W-entropy formulas and rigidity theorems on Wasserstein space over Riemannian manifolds

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German-Japanese Conference on Stochastic Analysis and Applications

Sendai, August 31-September 4, 2015

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Perelman's W-entropy

Let M be a compact manifold, $(g(t), f(t), \tau(t), t \in [0, T])$ be such that

$$\begin{aligned} \partial_t g &= -2Ric, \\ \partial_t f &= -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \\ \partial_t \tau &= -1. \end{aligned}$$

In 2002, Perelman introduced the W-entropy for the Ricci flow as

$$\mathcal{W}(g, f, \tau) = \int_M \left[\tau(R + |\nabla f|^2) + f - n\right] \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv,$$

and proved that

$$\frac{d}{dt}\mathcal{W}(g, f, \tau) = 2\tau \int_M \left|Ric + \nabla^2 f - \frac{g}{2\tau}\right|^2 \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv.$$

In particular, $\mathcal{W}(g, f, \tau)$ is nondecreasing in time and the monotonicity is strict unless that (M, g) is a shrinking Ricci soliton

$$Ric + \nabla^2 f = \frac{g}{2\tau}.$$

Ni's W-entropy formula for Laplace Beltrami

Recall Ni's W-entropy formula for the heat equation $\partial_t u = \Delta u$.

Theorem (Ni 2005)

Let (M, g) be a compact Riemannian manifold with a fixed metric. Let

$$u = \frac{e^{-f}}{(4\pi t)^{n/2}}$$

be a positive solution of

$$\partial_t u = \Delta u.$$

Let

$$W(u,t) = \int_M \left(t|\nabla f|^2 + f - n\right) \frac{e^{-f}}{(4\pi t)^{n/2}} dv.$$

Then

$$\frac{d}{dt} W(u,t) = -2 \int_M t \left(\left|\nabla^2 f - \frac{g}{2t}\right|^2 + Ric(\nabla f, \nabla f)\right) \frac{e^{-f}}{(4\pi t)^{n/2}} dv.$$

In particular, if $Ric \geq 0$, then $W(u,t)$ is decreasing in time t.

Outline

W-entropy for Witten Laplacian

Let M be a complete Riemannian manifold, $\phi \in C^2(M)$, $d\mu = e^{-\phi}dv$. The Witten Laplacian is defined by

$$L = \Delta - \nabla\phi \cdot \nabla.$$

For all $u, v \in C_0^\infty(M)$, we have

$$\int_M \langle \nabla u, \nabla v \rangle d\mu = -\int_M Luv d\mu = \int_M uLv d\mu.$$

The Bakry-Emery Ricci curvature associated with L is defined by

$$Ric(L) = Ric + \nabla^2\phi,$$

and the m-dimensional Bakry-Emery Ricci curvature associated with L is defined by

$$Ric_{m,n}(L) = Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m - n}.$$

Entropy for Witten Laplacian

Let u be a positive solution to the heat equation

$$\partial_t u = Lu.$$

Let

$$\mathrm{Ent}(u) = -\int_M u \log u d\mu.$$

Then, when M is compact or complete and with bounded geometry condition, it is well known that

$$\begin{aligned}
\frac{d}{dt}\mathrm{Ent}(u(t)) &= \int_M \frac{|\nabla u|^2}{u} d\mu, \\
\frac{d^2}{dt^2}\mathrm{Ent}(u(t)) &= -2\int_M [|\nabla^2 \log u|^2 + Ric(L)(\nabla \log u, \nabla \log u)] u d\mu.
\end{aligned}$$

Thus, if $Ric(L) \geq K$, then

$$\frac{d^2}{dt^2}\mathrm{Ent}(u(t)) \leq -2K \frac{d}{dt}\mathrm{Ent}(u(t)).$$

W-entropy formula for the Witten Laplacian

Let u be a positive solution of the heat equation $\partial_t u = Lu$. Let

$$H_m(u,t) := -\int_M u \log u d\mu - \frac{m}{2}\left(\log(4\pi t) + 1\right).$$

The Gaussian heat kernel on $\mathbb{R}^m$ is given by

$$u_m(x,t) = \frac{e^{\frac{-|x|^2}{4t}}}{(4\pi t)^{\frac{m}{2}}}$$

and its Boltzmann entropy is given by

$$H(u_m,t) = -\int_{\mathbb{R}^m} \log u_m(x) u_m(x) dx = \frac{m}{2}\left(\log(4\pi t) + 1\right).$$

Hence

$$H_m(u,t) = H(u,t) - H(u_m,t)$$

is the difference of the Boltzmann entropy for $\partial_t u = Lu$ on (M,μ) and the Boltzmann entropy for $\partial_t u = \Delta u$ on $(\mathbb{R}^m, dx)$. [Li2012]

Li-Yau Harnack inequality

Recall the Li-Yau Harnack inequality for Witten Laplacian.

