle|\ u\in\mathcal{F}_e, \int_{\mathbb{R}^d} u^2 d\mu=1\right\}\begin{cases} >1 & \text{(subcritical)}\\ =1 & \text{(critical)}\\ <1 & \text{(supercritical)}\end{cases}$$

where $\mathcal{F}_e$ is the extended Dirichlet space.

Preceding result -subcritical case-

Equivalent conditions (Takeda 2006)

- Subcriticality of μ i.e. $\inf \left\{ \mathcal{E}(u,u) \;\middle|\; u \in \mathcal{F}_e, \int_{\mathbb{R}^d} u^2 d\mu = 1 \right\} > 1$
- Gaugeability of Feynman-Kac semigroup i.e. $\sup_{x \in \mathbb{R}^d} \mathbb{E}_x[\exp(A^\mu_\infty)] < \infty$

Theorem 1 (W. 2012) -Stability of fundamental solution-

Suppose the positive Green-tight measure μ is of 0-order finite energy integral, i.e.

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} G(x,y)\mu(dx)\mu(dy) < \infty.$$

$p^\mu(t,x,y)$ satisfies $c_1 p(t,x,y) \le p^\mu(t,x,y) \le c_2 p(t,x,y)$ iff μ is subcritical.

Background of the problem

- If μ is critical or supercritical,
  $p^{\mu}(t,x,y)$ has different estimate from that of $p(t,x,y)$.

- The exact behavior of $p^{\mu}(t,x,y)$ for critical μ
  3-dimensional Brownian motion Grigor'yan (2006), Takeda (2007)

$$p^{\mu}(t,x,y) \asymp \frac{C}{t^{3/2}} \left(1 + \frac{\sqrt{t}}{1+|x|}\right) \left(1 + \frac{\sqrt{t}}{1+|y|}\right) \exp\left(-c\frac{|x-y|^2}{t}\right)$$

- The Feynman-Kac functional $\mathbb{E}_x[\exp(A_t^{\mu})] = \int_{\mathbb{R}^d} p^{\mu}(t,x,y)dy$
- μ is subcritical iff $\mathbb{E}_x[\exp(A_{\infty}^{\mu})] < \infty$.

If μ is not subcritical, how the Feynman-Kac functional diverges as $t \to \infty$?

Preceding result -supercritical case-

Equivalent conditions (Takeda and Tsuchida 2007)

- Supercriticality of μ i.e. $\inf\left\{\mathcal{E}(u,u) \;\middle|\; u\in\mathcal{F}_e, \int_{\mathbb{R}^d} u^2 d\mu = 1\right\} < 1$
- Positivity of the spectral bottom for the Schrödinger operator i.e.

$$C(\mu) = -\inf\left\{\mathcal{E}(u,u) - \int_{\mathbb{R}^d} u^2 d\mu \;\middle|\; u\in\mathcal{F}, \int_{\mathbb{R}^d} u^2(x)dx = 1\right\} > 0.$$

Via Fukushima's ergodic theorem,

$$\mathbb{E}_x[\exp(A_t^\mu)] \sim c_1 h(x)\exp(C(\mu)t) \quad (t\to\infty)$$ (Takeda 2008)

where $h(x)$ is the eigenfunction corresponding to the eigenvalue $-C(\mu)$

Preceding result -critical case-

If μ is critical, $C(\mu) = 0$.

Theorem 2 (Simon 1981, Cranston and Molchanov et al. 2009)
Suppose $\{X_t\}$ is transient Brownian motion on $\mathbb{R}^d$ and $\mu = V \cdot m$ for $V \in C_0^\infty(\mathbb{R}^d)$.

$$\mathbb{E}_x[\exp(A_t^\mu)] \sim \begin{cases} c_1 h(x) t^{\frac{1}{2}} & (d = 3) \\ c_2 h(x) t / \log t & (d = 4) \\ c_3 h(x) t & (d \geq 5) \end{cases}$$

where $A_t^\mu = \int_0^t V(X_s) ds$ and $h(x)$ satisfies $\left(\frac{\Delta}{2} + V\right) h = 0$.

Brownian motion is regarded as 2-stable process.
What happens if $\{X_t\}$ is the rotationally invariant α-stable process?

