Long-time Behavior of Lévy-type Processes: Transience, Recurrence and Ergodicity

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Outline

Motivation

An $\mathbb{R}^d$-valued, $d \geq 1$, Lévy process $(\{L_t\}_{t\geq 0}, \{\mathbb{P}^x\}_{x\in\mathbb{R}^d})$ is said to be transient if

$$\mathbb{E}^x\left[\int_0^\infty 1_{B_a(x)}(L_t)dt\right] < \infty \quad \text{for all } x \in \mathbb{R}^d \text{ and all } a > 0,$$

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and recurrent if

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Every Lévy process is either transient or recurrent.

Motivation

Equivalently, $\{L_t\}_{t\geq 0}$ is transient if, and only if,

$$\mathbb{P}^x\left(\int_0^\infty 1_{B_a(x)}(L_t)dt < \infty\right) = 1 \quad \text{for all } x \in \mathbb{R}^d \text{ and all } a > 0$$

or if, and only if,

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Similarly, $\{L_t\}_{t\geq 0}$ is recurrent if, and only if,

$$\mathbb{P}^x\left(\int_0^\infty 1_{B_a(x)}(L_t)dt = \infty\right) = 1 \quad \text{for all } x \in \mathbb{R}^d \text{ and all } a > 0$$

or if, and only if,

$$\mathbb{P}^x\left(\liminf_{t\longrightarrow\infty} |L_t - x| = \infty\right) = 1 \quad \text{for all } x \in \mathbb{R}^d.$$

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Every Lévy process $\{L_t\}_{t\geq 0}$ can be described in terms of its Fourier transform,

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where the characteristic exponent $q : \mathbb{R}^d \longrightarrow \mathbb{C}$ has the following representation

$$q(\xi) = i\langle \xi, b\rangle + \frac{1}{2}\langle \xi, C\xi\rangle + \int_{\mathbb{R}^d}\left(1 - e^{i\langle \xi, y\rangle} + i\langle \xi, y\rangle 1_{B_1(0)}(y)\right)\nu(dy).$$

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Chung-Fuchs criterion: A Lévy process $\{L_t\}_{t\geq 0}$ is transient if, and only if,

$$\int_{B_a(0)} \mathrm{Re}\left(\frac{1}{q(\xi)}\right) d\xi < \infty \quad \text{for some (all)} \quad a > 0.$$

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Goal: To extend the Chung-Fuchs criterion on a broader class of processes.

Preliminaries on Lévy-type Processes

A Feller process $(\{F_t\}_{t\geq 0}, \{\mathbb{P}^x\}_{x\in\mathbb{R}^d})$ with state space $\mathbb{R}^d$ is a strong Markov process whose associated operator semigroup $\{P_t\}_{t\geq 0}$,

$$P_t f(x) := \mathbb{E}^x[f(F_t)], \quad t \geq 0,\ x \in \mathbb{R}^d,\ f \in B_b(\mathbb{R}^d),$$

enjoys:

- $P_t(C_\infty(\mathbb{R}^d)) \subseteq C_\infty(\mathbb{R}^d)$ for all $t \geq 0$
- $\lim_{t\longrightarrow 0} ||P_t f - f||_\infty = 0$ for all $f \in C_\infty(\mathbb{R}^d)$.

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The infinitesimal generator $(\mathcal{A}, \mathcal{D}_\mathcal{A})$ of the semigroup $\{P_t\}_{t\geq 0}$ is given by the strong limit

$$\mathcal{A}f := \lim_{t\longrightarrow 0} \frac{P_t f - f}{t}$$

on the set $\mathcal{D}_\mathcal{A} \subseteq C_\infty(\mathbb{R}^d)$ of all $f \in C_\infty(\mathbb{R}^d)$ for which the above limit exists with respect to $||\cdot||_\infty$.