Theorem (Li JMPA2005, Math Ann2012)

Let M be a complete Riemannian manifold with $Ric_{m,n}(L) \geq 0$. Let u be a positive solution to the heat equation

$$\partial_t u = Lu.$$

Then the Li-Yau Harnack inequality holds

$$\frac{|\nabla u|^2}{u^2} - \frac{Lu}{u} \leq \frac{m}{2t},$$

i.e.,

$$L \log u + \frac{m}{2t} \geq 0.$$

Thus, under the condition $Ric_{m,n}(L) \geq 0$,

$$\frac{d}{dt} H_m(u,t) = \int_M \left(\frac{|\nabla u|^2}{u^2} - \frac{n}{2t} \right) u d\mu = -\int_M \left(L \log u + \frac{m}{2t} \right) u d\mu \leq 0.$$

W-entropy for the Witten Laplacian

Theorem (Li Math Ann2012, S. Li-Li PJM2015)

Let M be a compact or complete Riemannian manifold with bounded geometry condition. Let $u = \frac{e^{-f}}{(4\pi t)^{m/2}}$ be a positive solution of $\partial_t u = Lu$. Define

$$W(u, t) := \frac{d}{dt}(tH_m(u, t)).$$

Then

$$W(u, t) = \int_M \left(t|\nabla f|^2 + f - m\right) \frac{e^{-f}}{(4\pi t)^{m/2}} d\mu,$$

and

$$\begin{aligned} \frac{dW(u, t)}{dt} &= -2\int_M \left(t\left|\nabla^2 f - \frac{g}{2t}\right|^2 + Ric_{m,n}(L)(\nabla f, \nabla f)\right) u d\mu \\ &\quad - \frac{2}{m-n}\int_M t\left(\nabla\phi \cdot \nabla f + \frac{m-n}{2t}\right)^2 u d\mu. \end{aligned}$$

Warped product approach to W-entropy formula

Let $\widetilde{M} = M \times N$. Define

$$\widetilde{g} = g_M \oplus e^{-\frac{\phi}{m-n}} g_N.$$

Applying Ni's W-entropy formula to the heat equation on $(\widetilde{M}, \widetilde{g})$

$$\partial_t u = \Delta_{\widetilde{M}} u,$$

S. Li and Li (PJM2015) gave a new proof of the W-entropy formula for the Witten Laplacian, and proved the following

Proposition (S. Li-Li, PJM2015)

$$\left| \widetilde{\nabla}^2 f - -\frac{\widetilde{g}}{2\tau} \right|^2 = \left| \nabla^2 f - \frac{g}{2\tau} \right|^2 + \frac{2}{m-n} \left(\nabla \phi \cdot \nabla f + \frac{m-n}{2\tau} \right)^2 .$$

This gives a natural geometric interpretation for (RHS) in the W-entropy formula of the Witten Laplacian using the warped product metric.

A rigidity theorem for Perelman's W-entropy

Note that, under the assumption $Ric_{m,n}(L) \geq 0$, we have

$$\frac{d\mathcal{W}}{dt} = 0 \iff \begin{cases} \nabla^2_{ij} f = \frac{g_{ij}}{2t}, & \forall i, j = 1, \ldots, n, \\ Ric_{m,n}(L)(\nabla f, \nabla f) = 0, \\ \nabla \phi \cdot \nabla f + \frac{m-n}{2t} = 0, \end{cases}$$

$$\Longrightarrow \begin{cases} Ric_{m,n}(L)(\log u, \log u) = 0, \\ L \log u + \frac{m}{2t} = 0. \end{cases}$$

This is the case when

$$M = \mathbb{R}^n, \quad m = n, \quad \phi(x) = C, \quad u(x,t) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}.$$

Question

Can we prove a rigidity theorem for the W-entropy under the condition $Ric_{m,n}(L) \geq 0$ on n-dimensional complete Riemannian manifolds?

A rigidity theorem for Perelman's W-entropy

The following result gives an affirmative answer to the above question.

Theorem (Li Math Ann2012)

Under the same condition as above theorem, $Ric_{m,n}(L) \geq 0$. Then

$$\exists t = t_0 > 0 \text{ such that } \frac{d\mathcal{W}}{dt} = 0,$$

if and only if for all $t > 0$, and $x \in M$,

$$M = \mathbb{R}^n, \quad m = n, \quad \phi(x) = C, \quad u(x,t) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}.$$

Open problem

The above results hold in the case of $CD(0, m)$. After I proved the above results in 2009, many people in probability community and in geometry community asked me the following

Problem

What happens in the case of $CD(K, m)$ or $CD(K, \infty)$?

Problem

What happens in the cae of time dependent metrics and potentials ?