Growth order of the Feynman-Kac functional

Theorem 3 (Takeda and W. 2014)

Suppose $\{X_t\}$ is a transient symmetric α-stable process on $\mathbb{R}^d$ and Green-tight measure μ has compact support. Then,

$$\mathbb{E}_x[\exp(A_t^\mu)] \sim \frac{\alpha\Gamma(\frac{d}{2})\sin((\frac{d}{\alpha}-1)\pi)}{2^{1-d}\pi^{1-\frac{d}{2}}\Gamma(\frac{d}{\alpha})\langle\mu, h_0\rangle} h_0(x) t^{\frac{d}{\alpha}-1} \qquad (1 < d/\alpha < 2)$$

$$\mathbb{E}_x[\exp(A_t^\mu)] \sim \frac{\Gamma(\alpha+1)}{2^{1-d}\pi^{-\frac{d}{2}}\langle\mu, h_0\rangle} h_0(x) \frac{t}{\log t} \qquad (d/\alpha = 2)$$

$$\mathbb{E}_x[\exp(A_t^\mu)] \sim \frac{\langle\mu, h_0\rangle}{(h_0, h_0)} h_0(x) t \qquad (d/\alpha > 2)$$

where $h_0(x)$ is the ground state of $\mathcal{E}^\mu$ and $\langle\mu, h_0\rangle = \int_{\mathbb{R}^d} h_0(x)\mu(dx)$.

Remark

- By Takeda and Tsuchida (2007), $h_0(x) \asymp 1 \wedge |x|^{\alpha-d}$.
- Recently, this result has been extended to the measure of 0-order finite energy integral.

Methods for proof -difference between Theorems 2 and 3-

Asymptotic expansion for β-order resolvent kernel or corresponding operator

- Brownian motion : Hankel function
- α-stable process: Direct calculation using

$$p(t,x,y) = \left(2^{d-1}\pi^{\frac{d+1}{2}}\Gamma\left(\frac{d-1}{2}\right)\alpha\right)^{-1} t^{-\frac{d}{\alpha}} g\left(\frac{|x-y|}{t^{1/\alpha}}\right)$$

  and property of the function $g(w)$.

Functional space and operators

- Simon : $\mu = V \cdot m \quad K_\beta f(x) = \sqrt{V}(x)\int_{\mathbb{R}^d} G_\beta(x,y)\sqrt{V}(y)f(y)dy$

  perturbation theory for operators in $L^\infty(\mathbb{R}^d)$ $\to$ ambiguous !!
- Cranston, Molchanov et al : $L^2_{exp}(\mathbb{R}^d), C_{exp}(\mathbb{R}^d)$ etc. $\to$ too complicated !!
- Takeda-W. : $L^2(\mu)$ where $\mathcal{F}_e$ is compactly embedded
  $\to$ Based on Dirichlet form theory of the time-changed process.

Outline of the proof of Theorem 3

- Time changed process and Dirichlet form theory
  Compactness of the Green operator on $L^2(Y, \mu)$, where Y is the support of μ.

$$\mathcal{G}_\beta f(x) = \int_Y G_\beta(x, y) f(y) \mu(dy)$$

- Resolvent equation and Orthogonal decomposition

$$G^\mu_\beta \mu = (1 - \mathcal{G}_\beta)^{-1}(G_\beta \mu) = (1 - \gamma_\beta)^{-1}(h_\beta, G_\beta \mu)_\mu h_\beta + R_\beta$$

γ_β: Principal eigenvalue of $\mathcal{G}_\beta$, h_β: Principal eigenfunction with $\|h_\beta\|_\mu = 1$.

- Calculation of γ_β applying perturbation theory for compact operators
- Tauberian theorem : $G^\mu_\beta \mu \ (\beta \to 0)$ $\quad \int_0^t p^\mu_s \mu ds \ (t \to \infty)$

$$\mathbb{E}_x[\exp(A^\mu_t)] = p^\mu_t 1(x) = 1 + \int_0^t p^\mu_s \mu(x) ds$$

Killed processes and Time changed processes

- symmetric α-stable process: $\mathbb{M} = (\Omega, \mathscr{F}, \mathscr{F}_t, \{X_t\}, \{\mathbb{P}_x\})$
- β-killed process ($\beta \geq 0$): $\mathbb{M}^\beta = (\Omega, \mathscr{F}, \mathscr{F}_t, \{X_t\}, \{\mathbb{P}_x^\beta\})$
  Here $\mathbb{P}_x^\beta(\Lambda) = e^{-\beta t}\mathbb{P}_x(\Lambda), \quad \Lambda \in \mathscr{F}_t$
  Dirichlet form: $\mathcal{E}_\beta(u,u) = \mathcal{E}(u,u) + \beta \int_{\mathbb{R}^d} u^2(x)m(dx)$.
- Time changed process of $\mathbb{M}^\beta$ by μ: $\check{\mathbb{M}}^{\beta,\mu} = (\Omega, \mathscr{F}, \mathscr{F}_t, \{X_{\tau_t}\}, \{\mathbb{P}_x^\beta\})$
  Support of A_t^μ : $Y := \{x \in \mathbb{R}^d \mid \mathbb{P}_x(T=0) = 0\}, \quad T = \inf\{t \mid A_t^\mu > 0\}$
  Time change: $\tau_t = \{s > 0 \mid A_s^\mu > t\}$

