Systems of infinitely many Brownian motions with long ranged interaction

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Introduction 1

Infinite dimensional stochastic differential equations (ISDEs)

- $\{B^i\}_{i\in\mathbb{N}}$ are independent d-dimensional Brownian motions.
- $\Phi = \Phi(x)$; free potential.
- $\Psi = \Psi(x,y)$; interaction potential.

We study ISDEs of $\mathbf{X} = (X^i)_{i\in\mathbb{N}} \in C([0,\infty);(\mathbb{R}^d)^{\mathbb{N}})$:

ISDE (1)

$$dX_t^i = dB_t^i - \frac{\beta}{2}\nabla_x\Phi(X_t^i)dt - \frac{\beta}{2}\sum_{j\neq i}^{\infty}\nabla_x\Psi(X_t^i, X_t^j)dt$$

$$(X_0^i)_{i\in\mathbb{N}} = \mathbf{x} = (x_i)_{i\in\mathbb{N}}$$

- Existence (solutions, strong solutions)
- Uniqueness (in distribution, pathwise)

Introduction 2

Related results

1. Lang 1977,1978: Ψ : smooth, with compact support

2. Fritz 1987: singular interaction

3. T. 1996, Fradon-Roelly-T. 2002: Ψ : with hard core

4. Osada 2012: Existence of solutions for Ψ of Ruelle's class, and logarithmic potential (Dyson, Ginibre)

5. Osada-T(arXiv1412.8674): Existence and uniqueness of strong solutions for Ψ of Ruelle's class, logarithmic potential (Dyson, Ginibre)

Applications of 4 and 5:
Honda-Osada 2015 : logarithmic (Bessel),
Osada-T. (arXiv1408.0632) : logarithmic (Airy)

Introduction 3

Dyson's Brownian motion model [Dyson 62] is a one parameter family of the systems of one dimensional Brownian motions with long ranged repulsive interaction, whose strength is represented by a parameter $\beta > 0$. It soves the stochastic differential equation

$$X_j(t) = x_j + B_j(t) + \frac{\beta}{2} \sum_{\substack{k:1 \le k \le n \\ k \neq j}} \int_0^t \frac{ds}{X_j(s) - X_k(s)}, \ 1 \le j \le n \quad (1)$$

where $B_j(t), j = 1, 2, \ldots, n$ are independent one dimensional Brownian motions.We consider the case that $\beta = 2$ and call the model in the special case Dyson model .

Introduction 4

The Dyson model is realized by the following three processes:
(i) The process of eigenvalues of Hermitian matrix valued diffusion process in the Gaussian unitary ensemble (GUE).
(ii) The system of one-dimensional Brownian motions conditioned never to collide with each other.
(iii) The harmonic transform of the absorbing Brownian motion in a Weyle chamber of type A_{n-1}:

$$\mathbb{W}_n = \Big\{ \mathbf{x} = (x_1, x_2, \cdots, x_n) : x_1 < x_2 < \cdots < x_n \Big\}.$$

with harmonic function given by the Vandermonde determinant:

$$h_n(\mathbf{x}) = \prod_{1 \le j < k \le n} (x_k - x_j) = \det_{1 \le j, k \le n} \Big[x_k^{j-1} \Big].$$

Introduction 5

$n \times n$ Hermitian matrix valued process $(n \in \mathbb{N})$

$$M(t) = \begin{pmatrix} M_{11}(t) & M_{12}(t) & \cdots & M_{1n}(t) \\ M_{21}(t) & M_{22}(t) & \cdots & M_{2n}(t) \\ & & \cdots & \\ M_1(t) & M_{n2}(t) & \cdots & M_{nn}(t) \end{pmatrix}, \quad M_{\ell k}(t) = M_{k\ell}(t)^{\dagger}.$$