Outline

Theorem (S. Li-L. 2014)

Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that there exists a constant $K \geq 0$ such that

$$Ric_{m,n}(L) \geq -K.$$

Let u be a positive solution of $\partial_t u = Lu$. Then the Li-Yau-Hamilton Harnack inequality holds

$$\frac{\partial_t u}{u} - e^{-2Kt}\frac{|\nabla u|^2}{u^2} + e^{2Kt}\frac{m}{2t} \geq 0.$$

In particular, if $K = 0$, i.e., $Ric_{m,n}(L) \geq 0$, then the Li-Yau Harnack inequality holds

$$\frac{\partial_t u}{u} - \frac{|\nabla u|^2}{u^2} + \frac{m}{2t} \geq 0.$$

W-entropy and Harnack inequality

Let

$$H_{m,K}(u,t) = \mathrm{Ent}(u(t)) - \mathrm{Ent}(u_{m,K}(t))$$

where $u_{m,K}(t)$ is the density of the Gaussian distribution $N(0, \sigma_K^2(t))$ on $\mathbb{R}^m$, i.e.,

$$u_{m,K}(t,x) = \frac{1}{(4\pi\sigma_K^2(t))^{m/2}} \exp\left(-\frac{\|x\|^2}{4\sigma_K^2(t)}\right).$$

Note that

$$\mathrm{Ent}(u_{m,K}(t)) = \frac{m}{2}\left(\log(4\pi\sigma_K^2(t)) + 1\right).$$

By direct calculation, we have

$$\frac{d}{dt}H_{m,K}(u,t) = \int_M \left[\frac{|\nabla u|^2}{u^2} - m\frac{d}{dt}\log\sigma_K(t)\right] u d\mu.$$

W-entropy and Harnack inequality

Suppose that we can prove the following Harnack inequality

$$\frac{|\nabla u|^2}{u^2} - \alpha_K(t)\frac{\partial_t u}{u} \leq m\beta_K(t).$$

Taking $\sigma_K(t) \in C([0, \infty), \mathbb{R})$ be such that

$$\frac{d}{dt}\log \sigma_K(t) = \beta_K(t).$$

Then

$$\frac{d}{dt}H_{m,K}(u, t) = \int_M \left[\frac{|\nabla u|^2}{u^2} - \alpha_K(t)\frac{\partial_t u}{u} - m\beta_K(t)\right] ud\mu \leq 0.$$

W-entropy and Harnack inequality

In the case $CD(-K, m)$ holds, the Hamilton's Harnack inequality

$$\frac{|\nabla u|^2}{u^2} - \alpha(t)\frac{\partial_t u}{u} \leq m\beta(t)$$

holds with

$$\alpha(t) = e^{2Kt}, \quad \beta(t) = \frac{e^{4Kt}}{2t}.$$

Thus, under $CD(-K, m)$, we have

$$\frac{d}{dt}H_{m,K}(u,t) = \int_M \left[\frac{|\nabla u|^2}{u^2} - \frac{m}{2t}e^{4Kt} - e^{2Kt}\frac{\partial_t u}{u}\right] u d\mu.$$

Proposition (S. Li-Li arxiv204)

Under the $CD(-K, m)$ condition, i.e., $Ric_{m,n}(L) \geq -K$, we have

$$\frac{d}{dt}H_{K,m}(u,t) \leq 0.$$

W-entropy formula for Hamilton's Harnack quantity

Theorem (S. Li-Li arxiv2014)

Define

$$W_{m,K}(u,t) = \frac{d}{dt}(tH_{m,K}(u,t)).$$

Under the bounded geometry condition, we have

$$\begin{aligned}\frac{d}{dt}W_{m,K}(u,t) \quad = \quad & -2t\int_M \left|\nabla^2 \log u + \left(\frac{K}{2}+\frac{1}{2t}\right) g\right|^2 u d\mu \\ & -2t\int_M (Ric_{m,n}(L)+Kg)(\nabla \log u, \nabla \log u) u d\mu \\ & -\frac{2t}{m-n}\int_M \left|\nabla\phi\cdot\nabla \log u - \frac{(m-n)(1+Kt)}{2t}\right|^2 u d\mu \\ & -\frac{m}{2t}\left[e^{4Kt}(1+4Kt)-(1+Kt)^2\right].\end{aligned}$$

Monotonicity and rigidity theorem

Theorem (S. Li-Li arxiv2014)

Assume that $Ric_{m,n}(L) \geq -K$, then for all $t \geq 0$,

$$\frac{d}{dt} W_{m,K}(u,t) \leq -\frac{m}{2t}\left[e^{4Kt}(1+4Kt) - (1+Kt)^2\right].$$

Moreover, the equality holds at some $t = t_0 > 0$ if and only if

$$\begin{aligned} Ric_{m,n}(L) &= -Kg, \\ 2\nabla^2 f &= \left(\frac{1}{t} + K\right) g, \\ \nabla\phi \cdot \nabla f &= -\frac{(m-n)(1+Kt)}{2t}. \end{aligned}$$

W-entropy formula for Hamilton's Harnack quantity

The above result is new even in the non weighted case.

