Markov processes with jumps and nonlocal generators

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Feller semigroups $(T_t)_{t\geq 0}$ in $\mathbb{R}^n$

$$T_t : C_\infty(\mathbb{R}^n) \to C_\infty(\mathbb{R}^n), \qquad C_\infty(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n), \lim_{|x|\to\infty} f(x) = 0\}$$

s.t.

$$T_s \circ T_t = T_{s+t}$$
$$T_t u \to u \text{ as } t \to 0$$
$$0 \leq u \leq 1 \;\Rightarrow\; 0 \leq T_t u \leq 1$$

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$$\begin{aligned} &T_s \circ T_t = T_{s+t} \\ &T_t u \to u \text{ as } t \to 0 \\ &0 \leq u \leq 1 \;\Rightarrow\; 0 \leq T_t u \leq 1 \end{aligned}$$

Then $(A, D(A))$ generator of Feller semigroup iff

(i) $D(A) \subset C_\infty(\mathbb{R}^n)$ dense

(ii) $R(A - \lambda) = C_\infty(\mathbb{R}^n)$ for some $\lambda > 0$

(iii) A satisfies positive maximum principle:

$$\sup_{x\in\mathbb{R}^n} u(x) = u(x_0) \geq 0 \;\Rightarrow\; A(x_0) \leq 0$$

Theorem (Courrège): Assume $C_0^\infty(\mathbb{R}^n) \subset D(A)$. If A satisfies positive maximum principle, then A is a Lévy-type operator

$$Au(x) =$$

$$\text{2nd order diffusion operator} + \int_{\mathbb{R}^n \setminus \{0\}} \left(u(x+y) - u(x) - \frac{\langle y, \nabla u(x) \rangle}{1+|y|^2}\right) \mu(x, dy)$$

where $\mu(x, dy)$ kernel of Lévy measures: $\displaystyle\int_{\mathbb{R}^n \setminus \{0\}} |y|^2 \wedge 1 \, \mu(x, dy) < \infty$

Lévy-Khinchin formula

Equivalently: $A = -q(x, D)$ is a **pseudo differential operator**

$$Au(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x,\xi)\hat{u}(\xi)\, d\xi$$

where the symbol

$$q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$$

s.t. $\xi \mapsto q(x,\xi)$ is a continuous negative definite function.

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s.t. $\xi \mapsto q(x,\xi)$ is a continuous negative definite function. Here

$$q(x,\xi) = Q(x,\xi) + ib(x)\cdot\xi + c(x) + \int_{\mathbb{R}^n\setminus\{0\}} (1 - e^{iy\cdot\xi} + \frac{iy\cdot\xi}{1+|y|^2})\, \mu(x,dy)$$

where $Q(x,\cdot) \geq 0$ quadr. form, $b(x) \in \mathbb{R}^d$, $c(x) \geq 0$, $\mu(x,\cdot)$ kernel of Lévy measures

Translation invariant case

$$q(x,\xi) = \psi(\xi)$$

continuous negative definite function, independent of x.

In particular assume

- $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$
- ψ has no local part

$$\Rightarrow \qquad \psi(\xi) = \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(y \cdot \xi))\, \mu(dy)$$

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Corresponding semigroup (T_t) given by:
Convolution semigroup $(\mu_t)_{t \geq 0}$ of symmetric probability measures :

$$\mu_s * \mu_t = \mu_{s+t}, \qquad s, t, \geq 0$$
$$\mu_t \longrightarrow \mu_0 = \varepsilon_0 \text{ as } t \longrightarrow 0 \text{ vaguely}$$

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It follows: $(T_t)_{t\geq 0}$ extends to a contraction semigroup

$$(T_t^{(p)})_{t\geq 0} \text{ on } L^p(\mathbb{R}^n) \qquad \text{for } 1 \leq p < \infty$$

strongly continuous and sub-Markovian

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as well as to a Feller semigroup $(T_t^{(\infty)})_{t\geq 0}$ on $C_\infty(\mathbb{R}^n)$

Idea: Use L^p-theory to get better regularity results and embeddings for the nonlocal generator / semigroup

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More precisely use (r, p)-capacities: Assume $(T_t^{(p)})_{t\geq 0}$ is a sub-Markovian strongly cont. contraction semigroup on $L^p(\mathbb{R}^n)$ with generator $A^{(p)}$

Define Γ-transform of $(T_t^{(p)})$:

$$V_r^{(p)} u := \frac{1}{\Gamma(\frac{r}{2})} \int_0^\infty t^{r/2-1} e^{-t} T_t^{(p)} u, \qquad r \geq 0$$

$(V_r^{(p)})_{r\geq 0}$ is obtained from $(T_t^{(p)})_{t\geq 0}$ by subordination w.r.t the (modified) Γ-semigroup on $[0, \infty)$