(GUE) $B^{\mathrm{R}}_{k\ell}(t)$, $B^{\mathrm{I}}_{k\ell}(t)$,$1 \le k \le \ell \le n$: indep. BMs

$$M_{k\ell}(t) = \frac{1}{\sqrt{2}} B^{\mathrm{R}}_{k\ell}(t) + \frac{\sqrt{-1}}{\sqrt{2}} B^{\mathrm{I}}_{k\ell}(t), \quad 1 \le k < \ell \le n,$$

$$M_{kk}(t) = B^{\mathrm{R}}_{kk}(t) + x_k, \quad 1 \le k \le n,$$

Plan

1. Scaling limit of the Dyson model (Bulk, Soft edge).

2. Strong Markov property of the limit process.

3. Dirichlet form associated with the process

4. ISDE associated the process.

Preliminaries 1

The configuration space of unlabelled particles:

$$\mathfrak{M} = \Big\{ \xi : \xi \text{ is a nonnegative integer valued Radon measures in } \mathbb{R} \Big\}$$

$$= \Big\{ \xi(\cdot) = \sum_{j \in \mathbb{I}} \delta_{x_j}(\cdot) : \sharp\{ j \in \mathbb{I} : x_j \in K \} < \infty, \text{ for any } K \text{ compact} \Big\}$$

$\mathfrak{M}$ is a Polish space with the vague topology.

The configuration space of noncolliding systems:

$$\begin{aligned} \mathfrak{M}_0 &= \Big\{ \xi \in \mathfrak{M} : \xi(\{x\}) = 1, \text{ for any } x \in \text{supp } \xi \Big\} \\ &= \Big\{ \{x_j\} : \sharp\{ j : x_j \in K \} < \infty, \text{ for any } K \text{ compact} \Big\}. \end{aligned}$$

A configuration space $\mathcal{X}$ is relative compact, if

$$\sup_{\xi \in \mathcal{X}} \xi(K) < \infty, \quad \text{for any } K \subset \mathbb{R} \text{ compact}$$

The eigen values distribution on $M(t)$ at time $t = 1$, with $x_k = 0$, $1 \le k \le n$ is given by

$$m_2^n(d\boldsymbol{\lambda}_n) = \frac{1}{Z} \prod_{i<j} |\lambda_i - \lambda_j|^2 \exp\left\{ -\frac{1}{2} \sum_{i=1}^{n} |\lambda_i|^2 \right\} d\mathbf{x}_n.$$

(Bulk scaling limit) For the eigenvalues $\{\lambda_1^n, \dots, \lambda_n^n\}$

$$\mu_{\rm sin}^n = m_2^n(\{\sqrt{n}\lambda_1^n, \dots, \sqrt{n}\lambda_n^n\} \in \cdot) \to \mu_{\sf sin}, \quad \text{weakly as } n \to \infty.$$

(Soft edge scaling limit) For the eigenvalues $\{\lambda_1^n, \dots, \lambda_n^n\}$

$$\mu_{\rm Ai}^n = m_2^n(\{n^{1/6}(\lambda_1^n - 2\sqrt{n}), \dots, n^{1/6}(\lambda_n^n - \sqrt{n})\} \in \cdot) \\ \to \mu_{\rm Ai}, \quad \text{weakly as } n \to \infty.$$

Preliminaries 3

$\mu_{\sin}$ is the determinantal point process(DPP) with the *sine kernel*

$$K_{\sin}(x,y) \equiv \frac{\sin\{\pi(y-x)\}}{\pi(y-x)}, \quad x,y \in \mathbb{R}.$$

μ_{Ai} is the DPP with the *Airy kernel*

$$K_{\mathrm{Ai}}(x,y) \equiv \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}, \quad x,y \in \mathbb{R}.$$

The moment generating function of the DPP $\mu_\star$, $\star = \sin, \mathrm{Ai}$, is given by a Fredholm determinant

$$\int_{\mathfrak{M}} \exp\Big\{ \int_{\mathbb{R}} f(x)\xi(dx)\Big\}\mu_\star(d\xi) = \operatorname*{Det}_{(x,y)\in\mathbb{R}^2} \Big[\delta_x(y) + K_\star(x,y)\chi(y)\Big],$$

for $f \in \mathrm{C}_c(\mathbb{R})$, where $\chi(\cdot) = e^{f(\cdot)} - 1$.

Preliminaries 4

A processes $\Xi(t)$ is said to be determinantal if the multi-time moment generating function is given by the Fredholm determinant

$$\Psi^{\mathbf{t}}[\mathbf{f}] = \mathop{\mathrm{Det}}_{\substack{(s,t)\in\{t_1,t_2,\dots,t_M\}^2,\\ (x,y)\in\mathbb{R}^2}} \Big[\delta_{st}\delta_x(y) + \mathbb{K}(s,x;t,y)\chi_t(y)\Big], \qquad (2)$$

In other words, the multitime correlation functions are represented as

$$\rho\left(t_1, \mathbf{x}_{N_1}^{(1)}; \dots; t_M, \mathbf{x}_{N_M}^{(M)}\right) = \det_{\substack{1\le j\le N_m, 1\le k\le N_n\\ 1\le m,n\le M}} \left[\mathbb{K}(t_m, x_j^{(m)}; t_n, x_k^{(n)})\right].$$

The function $\mathbb{K}$ is called the correlation kernel of the process $\Xi(t)$.

