Stochastic Newton equation with absorbing area

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$X_t^\lambda, V_t^\lambda \in \mathbf{R}$: position and velocity of a particle at time t.

$$\begin{cases} dX_t^\lambda = V_t^\lambda dt \\ dV_t^\lambda = -bV_t^\lambda dt - \lambda \nabla g(X_t^\lambda)dt + \sigma(X_t^\lambda)dB_t, \\ (X_0^\lambda, V_0^\lambda) = (X_0, V_0), \end{cases} \tag{0.1}$$

with

$b > 0$: constant (coefficient of friction),

$\lambda \geq 1$: parameter (later, $\lambda \to \infty$),

$g \in C_0^\infty(\mathbf{R}; \mathbf{R})$: potential function,

$\sigma \in C^\infty(\mathbf{R}; \mathbf{R})$, positive uniformly

(X_0, V_0): initial condition, $|X_0|$ big enough.

"Hamiltonian $H(x, v) = \frac{1}{2}|v|^2 + \lambda g(x)$" + friction + randomized

Problem: Behavior of the particle as $\lambda \to \infty$.

Motivation (reason for $\lambda \to \infty$): Mechanical model of Brownian motion. Massive particle(s) in an ideal gas environment.

The Newtonian system with Hamiltonian: $H = \sum_{i=1}^{N} \frac{1}{2} M_i |V_i|^2 + \sum_{(x,v)} \frac{1}{2} m |v|^2 + \sum_{i=1}^{N} \sum_{(x,v)} U_i(X_i - x)$.

equivalently, (totally deterministic as long as the initial condition is given)

$$
\begin{cases}
\frac{d}{dt} X_i(t,\omega) = V_i(t,\omega), \quad i = 1, \cdots, N, \\
M_i \frac{d}{dt} V_i(t,\omega) = - \int_{\mathbf{R}^d \times \mathbf{R}^d} \mu_\omega(dx, dv) \\
\qquad \cdot \nabla U_i(X_i(t,\omega) - x(t,x,v,\omega)), \\
(X_i(0,\omega), V_i(0,\omega)) = (X_{i,0}, V_{i,0}), \\
\frac{d}{dt} x(t,x,v,\omega) = v(t,x,v,\omega), \\
m \frac{d}{dt} v(t,x,v,\omega) = - \sum_{i=1}^{N} \nabla U_i(x(t,x,v,\omega) - X_i(t,\omega)), \\
(x(0,x,v,\omega), v(0,x,v,\omega)) = (x,v).
\end{cases}
$$

with initial condition (x, v): Poisson point process, v and the density $\sim m^{-1/2}$,

Aim: Behavior of the massive particle(s) when $m \to 0$.

$P_m(d\omega)$: (probability on $Conf(\mathbf{R}^d \times \mathbf{R}^d)$),
the Poisson point process with intensity λ_m,

$$\lambda_m(dx, dv) = m^{\frac{d-1}{2}} \rho\Big(\frac{m}{2}|v|^2 + \sum_{i=1}^N U_i(x - X_{i,0})\Big) dx dv,$$

then (Kusuoka-L. (2010))

$$M_i V_i(t \wedge \sigma)$$

$$\approx \text{initial} + \text{martingale} + \text{differentiable term}$$

$$-m^{-1/2} \int_0^{t\wedge\sigma} \nabla_i \widetilde{U}(\vec{X}(s)) ds,$$

with |the jump of the martingale term| $\leq C m^{1/2}$.

$\widetilde{U}$: new potential for $\vec{X}$, (with no light particles as mediates)

Let $r_3 > 0$: $g(x) = 0(\forall |x| \geq r_3)$.

$\rightarrow$ In the domain $|x| > r_3$, λ has no effect

$\rightarrow$ In the limit $\lambda \rightarrow \infty$, we get the same SDE:

$$\begin{cases} dX_t^\lambda = V_t^\lambda dt \\ dV_t^\lambda = -bV_t^\lambda dt + \sigma(X_t^\lambda) dB_t, \end{cases}$$

The behavior at $|X_t| = r_3$ (and $V_t \cdot X_t < 0$):

(i) if $\exists \varepsilon_0 > 0$ s.t., $g(x) > 0$ for $|x| \in (r_3 - \varepsilon_0, r_3)$

$\rightarrow$ g gives a repulsion

$\rightarrow$ after $\lambda \rightarrow \infty$, we get a reflective diffusion

$\rightarrow$ Kusuoka (2004)

(ii) In this talk: $g(x) < 0$ right after entered $|x| \leq r_3$:

$\rightarrow$ g gives an attraction

$\rightarrow$ after $\lambda \rightarrow \infty$, $|V_t|$ becomes ∞.