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$\Rightarrow$ $(V_r^{(p)})_{r\geq 0}$ also sub-Markovian, strongly cont. contraction semigroup on $L^p(\mathbb{R}^n)$

In particular $$V_r^{(p)} \circ V_s^{(p)} = V_{r+s}^{(p)}$$

Abstract Bessel potential space

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&= \frac{1}{2\pi i} \int_\gamma \zeta^{-r/2} ((\zeta + 1)\mathrm{id} - A^{(p)})^{-1} u \, d\zeta
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Abstract Bessel potential space:

$$\begin{aligned} \mathcal{F}_{r,p} &:= V_r^{(p)}(L^p(\mathbb{R}^n)) \\ \|u\|_{\mathcal{F}_{r,p}} &:= \|v\|_{L^p} \quad \text{for } u = V_r^{(p)}(v) \end{aligned}$$

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(r, p)-capacity:

$$\begin{aligned} \mathrm{cap}_{r,p}(G) &= \inf\{\|u\|_{\mathcal{F}_{r,p}} : \ u \in \mathcal{F}_{r,p}, \ u \geq 1 \text{ a.e. on } G\} \qquad G \text{ open} \\ \mathrm{cap}_{r,p}(A) &= \inf\{\mathrm{cap}_{r,p}(G) : \ A \subset G, \ G \text{ open}\} \end{aligned}$$

Properties of $\mathcal{F}_{r,p}$?

Translation invariant case

$$\psi : \mathbb{R}^n \to \mathbb{R} \text{ continuous negative definite function}$$

$\to$ convolution semigroups $(T_t^{(p)})_{t \geq 0}$

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ψ-Bessel potential space $H_p^{\psi,r}$, $1 < p < \infty$

$$\|u\|_{H_p^{\psi,r}} = \|F^{-1}((1+\psi(\cdot))^{r/2}\hat{u}(\cdot)\|_{L^p}, \qquad u \in \mathcal{S}(\mathbb{R}^n)$$

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For $H_p^{\psi,r}$ explicit embeddings are known.
In particular

$$C_0^\infty(\mathbb{R}^n) \subset H_p^{\psi,r} \text{ dense for all } r \geq 0$$

$\Rightarrow \mathcal{F}_{r,p} = H_p^{\psi,r}$ regular

x-dependent symbols $q(x,\xi)$

Model case

$$q(x,\xi) = \sum_{i=1}^{N} b_i(x) \cdot \psi_i(\xi)$$

where

$\psi_i : \mathbb{R}^n \to \mathbb{R}$ continuous negative definite functions

$b_i > 0$ coefficient functions

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Consider small perturbations of constant coefficient case

Fix $x_0 \in \mathbb{R}^n$

$$q(x,\xi) = \underbrace{q(x_0,\xi)}_{=:\psi(\xi)} + \underbrace{\sum_{i=1}^{N} (b_i(x) - b_i(x_0)) \cdot \psi_i(\xi)}_{=:q_1(x,\xi) \text{ perturbation}}$$

$$q(x,D) = \psi(D) + q_1(x,D)$$

$-\psi(D)$ generates strongly continuous, sub-Markovian contraction semigroup on $L^p(\mathbb{R}^n)$

Assume

$$(*) \qquad \sup_{x,i} |b_i(x) - b_i(x_0)| < \varepsilon$$

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Banuelos, Bogdan '07:

$$m(\xi) = \frac{\int_{\mathbb{R}^d\setminus\{0\}} (1 - \cos(y,\xi))\Phi(y)\,\mu(dy)}{\int_{\mathbb{R}^d\setminus\{0\}} (1 - \cos(y,\xi))\,\mu(dy)}$$

μ Lévy-measure, Φ even, $|\Phi| \leq 1$

Then for $1 < p < \infty$

$$\|m(D)f\|_{L^p} \leq (p^* - 1) \cdot \|f\|_{L^p}, \quad p^* - 1 = \max(p-1, \frac{1}{p-1})$$

Now

$$\left.\begin{array}{l}\Psi_i(\xi) = \int_{\mathbb{R}^d\setminus\{0\}}(1-\cos(y,\xi))\,\mu_i(dy) \\ \Psi(\xi) = \int_{\mathbb{R}^d\setminus\{0\}}(1-\cos(y,\xi))\,\underbrace{\sum_i b_i(x_0)\mu_i(dy)}_{=:\mu(dy)}\end{array}\right\} \begin{array}{l}\mu_i << \mu \\ \Rightarrow \mu_i = \Phi_i\cdot\mu, |\Phi_i| \le \frac{1}{c_1} \\ (c_1 = \min_i\{b_i(x_0)\})\end{array}$$

$$\Rightarrow \quad \left\|\frac{\Psi_i}{\Psi}(D)\right\|_{L^p\to L^p} \le \frac{1}{c_1}(p^*-1)$$