Theorem (S. Li-Li arxiv2014)

Under the bounded geometry condition, we have

$$\begin{aligned}\frac{d}{dt}W_{n,K}(u,t) &= -2t\int_M \left|\nabla^2 \log u + \left(\frac{K}{2}+\frac{1}{2t}\right)g\right|^2 u d\mu \\ &\quad -2t\int_M (Ric+Kg)(\nabla \log u, \nabla \log u)u d\mu \\ &\quad -\frac{n}{2t}\left[e^{4Kt}(1+4Kt)-(1+Kt)^2\right].\end{aligned}$$

In particular, if $Ric \geq -K$, then for all $t \geq 0$,

$$\frac{d}{dt}W_{n,K}(u,t) \leq -\frac{n}{2t}\left[e^{4Kt}(1+4Kt)-(1+Kt)^2\right].$$

Moreover, the equality holds at some time $t = t_0 > 0$ if and only if

$$Ric = -Kg, \qquad 2\nabla^2 f = \left(\frac{1}{t}+K\right)g.$$

W-entropy formula under $CD(-K, m)$

By chaining the comparable model, we can obtain the following

Theorem (S. Li-Li PJM2015)

Under bounded geometry condition, define

$$W_{m,K}(u,t) = \int_M \left[t|\nabla f|^2 + f - m\left(1 + \frac{Kt}{2}\right)\right] u d\mu.$$

Then

$$\begin{aligned}\frac{d}{dt}W_{m,K}(u,t) &= -2t\int_M \left|\nabla^2 f - \left(\frac{1}{2t} + \frac{K}{2}\right) g\right|^2 u d\mu \\ &\quad -2t\int_M (Ric_{m,n}(L) + Kg)(\nabla f, \nabla f) u d\mu \\ &\quad -\frac{2t}{m-n}\int_M \left|\nabla\phi\cdot\nabla f - (m-n)\left(\frac{1}{2t} + \frac{K}{2}\right)\right|^2 u d\mu.\end{aligned}$$

This extends a previous result due to J. Li and Xu (AIM2010) for the case $L = \Delta$ and $m = n$.

Monotonicity and rigidity theorem

Theorem (S. Li-Li PJM 2015)

Suppose that $Ric_{m,n}(L) \geq -K$. *Then*

$$\frac{d}{dt} W_{m,K}(u, t) \leq 0.$$

Moreover, the equality holds at some time $t = t_0 > 0$ *if and only if*

$$\begin{aligned} Ric_{m,n}(L) &= -Kg, \\ 2\nabla^2 f &= \left(\frac{1}{t} + K\right) g, \\ \nabla\phi \cdot \nabla f &= (m-n)\left(\frac{1}{2t} + \frac{K}{2}\right). \end{aligned}$$

Thus, (M, g, ϕ) *is a quasi-Einstein manifold, and the potential* f *satisfies the soliton equation*

$$Ric_{m,n}(L) + 2\nabla^2 f = \frac{g}{t}.$$

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4. W-entropy for Langevin deformation on Wasserstein space

Theorem (Li 2013 arxiv SPA2015)

Let M be a complete Riemannian manifold. Suppose that there exists a constant $K \geq 0$ such that

$$Ric(L) \geq -K.$$

Let u be a positive and bounded solution to the heat equation

$$\partial_t u = Lu,$$

Then, the optimal Hamilton Harnack inequality holds

$$|\nabla \log u|^2 \leq \frac{2K}{e^{2Kt}-1} \log(A/u).$$

where $A = \sup\{u(t,x) : x \in M, t \geq 0\}$.

Corollary (Li 2013arxiv SPA2015)

Let M be a complete Riemannian manifold. Suppose that there exists a constant $K \geq 0$ such that

$$Ric(L) \geq -K.$$

Let u be a positive and bounded solution to the heat equation

$$\partial_t u = Lu,$$

The Hamilton Harnack inequality holds

$$|\nabla \log u|^2 \leq \left(\frac{1}{t} + 2K\right) \log(A/u).$$

The W-entropy formula for $CD(K,\infty)$ case

Theorem (S. Li-L. arxiv2014)

Let $u(\cdot,t)=P_tf$ be a positive solution to the heat equation $\partial_t u = Lu$ with $u(\cdot,0)=f\geq 0$. Suppose $Ric+\nabla^2\phi\geq K$, where $K\in\mathbb{R}$. Let

$$H_K(f,t)=D_K(t)\int_M (P_t(f\log f)-P_tf\log P_tf)d\mu,$$

where $D_K(t)=\frac{K}{1-e^{-2Kt}}$. Then, for all $t>0$, $\frac{d}{dt}H_K(f,t)\leq 0$, and

$$\frac{d^2}{dt^2}H_K(t)+2K\coth(Kt)\frac{d}{dt}H_K(t)\leq -2D_K(t)\int_M |\nabla^2\log P_tf|^2P_tfd\mu.$$