Theorem (Courrége 1965)

Let $(\mathcal{A}, \mathcal{D}_{\mathcal{A}})$ be a Feller generator which satisfies

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Then $\mathcal{A}|_{C_c^\infty(\mathbb{R}^d)}$ is a pseudo-differential operator, i.e., it has the representation

$$\mathcal{A}|_{C_c^\infty(\mathbb{R}^d)} f(x) = -\int_{\mathbb{R}^d} q(x,\xi) e^{i\langle \xi, x\rangle} \hat{f}(\xi) d\xi.$$

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The function $q : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$ is called the symbol of the pseudo-differential operator. It is measurable and locally bounded in (x, ξ) and continuous negative definite as a function of ξ.

Theorem (continued)

In particular, the function $\xi \longmapsto q(x,\xi)$ enjoys the Lévy-Khintchine representation

$$\begin{aligned} q(x,\xi) =& a(x) - i\langle \xi, b(x)\rangle + \frac{1}{2}\langle \xi, C(x)\xi\rangle \\ & - \int_{\mathbb{R}^d} \left(e^{i\langle \xi, y\rangle} - 1 - i\langle \xi, y\rangle 1_{B_1(0)}(y)\right) \nu(x, dy). \end{aligned}$$

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Further, we also assume:

(C2) $||q(\cdot,\xi)||_\infty \le c(1+|\xi|^2)$ for some $c \ge 0$,

(C3) $q(x,0) = a(x) = 0$ for all $x \in \mathbb{R}^d$.

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A Feller process satisfying **(C1)**, **(C2)** and **(C3)** is called a Lévy-type process.

Definition

An $\mathbb{R}^d$-valued, $d \geq 1$, Markov process $(\{M_t\}_{t\geq 0}, \{\mathbb{P}^x\}_{x\in\mathbb{R}^d})$ is called

- irreducible if there exists a σ-finite measure φ on $\mathcal{B}(\mathbb{R}^d)$ such that

$$\varphi(B) > 0 \Longrightarrow \mathbb{E}^x\left[\int_0^\infty 1_B(M_t)dt\right] > 0 \quad \text{for all} \quad x \in \mathbb{R}^d.$$

Transience and Recurrence of Lévy-type Processes

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- transient if it is irreducible and if there exists a countable covering $\{B_j\}_{j\in\mathbb{N}} \subseteq \mathcal{B}(\mathbb{R}^d)$ of $\mathbb{R}^d$ such that for each j there is a finite constant $C_j > 0$ such that

$$\mathbb{E}^x\left[\int_0^\infty 1_{B_j}(M_t)dt\right] \leq C_j \quad \text{for all} \quad x \in \mathbb{R}^d.$$

Definition (continued)

- recurrent if it is irreducible and if

$$\varphi(B) > 0 \Longrightarrow \mathbb{E}^x\left[\int_0^\infty 1_B(M_t)dt\right] = \infty \quad \text{for all} \quad x \in \mathbb{R}^d.$$

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Theorem (Tweedie 1994)

Every irreducible Markov process is either transient or recurrent.

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Theorem (Tweedie 1994)

Every irreducible Markov process is either transient or recurrent.

In the sequel, we consider only the so-called open-set irreducible Markov processes, i.e., Markov processes which admit an irreducibility measure with full support.

Transience and Recurrence of Lévy-type Processes

Theorem

Let $\{F_t\}_{t\geq 0}$ be an open-set irreducible Lévy-type process. Then, the following properties are equivalent:

(i) $\{F_t\}_{t\geq 0}$ is transient;

(ii) for some $x \in \mathbb{R}^d$ and some open neighborhood $O_x \subseteq \mathbb{R}^d$ of x,

$$\mathbb{E}^x\left[\int_0^\infty 1_{O_x}(F_t)dt\right] < \infty;$$

(iii) for some (all) $x \in \mathbb{R}^d$ and some (all) open bounded set $O \subseteq \mathbb{R}^d$,

$$\mathbb{P}^x\left(\int_0^\infty 1_O(F_t)dt = \infty\right) = 1;$$

(iv) for some (all) $x \in \mathbb{R}^d$,

$$\mathbb{P}^x\left(\lim_{t\longrightarrow\infty} |F_t| = \infty\right) 1.$$

Theorem (continued)