Dirichlet form on $L^2(Y;\mu)$: $(\check{\mathcal{E}}^\beta, \check{\mathcal{F}}^\beta)$

$$\check{\mathcal{F}}^\beta = \{\psi \in L^2(Y;\mu) \mid \exists u \in \mathcal{F}_e^\beta, \psi = u \text{ on } Y\} \quad \mathcal{F}_e^\beta : \mathcal{E}_\beta\text{-completion of } \mathcal{F}$$

$$\check{\mathcal{E}}^\beta(\psi,\psi) = \mathcal{E}_\beta(H_Y u, H_Y u), \quad H_Y u(x) = \mathbb{E}_x^\beta[u(X_{\sigma_Y})] = \mathbb{E}_x[e^{-\beta\sigma_Y}u(X_{\sigma_Y})]$$

We denote by $\mathcal{H}_\beta$ the corresponding generator, i.e. $\check{\mathcal{E}}^\beta(u,v) = (\mathcal{H}_\beta u, v)_\mu$.

Compactness of the Green operator $\mathcal{G}_\beta$

Green operator of $\check{\mathbb{M}}^{\beta,\mu}$: $\mathcal{G}_\beta f(x) = \int_Y G_\beta(x,y) f(y)\mu(dy) \quad f \in L^2(Y,\mu)$

Lemma 1 $\quad \mathcal{G}_\beta$ is a compact operator on $L^2(Y,\mu)$.

(Outline of proof)

- $\int_{\mathbb{R}^d} u^2 d\mu \le \|G\mu\|_\infty \mathcal{E}(u,u), \quad u \in \mathcal{F}_e$ (Stollmann and Voigt 1996)
- $\mathcal{F}_e^\beta$ is compactly embedded into $L^2(\mathbb{R}^d,\mu)$. (Takeda and Tsuchida 2007)
- $\check{\mathcal{F}}^\beta$ is a Hilbert space w.r.t. $\check{\mathcal{E}}^\beta$ and compactly embedded into $L^2(Y,\mu)$.

γ_β: Principal eigenvalue of $\mathcal{G}_\beta$, h_β: $\mathcal{G}_\beta h_\beta = \gamma_\beta h_\beta$ and $\|h_\beta\|_\mu = 1$.

Identification of $u \in \mathcal{F}_e^\beta$ and $\psi \in \check{\mathcal{F}}^\beta$

- Restriction map $r : \mathcal{F}_e^\beta \to \check{\mathcal{F}}^\beta \qquad r(u) = u|_Y$
- Extension map $e : \check{\mathcal{F}}^\beta \to \mathcal{F}_e^\beta \qquad e(\psi) = H_Y u, \quad (\psi = u \in \mathcal{F}_e^\beta$ μ-a.e. on $Y)$

In particular, the principal eigenfunction h_β satisfies

$$e(h_\beta)(x) = \frac{1}{\gamma_\beta}\int_Y G_\beta(x,y) h_\beta(y)\mu(dy)$$

Ground state and behavior of eigenfunction

Lemma 2 $\lim_{\beta \to 0} \gamma_\beta = \gamma_0 = 1$ and $h_\beta \to h_0$ ($L^2(\mu)$-strongly and $\mathcal{E}$-weakly).