$\exists r_1 \in (0, r_3)$ s.t.,

$g(x) < 0 (x \in (r_1, r_3))$,

$g(r_1) = 0$,

$g'(r_1) < 0$.

Related result: L. (2013):

the same potential (i.e., attracting) but with relative efficacy $\longrightarrow$ resulting in a stochastic process with two phases: a diffusion phase (for $|X_t| \geq r_3$) and a uniform motion phase (for $|X_t| \in (r_1, r_3)$).

In this talk: without relative efficacy

(In the limit $\lambda \to \infty$),

- V_t becomes ∞ right after it enters the domain $|X_t| \in (r_1, r_3)$,
- it could then never leave this domain.

  Indeed, let $H_t^\lambda := \frac{1}{2}|V_t^\lambda|^2 + \lambda g(X_t^\lambda)$, then by Ito's formula,

$$dH_t^\lambda = -b|V_t^\lambda|^2 dt + V_t^\lambda \sigma(X_t^\lambda) dt + \frac{1}{2}\sigma^2(X_t^\lambda) dt,$$

  hence H_t^λ becomes negative before its first hitting time to r_1.

So it is meaningless to consider the limit behavior of X_t^λ itself.

For any $f \in C_b(\mathbf{R})$, let

$$Y_t^{f,\lambda} := \int_0^t f(X_s^\lambda)ds$$

(Since f is bounded, $\Big\{$ the distribution of $\Big\{Y_t^{f,\lambda}; t \geq 0\Big\}; \lambda \geq 1\Big\}$ is tight).

Aim: The limit of the distribution of $\{Y_t^{f,\lambda}; t \in [0, \infty)\}$ as $\lambda \to \infty$.

W.l.o.g., let $X_0 = r_3$ and $V_0 < 0$.

A related known result (Sugiyama): $g(x) = ax^2$ $(a > 0)$, i.e.,

$$\begin{cases} dX_t^\lambda = V_t^\lambda dt \\ dV_t^\lambda = -bV_t^\lambda dt - 2\lambda a X_t^\lambda dt + \sigma(X_t^\lambda) dB_t, \\ (X_0^\lambda, V_0^\lambda) = (x_0, v_0), \end{cases}$$

Then

$$X_t^\lambda = \cdots \quad \text{, (3 lines)} \qquad V_t^\lambda = \cdots \quad \text{, (5 lines)}$$

so for any $T > 0$ and $f \in C_b^1(\mathbf{R})$,

$$\lim_{\lambda \to \infty} E\Big[\Big| \int_0^T f(X_t^\lambda) dt$$
$$-2 \int_{-x_0}^{x_0} \int_0^{\frac{\log x_0 - \log |x|}{b} \wedge \frac{T}{2}} f(x) h(u, x) du dx \Big|\Big] = 0,$$

here

$$h(u, x) = \frac{1}{\pi \sqrt{x_0^2 e^{-2bu} - x^2}}.$$

Consider the non-random case:

$$\begin{cases} dx_t^\lambda = v_t^\lambda dt, \\ dv_t^\lambda = -bv_t^\lambda dt - \lambda \nabla g(x_t^\lambda) dt, \\ (x_0^\lambda, v_0^\lambda) = (r_3, v), \end{cases}$$

with $v < 0$.
Same as the random case, the particle could never leave (r_1, r_3).

Problem: The range of the particle.

Let

$$h_t^\lambda := \frac{1}{2}|v_t^\lambda|^2 + \lambda g(x_t^\lambda), \qquad j_t^\lambda := \lambda^{-1} h_t^\lambda.$$

Then $j_t := \lim_{\lambda \to \infty} j_t^\lambda$ gives us the range of the particle (of the limit process) around time t.