Moreover, the equality holds if and only if (M,g,ϕ) is a Ricci soliton

$$Ric+\nabla^2\phi=Kg.$$

The W-entropy formula for $CD(K,\infty)$ case

Theorem (S. Li-L. arxiv2014)

Define the W-entropy by the revised Boltzmann entropy formula

$$W_K(f,t) = H_K(f,t) + \frac{\sinh(2Kt)}{2K}\frac{d}{dt}H_K(f,t).$$

Then, for all $K \in \mathbb{R}$, and for all $t > 0$, we have

$$\frac{d}{dt}W_K(f,t) + \frac{e^{2Kt}+1}{2}\int_M |\nabla^2 \log P_t f|^2 P_t f d\mu$$
$$= -\frac{e^{2Kt}+1}{2}\int_M (Ric(L) - K)(\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu.$$

In particular, if $Ric(L) \geq K$, then

$$\frac{d}{dt}W_K(f,t) + \frac{e^{2Kt}+1}{2}\int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \leq 0, \quad \forall t > 0.$$

Moreover, the equality holds if and only if $Ric + \nabla^2\phi = Kg$.

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Logarithmic Sobolev inequalities

Let M be a complete Riemannian manifold equipped with a family of time dependent metrics $g(t)$ and potentials $\phi(t)$.

Let

$$L = \Delta_{g(t)} - \nabla_{g(t)}\phi(t) \cdot \nabla_{g(t)}$$

be the time dependent Witten Laplacian on $(M, g(t), \phi(t))$.

Let $u(\cdot, t) = P_t f$ be a positive solution to the heat equation

$$\partial_t u = Lu,$$

with the initial condition $u(\cdot, 0) = f$, where $f \geq 0$ is a measurable function on M.

Logarithmic Sobolev inequalities

Theorem (S. Li-L. 2014)

Let M be a complete Riemannian manifold equipped with a K-super Perelman Ricci flow

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric(L) \geq -K.$$

where $K \geq 0$ is a constant independent of $t \in [0, T]$, $f \geq 0$. Then, the following logarithmic Sobolev inequality holds

$$P_t(f \log f) - P_t f \log P_t f \leq \frac{e^{2Kt} - 1}{2K} P_t \left(\frac{|\nabla f|^2}{f} \right), \quad \forall t \in [0, T],$$

and the reversal logarithmic Sobolev inequality holds

$$\frac{|\nabla P_t f|^2}{P_t f} \leq \frac{2K}{1 - e^{-2Kt}} \left(P_t(f \log f) - P_t f \log P_t f \right), \quad \forall t \in [0, T].$$

Theorem (S. Li-L. 2014)

Let M be a complete Riemannian manifold. Suppose that there exists a constant $K \geq 0$ such that

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric(L) \geq -K.$$

Let u be a positive and bounded solution to the heat equation

$$\partial_t u = Lu,$$

Then, the optimal Hamilton Harnack inequality holds

$$|\nabla \log u|^2 \leq \frac{2K}{e^{2Kt} - 1} \log(A/u).$$

where $A = \sup\{u(t,x) : x \in M, t \geq 0\}$.

Corollary (S. Li-L. 2014)

Let M be a complete Riemannian manifold. Suppose that there exists a constant $K \geq 0$ such that

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric(L) \geq -K.$$

Let u be a positive and bounded solution to the heat equation

$$\partial_t u = Lu,$$

The Hamilton Harnack inequality holds

$$|\nabla \log u|^2 \leq \left(\frac{1}{t} + 2K\right) \log(A/u).$$

Theorem (S. Li-L. 2014)

Let M be a compact Riemannian manifold, $\phi \in C^2(M)$. Suppose that there exists a constant $K \geq 0$ such that

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric_{m,n}(L) \geq -K.$$

Let u be a positive solution of $\partial_t u = Lu$. Then

$$\frac{\partial_t u}{u} - e^{-2Kt}\frac{|\nabla u|^2}{u^2} + e^{2Kt}\frac{m}{2t} \geq 0.$$

In particular, if $K = 0$, i.e., $\frac{1}{2}\frac{\partial g}{\partial t} + Ric_{m,n}(L) \geq 0$, then the Li-Yau Harnack inequality holds

$$\frac{\partial_t u}{u} - \frac{|\nabla u|^2}{u^2} + \frac{m}{2t} \geq 0.$$

W-entropy for time dependent Witten Laplacian

Theorem (S. Li-Li PJM2015)