Non-colliding Brownian motion

The Dyson model starting from n points all at the origin is determinantal with the correlation kernel

$$\mathbb{K}_n(s,x;t,y) = \begin{cases} \dfrac{1}{\sqrt{2s}} \displaystyle\sum_{k=0}^{n-1} \left(\frac{t}{s}\right)^{\frac{k}{2}} \varphi_k\left(\frac{x}{\sqrt{2s}}\right) \varphi_k\left(\frac{y}{\sqrt{2t}}\right), & s \le t, \\ \dfrac{-1}{\sqrt{2s}} \displaystyle\sum_{k=n}^{\infty} \left(\frac{t}{s}\right)^{\frac{k}{2}} \varphi_k\left(\frac{x}{\sqrt{2s}}\right) \varphi_k\left(\frac{y}{\sqrt{2t}}\right), & s > t, \end{cases}$$

where $h_k = \sqrt{\pi} 2^k k!$ and

$$\varphi_k(x) = \frac{1}{\sqrt{h_k}} e^{-x^2/2} H_k(x),$$

with the Hermite polynomials $H_k, k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$.

The *extended sine kernel*

(1) The bulk scaling limit[Spohn: 1987, Nagao-Forrester 1998]:

$$\mathbb{K}_n\left(\frac{2n}{\pi^2}+s,x;\frac{2n}{\pi^2}+t,y\right) \to \mathbf{K}_{\sin}(s,x;t,y), \quad n\to\infty,$$

$$\mathbf{K}_{\sin}(s,x;t,y) = \begin{cases} \displaystyle\int_0^1 du\, e^{\pi^2 u^2(t-s)/2}\cos\{\pi u(y-x)\} & \text{if } t>s \\ \frac{\sin\pi(x-y)}{\pi(x-y)} = K_{\sin}(x,y) & \text{if } t=s \\ \displaystyle -\int_1^\infty du\, e^{\pi^2 u^2(t-s)/2}\cos\{\pi u(y-x)\} & \text{if } t<s. \end{cases}$$

$x,y\in\mathbb{R}$.

The *extended Airy kernel*

(2) The soft-edge scaling limit [Prähofer-Spohn 2002, Johansson 2002]: Let $f_n(u) = 2n^{2/3} + n^{1/3}u - u^2/4$.

$$\mathbb{K}_n\Big(n^{1/3} + s, f_n(s) + x; n^{1/3} + t, f_n(t) + y\Big) \\ \to \mathbf{K}_{\mathrm{Ai}}(s, x; t, y), \quad N \to \infty.$$

$$\mathbf{K}_{\mathrm{Ai}}(s, x; t, y) \equiv \begin{cases} \displaystyle\int_0^\infty du\, e^{-u(t-s)/2} \mathrm{Ai}(u+x)\mathrm{Ai}(u+y) & \text{if } t > s, \\ \displaystyle\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} = K_{\mathrm{Ai}}(x, y) & \text{if } t = s, \\ \displaystyle -\int_{-\infty}^0 du\, e^{-u(t-s)/2} \mathrm{Ai}(u+x)\mathrm{Ai}(u+y) & \text{if } t < s, \end{cases}$$

$x, y \in \mathbb{R}$.
Where $\mathrm{Ai}(\cdot)$ is the Airy function.

Theorem 1(Strong Markov property)

We denote the determinatal process with the extended kernel $\mathbf{K}_\star$, $\star \in \{\sin, \mathrm{Ai}\}$ by $(\Xi_t, \mathbb{P}_\star)$.