Assume that g in (r_1, r_3) is single-well, i.e., $\exists r_2 \in (r_1, r_3)$, s.t.,

- $g(x)\Big|_{x\in(r_1,r_2)}$ is strictly decreasing (write the inverse: $g^{-1,1}$),
- $g(x)\Big|_{x\in(r_2,r_3)}$ is strictly increasing (write the inverse: $g^{-1,2}$)

+ some technical condition.

Then

- $j_t < 0$ for any $t > 0$,
- $dj_t = -2b\Big(j_t - A^g g(j_t)\Big)dt$, $j_0 = 0$, with

$$A^g f(x) := \frac{S_f(j)}{S_1(j)}, \qquad x \in (-\|g\|_\infty, 0),$$

$$S_f(j) := S_f^g(j) := \sqrt{2}\int_{g^{-1,1}(j)}^{g^{-1,2}(j)} \frac{f(y)}{\sqrt{j - g(y)}}\, dy.$$

Idea: balance of "time for each trip" and "decay of energy during each trip"

Come back to the random case.

Let

$$J_t^\lambda := \lambda^{-1} H_t^\lambda = \frac{1}{2}\lambda^{-1}|V_t^\lambda|^2 + g(X_t^\lambda).$$

($J_t := \lim_{\lambda\to\infty} J_t^\lambda$ gives us the range of the particle (of the limit process) around time t).

Theorem. Under the above assumptions, for any $f \in C_b(\mathbf{R})$, we have that when $\lambda \to \infty$, $\left\{(J_t^\lambda, Y_t^{f,\lambda}); t \in [0,\infty)\right\}$ converge to $\{(j_t, \int_0^t A^g f(j_s)ds); t \in [0,\infty)\}$ weakly in $(W, dist)$.

Here $W = C([0,\infty); \mathbf{R}^2)$ and for $\forall w_1, w_2 \in W$,

$dist(w_1, w_2) = \sum_{n=1}^\infty 2^{-n}\left(1 \wedge \left[\max_{t\in[0,n]} |w_1(t) - w_2(t)|\right]\right)$.

Remark: Non-random limit ONLY for $d = 1$

For $d \geq 2$: same limit for $|X_t|$, but random limit for the direction

Open problem. In our (motivating) mechanical model of Brownian Motion: Is b negative-definite?

The corresponding "limit" generator

$L_1 = \frac{1}{2}\sum_{k_1,k_2=1}^{N}\sum_{l_1,l_2=1}^{d} a_{k_1 l_1, k_2 l_2}(\vec{X})\frac{\partial^2}{\partial V_{k_1}^{l_1}\partial V_{k_2}^{l_2}} +$
$\sum_{k_1,k_2=1}^{N}\sum_{l_1,l_2=1}^{d} b_{k_1 l_1, k_2 l_2}(\vec{X}) V_{k_2}^{l_2}\frac{\partial}{\partial V_{k_1}^{l_1}} +$
$\sum_{k=1}^{N}\sum_{i=1}^{d} V_k^i \frac{\partial}{\partial X_k^i},$

with

$$\int_E \Big(\int_{-\infty}^{\infty} \nabla^2 U_i(\psi^0(t,x,v,\vec{X}) - X_i) z(t,x,v,\vec{X},\vec{V},-t)dt\Big)$$

$$\times\rho(\frac{1}{2}|v|^2)\nu(dx,dv) = \sum_{\ell=1}^{d}\sum_{j=1}^{N} b_{i\cdot;j\ell}(\vec{X})V_j^{\ell}.$$

$$E = \{(x,v)\in \mathrm{R}^d \times (\mathrm{R}^d\setminus\{0\});\ x\cdot v = 0\},$$

$z(t; x, v, \vec{X}, \vec{V}, a) \in \mathbb{R}^d$ denotes the solution of the following standard differential equation.

$$\begin{cases} \frac{d^2}{dt^2} Z(t) = -\sum_{i=1}^N \nabla^2 U_i(\psi^0(t, x, v, \vec{X}) - X_i)(Z(t) - (t+a)V_i), \\ \lim_{t \to -\infty} Z(t) = \lim_{t \to -\infty} \frac{d}{dt} Z(t) = 0. \end{cases}$$

(we have that $z(t; x, v, \vec{X}, \vec{V}, a)$ is a linear function of $\vec{V}$).