Analogously, the following properties are equivalent:

(i) $\{F_t\}_{t\geq 0}$ is recurrent;

(ii) for some (all) $x \in \mathbb{R}^d$,

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for all open neighborhoods $O_x \subseteq \mathbb{R}^d$ of x;

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$$\mathbb{P}^x\left(\int_0^\infty 1_O(F_t)dt = \infty\right) = 1;$$

(iv) for some (all) $x \in \mathbb{R}^d$,

$$\mathbb{P}^x\left(\liminf_{t\longrightarrow\infty} |F_t - x| = 0\right) = 1.$$

Idea of the proof relies on the following computation. For $x \in \mathbb{R}^d$ and $f \in L^1(\mathbb{R}^d)$, $f \geq 0$, we have

$$\begin{aligned}\mathbb{E}^x\left[\int_0^\infty f(F_t)dt\right] &= \lim_{q\longrightarrow 0}\mathbb{E}^x\left[\int_0^\infty e^{-qt}f(F_t)dt\right]\\ &= \lim_{q\longrightarrow 0}\int_0^\infty\int_{\mathbb{R}^d} e^{-qt+i\langle\xi,x\rangle}\hat{f}(\xi)\mathbb{E}^x\left[e^{i\langle\xi,F_t-x\rangle}\right]d\xi dt.\end{aligned}$$

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$$= \lim_{q \longrightarrow 0} \int_0^\infty \int_{\mathbb{R}^d} e^{-qt + i\langle \xi, x \rangle} \hat{f}(\xi) \mathbb{E}^x \left[e^{i\langle \xi, F_t - x \rangle} \right] d\xi dt.$$

We conclude:

- we need exponential lower and upper bound, in terms of the symbol, for $\mathbb{E}^x \left[e^{i\langle \xi, F_t - x \rangle} \right]$
- in the case of recurrence, the above computation has to be "x-independent"
- $\{F_t\}_{t \geq 0}$ has to be open-set irreducible.

Theorem (Schilling/Wang 2013)

Let $\{F_t\}_{t\geq 0}$ be a Lévy-type process with symbol $q(x,\xi)$ satisfying

$$\sup_{x\in\mathbb{R}^d} |\mathrm{Im}\, q(x,\xi)| \leq c \inf_{x\in\mathbb{R}^d} \mathrm{Re}\, q(x,\xi), \quad \xi \in \mathbb{R}^d,$$

for some $0 \leq c < 1$. Then,

$$\sup_{x\in\mathbb{R}^d} \left| \mathbb{E}^x \left[e^{i\langle \xi, F_t - x\rangle} \right] \right| \leq \exp\left[-\frac{t}{16} \inf_{z\in\mathbb{R}^d} \mathrm{Re}\, q(z, 2\xi) \right], \quad t \geq 0,\ \xi \in \mathbb{R}^d.$$

Theorem

Let $\{F_t\}_{t\geq 0}$ be a Lévy-type process with symbol $q(x,\xi)$. Then, for any $\xi \in \mathbb{R}^d$,

$$\inf_{x\in\mathbb{R}^d} \mathrm{Re}\,\mathbb{E}^x\left[e^{i\langle \xi, F_t - x\rangle}\right] \geq \exp\left[-4t\sup_{z\in\mathbb{R}^d}|q(z,\xi)|\right], \quad t\in[0,t(\xi)],$$

where $t(\xi) := \frac{\ln 2}{4\sup_{x\in\mathbb{R}^d}|q(x,\xi)|}$.