Theorem 1 [Katori-T 2011, Osada-T 2015]

Let $\star \in \{\sin, \mathrm{Ai}\}$. There exists a process $(\Xi(t), \mathbb{P}_\star^\xi)$, $\xi \in \mathfrak{X}_\star$ such that

$$\mathbb{P}_\star^\xi(\Xi(0) = \xi) = 1 \text{ and } \mathbb{P}_\star(\cdot) = \int_{\mathfrak{X}_\star} \mathbb{P}_\star^\xi(\cdot)\mu_\star(d\xi),$$

and there exists a measurable subset $\tilde{\mathfrak{X}}_\star$ of $\mathfrak{X}_\star$ such that

$$\mu_\star(\tilde{\mathfrak{X}}_\star) = 1 \text{ and } \mathbb{P}_\star^\xi(C([0,\infty), \mathfrak{X}_\star)) = 1, \ \xi \in \tilde{\mathfrak{X}}_\star.$$

Furthermore, the determinantal process $(\Xi(t), \mathbb{P}_\star)$ is a reversible diffusion process.

Dirichlet forms

A function f defined on the configuration space $\mathfrak{M}$ is local if $f(\xi) = f(\xi_K)$ for some compact set K.
A local function f is smooth if $f(\sum_{j=1}^{n} \delta_{x_j}) = \tilde{f}(x_1, x_2, \dots, x_n)$ with some smooth function $\tilde{f}$ on $\mathbb{R}^n$ with compact support. Put

$$\mathcal{D}_0 = \{f : f \text{ is local and smooth}\}.$$

We put

$$\mathbb{D}[f,g](\xi) = \frac{1}{2} \sum_{j=1}^{\xi(K)} \frac{\partial \tilde{f}(\mathbf{x})}{\partial x_j} \frac{\partial \tilde{g}(\mathbf{x})}{\partial x_j}$$

and for a probability measure μ we introduce the bilinear form

$$\mathcal{E}^{\mu}(f,g) = \int_{\mathfrak{M}} \mathbb{D}[f,g] d\mu, \quad f, g \in \mathcal{D}_{\infty},$$

$$\mathcal{D}_{\infty} = \{f \in \mathcal{D}_0 : \mathcal{E}^{\mu}(f,f) < \infty\}.$$

Hamiltonian

Let Φ be a free potential, Ψ be an interaction potential. For a given sequence $\{b_r\}$ of $\mathbb{N}$ we introduce a Hamiltonian on $S_r = \{x \in \mathbb{R}^d : |x| < b_r\}$:

$$H_r(\xi) = H_r^{\Phi,\Psi}(\xi) = \sum_{x_j \in I_r} \Phi(x_j) + \sum_{x_j, x_k \in I_r, j<k} \Psi(x_j, x_k)$$

Assume that such that there exist upper semicontinuous functions Φ_0 and $\Psi_0 : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ which are locally bounded from below and $\{x : \Psi_0(x) = \infty\}$ is compact, and there is a positive constant c such that

$$c^{-1}\Phi_0(x) \le \Phi(x) \le c\Phi(x), \quad c^{-1}\Psi_0(x) \le \Psi(x) \le c\Psi(x).$$

Quasi Gibbs measure

Definition (quasi Gibbs measure)

A probability measure μ is said to be a (Φ, Ψ)-quasi Gibbs measure if there exists an increasing sequence $\{b_r\}$ of $\mathbb{N}$ and measures $\{\mu^m_{r,k}\}$ on $\mathfrak{M}^m_r = \{\xi(S_r) = m\}$ such that for each $r, m \in \mathbb{N}$ satisfying

$$\mu^m_{r,k} \le \mu^m_{r,k+1}, \quad k \in \mathbb{N}, \quad \lim_{k\to\infty} \mu^m_{r,k} = \mu(\cdot \cap \mathfrak{M}^m_r), \mathit{weekly}$$

and that for all $r, m, k \in \mathbb{N}$ and for $\mu^m_{r,k}$-a.s. $\xi \in \mathfrak{M}^m_r$

$$c^{-1} e^{-H_r(\zeta)} \Lambda^m_r(d\zeta) \le \mu^m_{r,k}(\pi_{S_r} \in d\zeta | \xi_{S_r^c}) \le c e^{-H_r(\zeta)} \Lambda^m_r(d\zeta)$$

Here Λ^m_r is the restriction of Poisson random measure with intensity measure dx on $\mathfrak{M}^m_r$.