$$\psi(t, x, v; \vec{X}) := \lim_{s \to \infty} \varphi(t+s, x - sv, v; \vec{X}),$$

$$\begin{cases} \dfrac{d}{dt} \varphi^0(t, x, v; \vec{X}) = \varphi^1(t, x, v; \vec{X}) \\ \dfrac{d}{dt} \varphi^1(t, x, v; \vec{X}) = -\displaystyle\sum_{i=1}^N \nabla U_i(\varphi^0(t, x, v; \vec{X}) - X_i) \\ (\varphi^0(0, x, v; \vec{X}), \varphi^1(0, x, v; \vec{X})) = (x, v). \end{cases}$$

An important estimate for the proof: $\exists C > 0$, s.t.,

$$E\Big[\sup_{t\in[0,T]} |V_t^{\lambda}|^4\Big]^{1/4} \leq C\lambda^{\frac{1}{2}}, \qquad \lambda \geq 1.$$

Proof. By Ito's formula,

$dH_t^{\lambda} = -b|V_t^{\lambda}|^2 dt + V_t^{\lambda}\sigma(X_t^{\lambda})dB_t + \frac{1}{2}\sigma(X_t^{\lambda})^2 dt$, so

$$\begin{aligned}
|V_t^{\lambda}|^2 &= 2H_t^{\lambda} - 2\lambda g(X_t^{\lambda}) \\
&\leq 2H_0 + 2\int_0^t V_s^{\lambda}\sigma(X_s^{\lambda})dB_s + T\|\sigma\|_{\infty}^2 + 2\lambda\|g\|_{\infty}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& E\Big[\sup_{t\in[0,T]} |V_t^{\lambda}|^4\Big] \\
\leq\ & 2\Big(C_1 + C_2\lambda\Big)^2 + 2E\Big[\sup_{t\in[0,T]} \Big(2\int_0^t V_s^{\lambda}\sigma(X_s^{\lambda})dB_s\Big)^2\Big] \\
\leq\ & 2\Big(C_1 + C_2\lambda\Big)^2 + 32\|\sigma\|_{\infty}^2 T E\Big[\sup_{t\in[0,T]} |V_t^{\lambda}|^4\Big]^{1/2}.
\end{aligned}$$

In general, for any $c_1, c_2 \in \mathbf{R}^+$, we have that

$$x^2 \leq c_1 + c_2 x \Rightarrow x \leq \frac{c_2 + \sqrt{c_2^2 + 4c_1}}{2} \leq c_2 + \sqrt{c_1}.$$

Therefore, we get that

$$\begin{aligned} & E\Big[\sup_{t \in [0,T]} |V_t^{\lambda}|^4 \Big]^{1/2} \\ \leq \quad & 32 \|\sigma\|_{\infty}^2 T + \sqrt{2\Big(C_1 + C_2\lambda\Big)^2} \\ = \quad & 32 \|\sigma\|_{\infty}^2 T + \sqrt{2} C_1' + C_2'\lambda \end{aligned}$$

for any $\lambda \geq 1$.

Q.E.D.

Main idea for the dealing with randomness:

- For any $\{M(t)\}_{t\geq 0}$: continuous martingale, $\exists\{W(t)\}_{t\geq 0}$: BM, s.t., $M(t) = W\left(\langle M, M\rangle_t\right)$.

- For any standard BM $\left\{\overline{B}_t\right\}_{t\geq 0}$, we have the following:

  (i) $\lim_{a\to\infty} P\left(\left\{\inf_{u\geq 0}(\varepsilon u + \overline{B}_t) < -a\right\}\right) = 0$ for any $\varepsilon > 0$,

  (ii) $\lim_{a\to\infty} P\left(\left\{\overline{B}_s - \varepsilon s \geq 0 \text{ for some } s \geq a\right\}\right) = 0$ for any $\varepsilon > 0$,

  (iii) $P\left(\limsup_{\varepsilon\to 0}\left\{\sup_{0\leq s\leq t+s\leq T, t\leq\varepsilon} \frac{|\overline{B}_{t+s} - \overline{B}_s|}{\sqrt{2\varepsilon \log c1\varepsilon}}\right\} = 1\right) = 1.$

Thank you!