Let M be a compact manifold, $\{g(t), \phi(t), t \in [0, T]\}$ satisfies

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\mathrm{Tr}\frac{\partial g}{\partial t}.$$

Let

$$u(x,t) = \frac{e^{-f(x,t)}}{(4\pi t)^{m/2}}$$

be the solution of the heat equation $\partial_t u = Lu$. Then

$$\frac{d\mathcal{W}(u,t)}{dt} = -2\int_M t\left[\left|\nabla^2 f - \frac{g}{2t}\right|^2 + \left(\frac{1}{2}\frac{\partial g}{\partial t} + Ric_{m,n}(L)\right)(\nabla f, \nabla f)\right] ud\mu$$
$$-\frac{2}{m-n}\int_M t\left(\nabla\phi\cdot\nabla f + \frac{m-n}{2t}\right)^2 ud\mu.$$

W-entropy formula on Perelman's super m-Ricci flow

Corollary (S. Li-Li PJM2015)

Let M be a compact manifold. Suppose that $g(t)$ is a Perelman's super m-Ricci flow

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric_{m,n}(L) \geq 0,$$

and $f(t)$ satisfies the conjugate equation

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\mathrm{Tr}\frac{\partial g}{\partial t}.$$

Let u be a positive solution of the heat equation $\partial_t u = Lu$. Then

$$\frac{dW(u,t)}{dt} \leq 0.$$

W-entropy formula for geodesic flow on Wasserstein space

Let M be a compact or complete Riemannian manifold, $\phi \in C^2(M)$. Consider the geodesic flow on the Wasserstein space over (M, μ) equipped with Otto's infinite dimensional Riemannian metric

$$\frac{\partial \rho}{\partial t} + \nabla_\phi^*(\rho \nabla f) = 0,$$
$$\frac{\partial f}{\partial t} + \frac{1}{2}|\nabla f|^2 = 0.$$

Let

$$H_m(\rho, t) := -\int_M \rho \log \rho d\mu - \frac{m}{2}\left(\log(4\pi t^2) + 1\right),$$

and define the W-entropy for the Witten Laplacian by

$$W(\rho, t) := \frac{d}{dt}(t H_m(\rho, t)).$$

W-entropy formula along the optimal transportation

Theorem (S. Li-X.D. Li 2012, 2015)

Let M be a compact (or complete) Riemannian manifold, $(\rho(t), f(t))$ be the smooth solution to the above equations (with suitable growth condition). Then

$$\frac{dW(\rho,t)}{dt} = -\int_M t\left[\left|\nabla^2 f - \frac{g}{t}\right|^2 + Ric_{m,n}(L)(\nabla f, \nabla f)\right]\rho d\mu$$
$$-\frac{1}{m-n}\int_M t\left(\nabla f\cdot\nabla\phi + \frac{m-n}{t}\right)^2 \rho d\mu.$$

The rigidity model is $N(0, t^2)$ on $M = \mathbb{R}^n$, i.e., $m = n$, and

$$\overline{\rho}(t) = \frac{1}{(4\pi t^2)^{m/2}} e^{-\frac{|x|^2}{4t^2}}, \quad \overline{f}(t) = \frac{|x|^2}{2t^2}.$$

Lott-Villani's theorem

As a corollary of our W-entropy formula on the Wasserstein space, we can recapture the following result due to Lott-Villani.

Theorem (Lott-Villani Ann. Math. 2009, Lott 2009)

Let M be a compact Riemannian manifold. Suppose $Ric \geq 0$. Then

$$t\mathrm{Ent}(\rho(t)) + nt\log t$$

is convex along the geodesic $(\rho(t), f(t))$ on $(P_2(M), dv)$.

Langevin deformation on Wasserstein space

Problem

How to explain this similarity between the W-entropy formula for the Witten Laplacian and for the optimal transport problem ?

- The vanishing viscosity limit using the Cole-Hopf transformation does not provide a good answer to this problem.
- Inspired by J.-M. Bismut's work, S. Li and Li (2013) introduced a deformation of geometric flows on the Wasserstein space, which interpolates the heat equation on the underlying manifold M, and the geodesic flow on the Wasserstein space over M.
- A discussion with C. Villani on 31 May 2013.