Log derivative

Definition (log derivative)

We call $\mathbf{d}^\mu \in L^1_{loc}(\mathbb{R}^d \times \mathfrak{M}, \mu^1)$ the log derivative of μ if

$$\int_{\mathbb{R}^d \times \mathfrak{M}} \mathbf{d}^\mu(x,\eta) f(x,\eta) d\mu^1(x,\eta) = -\int_{\mathbb{R}^d \times \mathfrak{M}} \nabla_x f(x,\eta) d\mu^1(x,\eta),$$

is satisfied for $f \in C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}_0$.

Here μ^k is the Campbell measure of μ:

$$\mu^k(A \times B) = \int_A \mu_{\mathbf{x}}(B) \rho^k(\mathbf{x}) d\mathbf{x}, \quad A \in \mathcal{B}(\mathbb{R}^k), B \in \mathcal{B}(\mathfrak{M}).$$

and $\mu_{\mathbf{x}}$ is the reduced Palm measure conditioned at $\mathbf{x} \in (\mathbb{R}^d)^k$

$$\mu_{\mathbf{x}} = \mu(\cdot - \sum_{j=1}^k \delta_{x_j} | \xi(x_j) \geq 1 \text{ for } j = 1, 2, \ldots, k).$$

General theorem

Theorem [Osada 2012,2013a]

(i) If μ is a (Φ,Ψ)-quasi Gibbs measure, then $(\mathcal{E}^\mu,\mathcal{D}_\infty)$ is closable and its closure is a quasi regular Dirichlet form. Then there exists the diffusion process $(\Xi(t),\mathbf{P}_\mu^\xi)$ associated with the Dirichlet form.
(ii) Assume that there exists a log derivative $\mathbf{d}^\mu$ (and some conditions). There exists $\widehat{\mathfrak{M}}\subset\mathfrak{M}$ such that $\mu(\widehat{\mathfrak{M}})=1$, and for any $\xi=\sum_{j\in\mathbb{N}}\delta_{x_j}\in\widehat{\mathfrak{M}}$, there exists $\mathbb{R}^{\mathbb{N}}$-valued continuous process $\mathbf{X}(t)=(X_j(t))_{j=1}^\infty$ satisfying $\mathbf{X}(0)=\mathbf{x}=(x_j)_{j=1}^\infty$ and

$$dX_j(t)=dB_j(t)+\frac{1}{2}\mathbf{d}^\mu\left(X_j(t),\sum_{k:k\neq j}\delta_{X_k(t)}\right)dt,\quad j\in\mathbb{N}.$$

ISDEs

DPP with the sine kernel: [Osada 12,13a] μ_{sin} is a quasi-Gibbs state and the associated labeled process solves ISDE(sin):

$$dX_j(t) = dB_j(t) + \sum_{\substack{k \in \mathbb{N} \\ k \neq j}} \frac{dt}{X_j(t) - X_k(t)}, \quad j \in \mathbb{N}.$$

DPP with the Airy kernel : [Osada13b, Osada-T arXiv:1408.0632] μ_{Ai} is a quasi-Gibbs state and the associated labeled process solves ISDE(Ai):

$$\begin{aligned} dX_j(t) &= dB_j(t) \\ &+ \lim_{L \to \infty} \left\{ \sum_{\substack{k \in \mathbb{N}, k \neq j \\ |X_k(t)| \leq L}} \frac{1}{X_j(t) - X_k(t)} - \int_{|y| \leq L} \frac{\widehat{\rho}(y)}{-y} dy \right\} dt, \ j \in \mathbb{N}, \end{aligned}$$

where $\widehat{\rho}(x) = \dfrac{\sqrt{-x}\mathbf{1}(x < 0)}{\pi}$.

Theorem 2

Theorem 2 [Osada-T 2015]

Let $\star \in \{\mathsf{sin}, \mathrm{Ai}\}$. Put $\mathbf{P}_{\star}^{\mu_\star} = \int_{\mathfrak{M}} \mu_\star(d\xi)\mathbf{P}_{\star}^{\xi}$.
(i) A labeled process $\mathbf{X} = (X_j)_{j\in\mathbb{N}}$ associated with the determinantal process $(\Xi(t), \mathbb{P}_\star)$ solves the ISDE $(\star)$.
(ii) Process $(\Xi(t), \mathbb{P}_\star)$ coincides with process $(\Xi(t), \mathbf{P}_{\star}^{\mu_\star})$ in distribution. In particular, process $(\Xi(t), \mathbb{P}_\star)$ is associated with the Dirichlet form $(\mathcal{E}_\star, \mathcal{D}_\star)$.