Transience and Recurrence of Lévy-type Processes

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Theorem (Schilling/Wang 2013)

Let $\{F_t\}_{t\geq 0}$ be an open-set irreducible Lévy-type process with symbol $q(x,\xi)$ satisfying the sector condition. Then, $\{F_t\}_{t\geq 0}$ is transient if

$$\int_{B_a(0)} \frac{d\xi}{\inf_{x\in\mathbb{R}^d} \mathrm{Re}\, q(x,\xi)} < \infty \quad \text{for some } a > 0. \tag{1}$$

Theorem

Let $\{F_t\}_{t\geq 0}$ be an open-set irreducible Lévy-type process with symbol $q(x,\xi)$, satisfying

$$\liminf_{\alpha\longrightarrow 0}\int_{\mathbb{R}^d}\left(\int_{t(\xi)}^{\infty}e^{-\alpha t}\mathrm{Re}\,\mathbb{E}^0[e^{i\langle\xi,F_t\rangle}]dt\right)\frac{\sin^2\left(\frac{a\xi_1}{2}\right)}{\xi_1^2}\cdots\frac{\sin^2\left(\frac{a\xi_d}{2}\right)}{\xi_d^2}d\xi>-\infty$$

for all $a>0$ small enough. Then, $\{F_t\}_{t\geq 0}$ is recurrent if

$$\int_{B_a(0)}\frac{d\xi}{\sup_{x\in\mathbb{R}^d}|q(x,\xi)|}=\infty \quad \text{for some } a>0. \tag{2}$$

Proposition

Let $\{F_t\}_{t\geq 0}$ be a Lévy-type process with generator $(\mathcal{A}, \mathcal{D}_{\mathcal{A}})$ and symbol $q(x,\xi)$, satisfying:

(i) $C_c^\infty(\mathbb{R}^d)$ is an operator core for $(\mathcal{A}, \mathcal{D}_{\mathcal{A}})$;

(ii) $q(x,\xi) = q(-x,-\xi)$ for all $x, \xi \in \mathbb{R}^d$;

(iii) the sector condition holds and $\int_{\mathbb{R}^d} \exp\left[-t \inf_{x\in\mathbb{R}^d} \mathrm{Re}\, q(x,\xi)\right] d\xi < \infty$ for all $t > 0$;

(iv) the corresponding transition density function (which exists due to (iii)) satisfies

$$p(t,0,0) = \sup_{y\in\mathbb{R}^d} p(t,0,y), \quad t > 0.$$

Then, $\mathbb{E}^0[e^{i\langle\xi,F_t\rangle}] \geq 0$ for all $t \geq 0$ and all $\xi \in \mathbb{R}^d$.

Theorem

If $d \geq 3$ and

$$\liminf_{|\xi|\longrightarrow 0} \frac{\inf_{x\in\mathbb{R}^d}\left(\langle\xi, C(x)\xi\rangle + \int_{\{|y|\leq\frac{\pi}{2|\xi|}\}}\langle\xi, y\rangle^2\nu(x, dy)\right)}{|\xi|^2} > 0,$$

then $q(x,\xi)$ satisfies (1).

Transience and Recurrence of Lévy-type Processes

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then $q(x,\xi)$ satisfies (1). If $d = 1, 2$, $q(x,\xi) = \mathrm{Re}\, q(x,\xi)$ for all $x, \xi \in \mathbb{R}^d$ and

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|y|^2\nu(x,dy) < \infty,$$

then $q(x,\xi)$ satisfies (2).

Theorem

Let $\{F_t\}_{t\geq 0}$ be a Feller-Dynkin diffusion with symbol given by

$$q(x,\xi) = \frac{1}{2}\langle \xi, C(x)\xi\rangle.$$

Assume that there exists $c > 0$ such that

$$\inf_{x\in\mathbb{R}^d}\langle \xi, C(x)\xi\rangle \geq c|\xi|^2, \quad \xi \in \mathbb{R}^d.$$

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- If $d = 1, 2$, then $q(x,\xi)$ satisfies (2).