(Outline of the proof)

- λ_β: minimum eigenvalue of $\mathcal{H}_\beta$ / $\gamma_\beta = \lambda_\beta^{-1}$: principal eigenvalue of $\mathcal{G}_\beta$
$$\lambda_0 = \inf \left\{ \check{\mathcal{E}}(\psi, \psi) \mid \int_Y \psi^2 d\mu = 1 \right\} = \check{\mathcal{E}}(h_0, h_0) = \mathcal{E}(h_0, h_0) = \int_{\mathbb{R}^d} h_0^2 d\mu = 1$$
- $\{u_n\}$: approximation sequence for $h_0 \in \mathcal{F}_e$ with $\|u_n\|_\mu = 1$.
- $\sup_{0 \le \beta \le 1} \mathcal{E}(h_\beta, h_\beta) \le \sup_{0 \le \beta \le 1} \mathcal{E}_\beta(h_\beta, h_\beta) = \sup_{0 \le \beta \le 1} \lambda_\beta \le \lambda_1$
- $\exists \tilde{h}_0$ s.t. $h_{\beta_k} \to \tilde{h}_0$ $\mathcal{E}$-weakly and $L^2(\mathbb{R}^d, \mu)$-strongly.
- $\mathcal{E}(\tilde{h}_0, \tilde{h}_0) \le \liminf_{k \to \infty} \mathcal{E}(h_{\beta_k}, h_{\beta_k}) \le \lim_{k \to \infty} \lambda_{\beta_k} = \lim_{\beta \to 0} \lambda_\beta$ (Banach-Steinhaus)
- $\lambda_{\beta_k} = \mathcal{E}_{\beta_k}(h_{\beta_k}, h_{\beta_k}) \le \mathcal{E}_{\beta_k}(u_l, u_l) \Rightarrow \lim_{\beta \to 0} \lambda_\beta \le \mathcal{E}(u_l, u_l) \to \mathcal{E}(h_0, h_0)$.
- $\tilde{h}_0 = h_0$ and $\lambda_0 = \lim_{\beta \to 0} \lambda_\beta$.

Orthogonal decomposition

Lemma 3 By the resolvent equation and orthogonal decomposition for $\mathcal{G}_\beta$,

$$G^\mu_\beta \mu = (1-\mathcal{G}_\beta)^{-1} G_\beta \mu = (1-\gamma_\beta)^{-1}(G_\beta \mu, h_\beta)_\mu h_\beta + R_\beta,$$

It follows that $R_\beta \in \mathcal{F}_e$ and $\sup_{\beta \geq 0} \mathcal{E}(R_\beta, R_\beta) < \infty$.

(Outline of proof)

- $g_\beta := (1-P_\beta) G_\beta \mu \quad P_\beta f = (f, h_\beta)_\mu h_\beta$
- λ'_β: the second smallest eigenvalue of $\mathcal{H}_\beta \quad \lambda_\beta < \lambda'_\beta$ and $\lambda'_0 \leq \lambda'_\beta$.

$$\mathcal{E}_\beta(R_\beta, R_\beta) = \check{\mathcal{E}}^\beta(R_\beta, R_\beta) = (\mathcal{H}_\beta R_\beta, R_\beta)_\mu = \int_{\lambda'_\beta}^\infty \frac{\lambda}{(1-\lambda^{-1})^2} d(E_\lambda g_\beta, g_\beta) \leq$$

$$\left(\frac{\lambda'_0}{\lambda'_0 - 1}\right)^2 \check{\mathcal{E}}^\beta(G_\beta \mu, G_\beta \mu) = \left(\frac{\lambda'_0}{\lambda'_0 - 1}\right)^2 \mathcal{E}_\beta(G_\beta \mu, G_\beta \mu) \leq c_1 \int_{\mathbb{R}^d} G\mu(x)\mu(dx)$$

Asymptotic expansion of resolvent kernel

- $1 < d/\alpha < 2$

$$G_\beta(x,y) = G(x,y) - \frac{2^{1-d}\pi^{1-\frac{d}{2}}}{\alpha\Gamma(d/2)\sin((d/\alpha - 1)\pi)}\beta^{\frac{d}{\alpha}-1} + E_\beta(x,y)$$

$$0 \le E_\beta(x,y) \le c_1\beta|x-y|^{2\alpha-d}$$

- $d/\alpha = 2$

$$G_\beta(x,y) = G(x,y) - \frac{2^{1-d}\pi^{-\frac{d}{2}}}{\Gamma(\alpha+1)}\beta\log\beta^{-1} + E_\beta(x,y)$$

$$|E_\beta(x,y)| \le c_2\beta(1 + |\log|x-y|| + \beta|x-y|^\alpha)$$

- $d/\alpha > 2$

$$G_\beta(x,y) = G(x,y) - \beta\int_0^\infty tp(t,x,y)dt + E_\beta(x,y)$$

$$0 \le E_\beta(x,y) \le \begin{cases} c_1\beta^{\frac{d}{\alpha}-1} & (2 < d/\alpha < 3) \\ c_1\beta^2(1 + |\log|x-y|| + \beta|x-y|^\alpha) & (d/\alpha = 3) \\ c_1\beta^2|x-y|^{3\alpha-d} & (d/\alpha > 3) \end{cases}$$