Interpolation between geodesics and gradient flow

Our work is based on the following well-known observation, inspired by J-M. Bismut's work on the deformation of the Witten Laplacian and geodesic flow on the cotangent bundle,

Proposition (S. Li-Li 2013)

Let M be a complete Riemannian manifold, $V \in C^2(M)$. Let (ρ_t, v_t) be defined as follows

$$\begin{aligned} \dot{\rho} &= \frac{v}{c}, \\ \dot{v} &= -\frac{v}{c^2} + \frac{\nabla V(\rho)}{c}. \end{aligned}$$

Then ρ_t satisfies the Langevin equation

$$c^2 \ddot{\rho} = -\dot{\rho} + \nabla V(\rho).$$

Interpolation between geodesics and gradient flow

Now in the Wasserstein space $\mathcal{P}_2(M, \mu)$ over M, define

$$H(\rho, \dot\rho) = \frac{1}{2}\|\dot\rho\|^2 + \int_M \rho \log \rho d\mu.$$

Inspired by J.-M. Bismut, S. Li and Li (2013) introduced the following geometric flows on $\mathcal{P}_2(M, \mu)$:

$$v = -c\nabla_\mu^* \cdot (\rho\nabla\phi),$$

This yields

$$\begin{aligned} \partial_t\rho + \nabla_\mu^* \cdot (\rho\nabla\phi) &= 0, \\ c^2(\partial_t\phi + \frac{1}{2}|\nabla\phi|^2) &= -\phi + \log\rho + 1. \end{aligned}$$

When $c = 0$, $\phi = \log\rho + 1$, ρ satisfies the backward heat equation

$$\partial_t\rho = -L\rho,$$

and when $c = \infty$, (ρ, ϕ) is a geodesic flow on $P_2(M, \mu)$

$$\begin{aligned} \partial_t\rho + \nabla_\mu^* \cdot (\rho\nabla\phi) &= 0, \\ \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 &= 0. \end{aligned}$$

Link with compressible Euler equation with damping

Let $u = \nabla\phi$, then u satisfies the compressible Euler equation with damping

$$\partial_t u + u \cdot \nabla u = -\frac{1}{c^2} u + \frac{1}{c^2} \nabla \log \rho.$$

Theorem (S. Li-Li 2015)

Let M be a compact Riemannian manifold, $\rho_0 > 0, \phi_0 \in C^\infty(M)$. Then the Cauchy problem to the Langevin deformation flow equation has a unique local smooth solution $(\rho(t), \phi(t))$ on $[0, T] \times M$ with initial data (ρ_0, ϕ_0).

Theorem (S. Li-Li 2015)

Let M be a compact Riemannian manifold. Then for small initial data (ρ_0, ϕ_0) (in suitable Sobolev norm), there is a unique global smooth solution to the Cauchy problem of the Langevin deformation flow equation.

Interpolation between geodesics and gradient flow

Theorem (S. Li-Li 2013)

For $c \in [0, \infty)$, we have

$$\frac{d^2}{dt^2} H(\rho, \dot{\rho}) = 2 \int_M [c^{-2} |\nabla \phi - \rho^{-1} \phi|^2 + |\mathrm{Hess} \phi|^2 + (\mathrm{Ric} + \nabla^2 f)(\nabla \phi, \nabla \phi)] \rho d\mu.$$

When $c = 0$, $\phi = \log \rho + 1$, $\partial_t \rho = -L\rho$, and

$$\frac{d^2}{dt^2} \mathrm{Ent}(\rho(t)) = 2 \int_M [|\mathrm{Hess} \phi|^2 + (\mathrm{Ric} + \nabla^2 f)(\nabla \phi, \nabla \phi)] \rho d\mu.$$

When $c = \infty$, (ρ, ϕ) is a geodesic flow on $P_2(M, \mu)$, and

$$\frac{d^2}{dt^2} \mathrm{Ent}(\rho(t)) = \int_M [|\mathrm{Hess} \phi|^2 + (\mathrm{Ric} + \nabla^2 f)(\nabla \phi, \nabla \phi)] \rho d\mu.$$

W-entropy formula for deformation of flows

Problem (S. Li-Li 2013)

Can we prove an analogue of the W-entropy formula for the deformation flow, and prove a rigidity theorem under suitable curvature-dimension condition?

Theorem (S. Li-Li 2015)

For any $c > 0$, we have

$$\begin{aligned}\left(\frac{d^2}{dt^2} + \frac{1}{c^2}\frac{d}{dt}\right)\mathrm{Ent}(\rho(t)) &= \int_M [|\mathrm{Hess}\phi|^2 + Ric(L)(\nabla\phi, \nabla\phi)]\rho d\mu \\ &\quad + \frac{1}{c^2}\int_M \frac{|\nabla\rho|^2}{\rho} d\mu. \\ \left(\frac{d^2}{dt^2} + \frac{2}{c^2}\frac{d}{dt}\right)H(\rho(t), \phi(t)) &= \int_M [|\mathrm{Hess}\phi|^2 + Ric(L)(\nabla\phi, \nabla\phi)]\rho d\mu \\ &\quad + \frac{2}{c^2}\int_M \frac{|\nabla\rho|^2}{\rho} d\mu.\end{aligned}$$

W-entropy formula for deformation of flows

Theorem (S. Li-Li 2015)