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$\Rightarrow$ for ε small

$$\|q_1(x,D)u\|_{L^p} < \frac{1}{2}\|\psi(D)u\|_{L^p} + c\|u\|_{L^p}$$

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$\Rightarrow$

$-q(x,D)$ generates strongly continuous contraction semigroup $(\tilde{T}_t)_{t\ge 0}$ on $L^p(\mathbb{R}^n)$

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Assume symmetry
for that purpose for $i = 1, \ldots, N$
(S) assume $\mathbb{R}^n = V_i^1 \oplus V_i^2$, s.t. $\begin{matrix} b_i : V_i^1 \to \mathbb{R} \\ \psi_i : V_i^2 \to \mathbb{R} \end{matrix}$

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$\Rightarrow$ $(\tilde{T}_t)_{t \geq 0}$ sub-Markovian

Bessel potential spaces

Consider Bessel potential spaces $\mathcal{F}^q_{r,p}$ corresponding to $-q(x,D)$

Theorem : Assume (*) with ε sufficiently small and (S). Then

$$\mathcal{F}^q_{r,p} = H^{\psi,r}_p$$

In particular $\mathcal{F}^q_{r,p}$ is regular.

Hence

- every $u \in \mathcal{F}^q_{r,p}$ admits an (r,p)-quasi-continuous modification, unique up to (r,p)-quasi everywhere equality
- $\mathcal{F}^q_{r,2}$, $r \leq 1$ has the contraction property

Sketch of proof

$\mathcal{F}^q_{r,p} = V^{(p)}_r(L^P(\mathbb{R}^n))$

First from the semigroup property of $V^{(p),q}_r$:

$$V^{(p),q}_r(\mathcal{F}^q_{s,p}) = V^{(p),q}_{r+s}(L^p(\mathbb{R}^n)) = \mathcal{F}^q_{r+s,p}$$

Suffices to consider $r < 2$

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Suffices to consider $r < 2$
Then

$$\begin{aligned} V^{(p),q}_r u &= ((id + q(x,D))^{-r/2} u \\ &= \frac{1}{2\pi i}\int_\gamma \zeta^{-r/2}((\zeta+1)\mathrm{id} + q(x,D))^{-1} u \, d\zeta \\ &= \frac{1}{2\pi i}\int_\gamma \zeta^{-r/2} R^q_{\zeta+1} u \, d\zeta \\ &= \frac{\sin\frac{r}{2}\pi}{\pi}\int_0^\infty \lambda^{-r/2} R^q_{\lambda+1} u \, d\lambda \end{aligned}$$

Calculate resolvent R_λ^q of perturbed operator $-q(x, D)$:
Solve

$$(\lambda + q(x, D))u = f$$

$$\begin{aligned}(\lambda + q(x, D))u &= (\lambda + \psi(D) + q_1(x, D)) \circ R_\lambda \circ (\lambda + \psi(D))u \\ &= (\mathrm{id} + q_1(x, D) \circ R_\lambda)(\lambda + \psi(D))u\end{aligned}$$

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Now

$$q_1(x, D) \circ R_\lambda = \sum_{i=1}^{N} (b_i(x) - b_i(x_0)) \cdot \psi_i(D) \circ R_\lambda$$

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We know

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$$\left\| \frac{\Psi_i}{\Psi}(D) \right\|_{L^p \to L^p} \leq c$$

Moreover

$$\left\| \frac{\Psi(D)}{\lambda + \Psi(D)} \right\|_{L^p \to L^p} = \left\| \mathrm{Id} - \frac{\lambda}{\lambda + \Psi(D)} \right\|_{L^p \to L^p} \leq 2$$

$\Rightarrow$ for ε sufficiently small: $\quad \|q_1(x,D)\circ R_\lambda\|_{L^p\to L^p} < 1$
$\Rightarrow$ $\mathrm{id} + q_1(x,D)\circ R_\lambda$ invertible in $L^p(\mathbb{R}^n)$

Hence $u = R_\lambda \circ (\mathrm{id} + q_1(x,D)\circ R_\lambda)^{-1} f$

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$\Rightarrow$

$$R_\lambda^q = R_\lambda \circ (\mathrm{id} + q_1(x,D)\circ R_\lambda)^{-1}$$

which gives

$$\begin{aligned} V_r^{(p),q} u &= \underbrace{\frac{\sin\frac{r}{2}\pi}{\pi}\int_0^\infty \lambda^{-r/2} R_{\lambda+1}}_{V_r^{(p)}} \circ (\mathrm{id} + q_1(x,D)\circ R_\lambda)^{-1} u \, d\lambda \\ &= V_r^{(p)} \circ (\mathrm{id} + q_1(x,D)\circ R_\lambda)^{-1} u \end{aligned}$$