Asymptotic expansion of γ_β

If μ has compact support, we can apply the expansion directly to $\mathcal{G}_\beta$.
(Example for $1 < d/\alpha < 2$)

$$G_\beta(x,y) = G(x,y) - c_0\beta^{\frac{d}{\alpha}-1} + E_\beta(x,y), \quad E_\beta(x,y) \le c_1\beta|x-y|^{2\alpha-d}$$

$$\mathcal{G}_\beta f(x) = \int_Y G_\beta(x,y)f(y)\mu(dy), \quad \mathcal{G}_\beta = \mathcal{G}_0 - c_0\beta^{\frac{d}{\alpha}-1}\mathcal{D}_1 + \mathcal{D}_2$$

The operator norm of $\mathcal{D}_2$ is less than $c_2\beta$ and $\mathcal{D}_1$ is a bounded operator.
We can apply one-order perturbation theory for $\mathcal{G}_0 - c_0\beta^{\frac{d}{\alpha}-1}\mathcal{D}_1$.

Lemma 4 The principal eigenvalue γ_β satisfies

$$\gamma_\beta = \gamma_0 - \frac{2^{1-d}\pi^{1-\frac{d}{2}}(\sqrt{V}, \sqrt{V}h_0)^2}{\alpha\Gamma(\frac{d}{2})\sin((\frac{d}{\alpha}-1)\pi)(\sqrt{V}h_0, \sqrt{V}h_0)}\beta^{\frac{d}{\alpha}-1} + o(\beta^{\frac{d}{\alpha}-1}), \quad (1 < d/\alpha < 2)$$

$$\gamma_\beta = \gamma_0 - \frac{2^{1-d}\pi^{-\frac{d}{2}}(\sqrt{V}, \sqrt{V}h_0)^2}{\Gamma(1+\alpha)(\sqrt{V}h_0, \sqrt{V}h_0)}\beta\log\beta^{-1} + o(\beta\log\beta^{-1}), \quad (d/\alpha = 2)$$

$$\gamma_\beta = \gamma_0 - \frac{(h_0, h_0)}{(\lambda_\mu\sqrt{V}h_0, \lambda_\mu\sqrt{V}h_0)}\beta + o(\beta), \quad (d/\alpha > 2)$$

Behavior of $G_\beta^\mu \mu$ as $\beta \to 0$ -weak convergence-

$$G_\beta^\mu \mu = (1-\gamma_\beta)^{-1}(G_\beta \mu, h_\beta)_\mu h_\beta + R_\beta, \qquad \sup_{\beta \geq 0} \mathcal{E}_\beta(R_\beta, R_\beta) < \infty.$$

Recalling that $h_\beta \to h_0$ $\mathcal{E}$-weakly by Lemma 2, we have

$$k_\beta G_\beta^\mu \mu \to C_{d,\alpha}^{-1} \langle G_0 \mu, h_0 \rangle_\mu h_0 \quad \mathcal{E}\text{-weakly} \quad k_\beta = \begin{cases} \beta^{\frac{d}{\alpha}-1} & (1 < d/\alpha < 2) \\ \beta \log \beta^{-1} & (d/\alpha = 2) \\ \beta & (d/\alpha > 2) \end{cases}$$

By Stollmann and Voigt (1996), $\int_{\mathbb{R}^d} u^2 d\nu \leq \|G\nu\|_\infty \mathcal{E}(u,u)$ for Green-tight ν.
In particular, $\mathcal{E}$-weakly convergence implies $L^2(\nu)$-weakly one and

$$k_\beta G_\beta^\mu \mu \to C_{d,\alpha}^{-1} \langle G_0 \mu, h_0 \rangle_\mu h_0 \quad L^2(\nu)\text{-weakly}$$

From weakly-convergence to strongly-convergence

$$k_\beta G_\beta^\mu \mu \to C_{d,\alpha}^{-1}(G_0\mu, h_0)_\mu h_0 \quad L^2(\nu)\text{-}weakly$$

If $\nu(\mathbb{R}^d) < \infty, 1 \in L^2(\nu)$ and $\langle \nu, k_\beta G_\beta^\mu \mu \rangle \to C_{d,\alpha}^{-1}(G_0\mu, h_0)_\mu \langle \nu, h_0 \rangle$.