Transience and Recurrence of Lévy-type Processes

Theorem

Let $\{F_t\}_{t\geq 0}$ be a stable-like process with symbol given by

$$q(x,\xi) = \gamma(x)|\xi|^{\alpha(x)},$$

where $\alpha : \mathbb{R}^d \longrightarrow (0,2)$ and $\gamma : \mathbb{R}^d \longrightarrow (0,\infty)$, $\alpha, \gamma \in C_b^1(\mathbb{R}^d)$, are such that

$$0 < \inf_{x\in\mathbb{R}^d} \alpha(x) \leq \sup_{x\in\mathbb{R}^d} \alpha(x) < 2 \quad \text{and} \quad \inf_{x\in\mathbb{R}^d} \gamma(x) > 0.$$

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- If $d \geq 2$, then $q(x,\xi)$ satisfies (1).
- If $d = 1$ and $\limsup_{|x|\longrightarrow\infty} \alpha(x) < 1$, then $q(x,\xi)$ satisfies (1).

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- If $d \geq 2$, then $q(x,\xi)$ satisfies (1).
- If $d = 1$ and $\limsup_{|x|\longrightarrow\infty} \alpha(x) < 1$, then $q(x,\xi)$ satisfies (1).
- If $d = 1$ and $\liminf_{|x|\longrightarrow\infty} \alpha(x) \geq 1$, then $q(x,\xi)$ satisfies (2).

Transience and Recurrence of Lévy-type Processes

In the sequel, we assume $d = 1, 2$ and $\xi \longmapsto q(x, \xi)$ is radial for all $x \in \mathbb{R}^d$. Equivalently,

(i) $b(x) = 0$ for all $x \in \mathbb{R}^d$;

(ii) $C(x) = c(x)I$ for some Borel function $c : \mathbb{R}^d \longrightarrow [0, \infty)$;

(iii) $\nu(x, dy) = \nu(x, Rdy)$ for all $x \in \mathbb{R}^d$ and all rotations R of $\mathbb{R}^d$.

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In the sequel, we assume $d = 1, 2$ and $\xi \longmapsto q(x, \xi)$ is radial for all $x \in \mathbb{R}^d$. Equivalently,

(i) $b(x) = 0$ for all $x \in \mathbb{R}^d$;

(ii) $C(x) = c(x)I$ for some Borel function $c : \mathbb{R}^d \longrightarrow [0, \infty)$;

(iii) $\nu(x, dy) = \nu(x, Rdy)$ for all $x \in \mathbb{R}^d$ and all rotations R of $\mathbb{R}^d$.

Theorem

Let $\{F_t^1\}_{t\geq 0}$ and $\{F_t^2\}_{t\geq 0}$ be Lévy-type processes with symbols $q_1(x, \xi)$ and $q_2(x, \xi)$ and Lévy measures $\nu_1(x, dy)$ and $\nu_2(x, dy)$.

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$$\sup_{x\in\mathbb{R}^d} \int_0^\infty y^2 |\nu_1(x, dy) - \nu_2(Rx, dy)| < \infty,$$

then $q_1(x, \xi)$ satisfies (1) (resp. (2)) if, and only if, $q_2(x, \xi)$ satisfies (1) (resp. (2)).

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$$\int_{\rho}^{\infty}\left(r^{d-1}\sup_{x\in\mathbb{R}^d}\int_0^r y\nu(x,B_y^c(0))dy\right)^{-1}dr=\infty \tag{3}$$

implies (2),

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implies (2), and (1) implies

$$\int_\rho^\infty \left(r^{d-1} \inf_{x\in\mathbb{R}^d} \int_0^r y\nu(x, B_y^c(0))dy \right)^{-1} dr < \infty \tag{4}$$

for all $\rho > 0$ large enough.

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for all $\rho > 0$ large enough. In addition, if there exists $y_0 \geq 0$ such that $y \longmapsto \nu(x, B_y^c(0))$ is convex on (y_0, ∞), then (1) holds true if, and only if, (4) holds true, and (2) holds true if, and only if, (3) holds true.

Theorem

Under the assumptions of the previous theorem, $q(x,\xi)$ satisfies (1) if for some $\rho \geq y_0$,

$$\int_{\rho}^{\infty} \frac{dr}{r^{2d+1}\inf_{x\in\mathbb{R}} n(x,r)} < \infty,$$

where $n : \mathbb{R}\times\mathbb{R} \longrightarrow [0,\infty)$ is a Borel function such that $\nu(x,dy) = n(x,|y|)dy$ on $\mathcal{B}(B^c_{y_0}(0))$ and $y \longmapsto n(x,y)$ is nonincreasing on (y_0,∞) for all $x \in \mathbb{R}^d$.