The coincidence of processes enable us to examine them from various points of view. From the algebraic construction by their space-time correlation functions, we obtain *quantitative* information of the processes such as the moment generating functions; while from the analytic construction through Dirichlet form theory, we deduce many *qualitative* properties of sample paths by means of the ISDE representation.

Polynomial functions

Let $\star \in \{\sin, \mathrm{Ai}\}$. Theorem 1 implies that the Dirichlet form $(\hat{\mathcal{E}}_\star, \hat{\mathcal{D}}_\star)$ associated with process $(\Xi(t), \mathbb{P}_\star)$ is quasi-regular. A function f on the configuration space $\mathfrak{M}$ is said to be polynomial if it is written in the form

$$f(\xi) = F\left(\int_{\mathbb{R}} \phi_1(x)\xi(dx), \int_{\mathbb{R}} \phi_2(x)\xi(dx), \dots, \int_{\mathbb{R}} \phi_k(x)\xi(dx)\right)$$

with polynomial function F on $\mathbb{R}^k, k \in \mathbb{N}$, and smooth functions $\phi_j, 1 \le j \le k$ on $\mathbb{R}$ with compact supports.
Let $\mathcal{P}$ be the set of all polynomial functions on $\mathfrak{M}$.

In Proposition 7.2 [Katori-T07] it was proved that

$$\hat{\mathcal{E}}_\star(f, g) = \mathcal{E}_\star(f, g), \quad f, g \in \mathcal{P}. \tag{3}$$

Cores of Dirichlet forms

By [Osada-T 2014] it is proved that $(\mathcal{E}_\star, \mathcal{D}_\star)$ and $(\hat{\mathcal{E}}_\star, \hat{\mathcal{D}}_\star)$ are both closed extensions of $(\mathcal{E}_\star, \mathcal{P})$, and the former is the smallest one.
These relations are generalized to k-labeled dynamics.
By the same procedure in [Osada 2012] we obtain that a labeled process $\mathbf{X} = (X_j)_{j\in\mathbb{N}}$ associated with $(\Xi(t), \mathbb{P}_\star)$ solves for $\star = \sin$

$$dX_j(t) = dB_j(t) + \sum_{\substack{k\in\mathbb{N} \\ k\neq j}} \frac{dt}{X_j(t) - X_k(t)}, \quad j \in \mathbb{N},$$

and for $\star = \mathrm{Ai}$

$$\begin{aligned} dX_j(t) &= dB_j(t) \\ &+ \lim_{L\to\infty} \Bigg\{ \sum_{\substack{k\in\mathbb{N}, k\neq j \\ |X_k(t)|\leq L}} \frac{1}{X_j(t) - X_k(t)} - \int_{|y|\leq L} \frac{\widehat{\rho}(y)}{-y} dy \Bigg\} dt, \ j \in \mathbb{N}. \end{aligned}$$

Theorem 3 (uniqueness of strong solutions)

The coincidence of two diffusion processes are derived from the following theorem.

Theorem 3 [Osada-T arXiv:1412.8674]

Let $\star \in \{\text{sin}, \mathrm{Ai}\}$. ISDE($\star$) has a strong solution satisfying the $\mu_\star$-absolute continuity condition, and strong uniqueness holds for ISDE($\star$) with the $\mu_\star$-absolutely continuity condition, where we call a solution $\mathbf{X}$ satisfies μ-absolute continuity condition if it satisfies

$$\mathsf{P}_\mu \circ \Xi_t^{-1} \prec \mu \text{ for } \forall t > 0.$$

Finite system

Remind the eigenvalue distribution of GUE

$$m_2^n(d\boldsymbol{\lambda}_n) = \frac{1}{Z}\prod_{i<j}|\lambda_i - \lambda_j|^2 \exp\left\{-\frac{1}{2}\sum_{i=1}^n |\lambda_i|^2\right\} d\mathbf{x}_n.$$

and its scaling limit convergence as $n \to \infty$:

$$\mu_{\mathrm{sin}}^n = m_2^n(\{\sqrt{n}\lambda_1^n, \dots, \sqrt{n}\lambda_n^n\} \in \cdot) \to \mu_{\sin}.$$

$$\mu_{\mathrm{Ai}}^n = m_2^n(\{n^{1/6}(\lambda_1^n - 2\sqrt{n}), \dots, n^{1/6}(\lambda_n^n - \sqrt{n})\} \in \cdot) \to \mu_{\mathrm{Ai}}.$$