$$\langle \nu, k_\beta G_\beta^\mu \mu \rangle = k_\beta \langle \nu, G_\beta^\mu \mu \rangle = k_\beta \int_0^\infty e^{-\beta t} \langle \nu, p_t^\mu \mu \rangle dt \to C_{d,\alpha}^{-1}(G_0\mu, h_0)_\mu \langle \nu, h_0 \rangle$$

In particular, for $\nu(dy) = p^\mu(\epsilon, x, y)m(dy)$, we have $\langle \nu, h_0 \rangle = h_0$ and

$$k_\beta \int_0^\infty e^{-\beta t} p_{t+\epsilon}^\mu \mu(x) dt \to C_{d,\alpha}^{-1}(G_0\mu, h_0)_\mu h_0(x) \quad \beta \to 0$$

By the Tauberian theorem,

$$\frac{1}{k_{1/t}} \int_0^t p_{s+\epsilon}^\mu \mu(x) ds \to (C_{d,\alpha}\Gamma(d/\alpha \wedge 2))^{-1}(G_0\mu, h_0)_\mu h_0(x) \quad t \to \infty$$

$\dfrac{k_{1/(t+\epsilon)}}{k_{1/t}} \to 1$ and $\displaystyle\int_0^\epsilon p_t^\mu \mu(x) dt < \infty$ imply the main result.

Extension to non-compact measure

Lemma 1

Green operator of $\check{\mathbb{M}}^{\beta,\mu}$ is given by $\mathcal{G}_\beta f(x) = \int_Y G_\beta(x,y) f(y)\mu(dy)$.

$\mathcal{G}_\beta$ is a compact operator on $L^2(Y,\mu)$.

Lemma 2

Let γ_β be the principal eigenvalue of $\mathcal{G}_\beta$ and denote by h_β the corresponding eigenfunction. $\lim_{\beta\to 0} \gamma_\beta = \gamma_0 = 1$ and $h_\beta \to h_0$ ($L^2(\mu)$-strongly and $\mathcal{E}$-weakly).

Lemma 3 Consider the orthogonal decomposition

$$G^\mu_\beta \mu = (1-\mathcal{G}_\beta)^{-1} G_\beta \mu = (1-\gamma_\beta)^{-1}(G_\beta\mu, h_\beta)_\mu h_\beta + R_\beta.$$

$R_\beta \in \mathcal{F}_e$ and $\sup_{\beta\geq 0} \mathcal{E}(R_\beta, R_\beta) < \infty$ if μ is of 0-order finite energy integral.

Lemma 4 (Modification Needed !!)

For the expansion of $G_\beta(x,y)$, the error term $E_\beta(x,y)$ may diverge as $|x-y| \to \infty$ and we cannot obtain the asymptotic expansion of $\mathcal{G}_\beta$ directly.

Modification of Lemma 4

- Upper estimate for $G_\beta(x,y)$

$$\sup_{x\in\mathbb{R}^d}\int_{K_\epsilon} G(x,y)\mu(dy) < \epsilon,\ \ G^\epsilon_\beta(x,y) = \begin{cases} G_\beta(x,y) & (x,y\in K_\epsilon = \{x \mid |x| \le R_\epsilon\} \\ G(x,y) & (\text{otherwise}) \end{cases}$$

Consider the principal eigenvalue of $\mathcal{G}^\epsilon_\beta f(x) = \int_Y G^\epsilon_\beta(x,y) f(y)\mu(dy)$

- Lower estimate for $G_\beta(x,y)$

$$G_\beta(x,y) \ge G(x,y) - \frac{2^{1-d}\pi^{1-\frac{d}{2}}}{\alpha\Gamma(d/2)\sin((d/\alpha-1)\pi)}\beta^{\frac{d}{\alpha}-1} \quad (1 < d/\alpha < 2)$$

$$G_\beta(x,y) \ge G(x,y) - \frac{2^{1-d}\pi^{-\frac{d}{2}}}{\Gamma(\alpha+1)}\beta\log\beta^{-1} - c_1\beta \quad (d/\alpha = 2)$$

$$G_\beta(x,y) \ge G(x,y) - \beta\int_0^\infty tp(t,x,y)dt \quad (d/\alpha > 2).$$

Lower estimate for $(\mathcal{G}_\beta h_0, h_0)_\mu$ for ground state $h_0(x)$

Lemma 4′ $\exists ! C_{d,\alpha} > 0$ s.t. $\displaystyle\lim_{\beta\to 0}\frac{1-\gamma_\beta}{k_\beta} = C_{d,\alpha}$

Thank you for your attention !!