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- $y \longmapsto \nu_1(x, B^c_y(0))$ is convex on (y_0,∞) for all $x \in \mathbb{R}^d$;
- $\nu_1(x, B^c_y(0)) \geq \nu_2(x, B^c_y(0))$ for all $y \geq y_0$ and all $x \in \mathbb{R}^d$.

Transience and Recurrence of Lévy-type Processes

Theorem (continued)

Then, for all $a > 0$ small enough,

$$\int_{B_a(0)} \frac{d\xi}{\inf_{x\in\mathbb{R}^d} q_2(x,\xi)} < \infty$$

implies

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Theorem (continued)

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and, for all $a > 0$,

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Ergodicity of Lévy-type Processes

Let $(\Omega, \mathcal{F}, \{\mathbb{P}^x\}_{x\in\mathbb{R}^d}, \{\mathcal{F}_t\}_{t\geq 0}, \{\theta_t\}_{t\geq 0}, \{M_t\}_{t\geq 0})$ be a Markov process on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

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$$\int_{\mathbb{R}^d} \mathbb{P}^x(M_t \in B)\pi(dx) = \pi(B), \quad t \geq 0,\ B \in \mathcal{B}(\mathbb{R}^d).$$

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A set $B \in \mathcal{F}$ is said to be shift-invariant if $\theta_t^{-1} B = B$ for all $t \geq 0$. The shift-invariant σ-algebra $\mathcal{I}$ is a collection of all such shift-invariant sets.

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Definition

A Markov proocess $\{M_t\}_{t\geq 0}$ is said to be ergodic if it possesses an invariant probability measure $\pi(dx)$ and if $\mathcal{I}$ is trivial with respect to $\mathbb{P}^\pi(d\omega)$.

Ergodicity of Lévy-type Processes

Definition

A Markov process $\{M_t\}_{t\geq 0}$ is said to be strongly ergodic if it possesses an invariant probability measure $\pi(dx)$ and if

$$\lim_{t\longrightarrow\infty} ||\mathbb{P}^x(M_t \in \cdot) - \pi(\cdot)||_{TV} = 0, \quad x \in \mathbb{R}^d.$$

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Theorem (Azéma/Kaplan-Duflo/Revuz 1967)

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Theorem (Azéma/Kaplan-Duflo/Revuz 1967)

Every recurrent Markov process possesses a unique (up to constant multiplies) invariant measure.

A recurrent Markov process which possesses a finite invariant measure is called positive recurrent, otherwise it is called null recurrent.

Ergodicity of Lévy-type Processes

Theorem

Let $\{F_t\}_{t\geq 0}$ be an open-set irreducible Lévy-type process. Then the following properties are equivalent:

- $\{F_t\}_{t\geq 0}$ is positive recurrent;
- $\{F_t\}_{t\geq 0}$ is ergodic;
- $\{F_t\}_{t\geq 0}$ is strongly ergodic.

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- $\{F_t\}_{t\geq 0}$ is strongly ergodic.

Theorem (Behme/Schnurr 2014)

Let $\{F_t\}_{t\geq 0}$ be a Lévy-type processes with generator $(\mathcal{A}, \mathcal{D}_{\mathcal{A}})$ and symbol $q(x,\xi)$. Assume that $C_c^\infty(\mathbb{R}^d)$ is an operator core for $(\mathcal{A}, \mathcal{D}_{\mathcal{A}})$. Then, a probability measure $\pi(dx)$ on $\mathcal{B}(\mathbb{R}^d)$ is invariant for $\{F_t\}_{t\geq 0}$ if, and only if,

$$\int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} q(x,\xi)\pi(dx) = 0, \quad \xi \in \mathbb{R}^d.$$

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Thank you for your attention!