Let $(\Xi(t), \mathbf{P}_\star^{\xi^n})$, $\star \in \{\sin, \mathrm{Ai}\}$ be the diffusion process associated with the quasi-regular Dirichlet form $(\mathcal{E}_\star^n, \mathcal{D}_\star^n)$, the closure of the bilinear form $(\mathcal{E}_{\mu_\star^n}, \mathcal{P})$.

labeled map

We take a labeled map $\mathfrak{l} : \mathfrak{M} \to \mathbb{R}^{\mathbb{N}} \oplus \bigoplus_{n=0}^{\infty} \mathbb{R}^n$ such that $\mathfrak{l}(\xi) = (\mathfrak{l}_j(\xi))_{j\in\mathbb{N}}$ if $\xi(\mathbb{R}) = \infty$ and $\mathfrak{l}(\xi) = (\mathfrak{l}_j(\xi))_{j=1}^{n}$ if $\xi(\mathbb{R}) = n \in \mathbb{N}$, and

$$|\mathfrak{l}_j(\xi)| \le |\mathfrak{l}_{j+1}(\xi)|, \quad 1 \le j < \xi(\mathbb{R}).$$

For the label $\mathfrak{l}$ and $\xi \in \mathfrak{M}$ we set

$$\xi_n^{\mathfrak{l}} = \sum_{j=1}^{n} \delta_{\mathfrak{l}_j(\xi)} \quad \text{for } n \in \mathbb{N} \text{ with } n < \xi(\mathbb{R}),$$

and $\xi_n^{\mathfrak{l}} = \xi$ for $n = \xi(\mathbb{R}) < \infty$.
Let $\mathbf{X} = (X_j)_{j\in\mathbb{N}}$ and $\mathbf{X}^n = (X_j^n)_{j=1}^{n}$ be the labeled processes associated with $(\Xi(t), \mathbf{P}_{\star}^{\xi})$ and $(\Xi(t), \mathbf{P}_{\star}^{\xi_n})$, respectively. Note that $\mathbf{X}(0) = \mathfrak{l}(\xi) \equiv \mathbf{x}$ and $\mathbf{X}^n(0) = (\mathfrak{l}_j(\xi))_{j=1}^{n} \equiv \mathbf{x}^n$. We have then the following as a corollary of Theorem 3.

Corollary 4

Let $\star \in \{\mathrm{sin}, \mathrm{Ai}\}$. (i) For $\mu_\star$ a.s. ξ,

$$(\Xi(t), \mathbf{P}_\star^{\xi_n^{\mathfrak{l}}}) \to (\Xi(t), \mathbf{P}_\star^{\xi}), \quad n \to \infty,$$

weakly on the path space $C([0,\infty), \mathfrak{M})$.
(ii) For $\mu_\star \circ \mathfrak{l}^{-1}$ a.s. $\mathbf{x}$, and $m \in \mathbb{N}$

$$(X_1^n(t), X_2^n(t), \dots, X_m^n(t)) \to (X_1(t), X_2(t), \dots, X_m(t)), \quad n \to \infty,$$

weakly on the path space $C([0,\infty), \mathbb{R}^m)$.

Remarks

Suppose that μ and μ_N ($N \in \mathbb{N}$) are probability measures on $\mathfrak{M}$, and $(\Xi(t), P)$ and $(\Xi(t), P^N)$ are diffusion processes associated with the Dirichlet spaces given by the closures of $(\mathcal{E}_\mu, \mathcal{P}, L^2(\mathfrak{M}, \mu))$ and $(\mathcal{E}^a_{\mu_N}, \mathcal{P}, L^2(\mathfrak{M}, \mu^N))$, respectively. Let us consider the problem on the weak convergence of stationary processes. That is,

$$\mu_N \to \mu, \ N \to \infty \ \Rightarrow (\Xi(t), P^N) \to (\Xi(t), P), \ N \to \infty.$$

If the measures μ_N and μ are singular each other, then such a convergence is not covered by a general theorem of convergence of diffusions associated with Dirichlet forms. We remark that Corollary 4 (i) gives examples of such a convergence even if the measures μ_N and μ are singular each other. Recently, Kawamoto-Osada[preprint] also showed Corollary 4 (ii) using a different method.

Thank you for your attention