Define

$$\begin{aligned} W_{H,c}(t) &= H(\rho(t),\phi(t)) + \frac{c^2(1-e^{\frac{2t}{c^2}})}{2}\frac{d}{dt}H(\rho(t),\phi(t)),\\ W_c(t) &= \mathrm{Ent}(\rho(t)) + c^2(1-e^{\frac{t}{c^2}})\frac{d}{dt}\mathrm{Ent}(\rho(t)). \end{aligned}$$

Note that, as $c \to \infty$, $c^2(1-e^{\frac{t}{c^2}}) \to t$. *Then*

$$\begin{aligned} \frac{d}{dt}W_{H,c}(t) &= (1-e^{\frac{2t}{c^2}})\int_M \frac{|\nabla\rho|^2}{\rho}d\mu + \int_M [|\mathrm{Hess}\phi|^2 + Ric(L)(\nabla\phi,\nabla\phi)]\rho d\mu,\\ \frac{d}{dt}W_c(t) &= (1-e^{\frac{t}{c^2}})\int_M \frac{|\nabla\rho|^2}{\rho}d\mu + \int_M [|\mathrm{Hess}\phi|^2 + Ric(L)(\nabla\phi,\nabla\phi)]\rho d\mu. \end{aligned}$$

In particular, if $Ric(L) \geq 0$*, then for all* $c > 0$*, we have*

$$\frac{d}{dt}W_{H,c}(t) \leq 0, \quad \frac{d}{dt}W_c(t) \leq 0, \quad \forall t \geq 0.$$

The model: deformation of flows on $P_2(\mathbb{R}^m, dx)$

Let $V(u) = -\frac{1}{2}\log u$, $u > 0$. Then $V'(u) = -\frac{1}{2u}$. Consider the Newton-Langevin equation on $T^*\mathbb{R}^+ = \mathbb{R}^+ \times \mathbb{R}$

$$c^2(\ddot{u} + \dot{u}) = -\frac{1}{2u}.$$

Note that $V(u) = -\frac{1}{2}\log u$ is locally Lipschitz on $(0, +\infty)$. By Picard theorem, for any given $T > 0$, and for given $u(T) > 0$ and $\dot{u}(T) \in \mathbb{R}$, there exists a unique solution $u(t)$ on an interval $[T - \delta, T]$ for some $\delta > 0$.

Let $\beta : [T - \delta, T] \to \mathbb{R}$ be a smooth solution to the followings ODE

$$c^2\dot{\beta} + \beta \quad = \quad -m\log u - \frac{m}{2}\log(4\pi) + 1.$$

The model on $P_2(\mathbb{R}^m, dx)$

Theorem (S. Li-Li 2014)

Let $\alpha(t) = \frac{u'}{u}$ *and define*

$$\begin{aligned}\phi_m(x,t) &= \frac{\alpha(t)}{2}\|x\|^2 + \beta(t),\\ \rho_m(x,t) &= \frac{1}{(4\pi u^2(t))^{m/2}} e^{-\frac{\|x\|^2}{4u^2(t)}}.\end{aligned}$$

Then (ρ_m, ϕ_m) *is a solution of the equations of the deformation flow on*

$$\begin{aligned}\partial_t\rho + div(\rho\nabla\phi) &= 0,\\ c^2\left(\partial_t\phi + \frac{1}{2}|\nabla\phi|^2\right) &= -\phi + \log\rho + 1.\end{aligned}$$

By calculation, we have

$$\mathrm{Ent}(\rho_m(t)) = -\frac{m}{2}[1 + \log(4\pi u^2(t))].$$

W-entropy formula for deformation of flows

Let $m > n$, $(\rho(t), \phi(t))$ be the deformation of flows on $T^*P_2(M, \mu)$

$$\begin{aligned} \partial_t \rho + \nabla_\mu^* \cdot (\rho \nabla \phi) &= 0, \\ c^2\left(\partial_t \phi + \frac{1}{2}|\nabla \phi|^2\right) &= -\phi + \log \rho + 1. \end{aligned}$$

Define

$$H_m(\rho(t)) = \mathrm{Ent}(\rho(t)) - \mathrm{Ent}(\rho_m(t)).$$

Theorem (S. Li-Li 2015)

$$\begin{aligned} & \frac{d^2}{dt^2} H_m(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right) \frac{d}{dt} H_m(\rho(t)) \\ = & \int_M \left[|\mathrm{Hess}\phi - \alpha(t) g|^2 + Ric_{m,n}(L)(\nabla\phi, \nabla\phi) \right] \rho d\mu \\ & \quad + (m-n) \int_M \left| \alpha(t) + \frac{\nabla\phi \cdot \nabla f}{m-n} \right|^2 \rho d\mu + \frac{1}{c^2} \int_M \frac{|\nabla \rho|^2}{\rho} d\mu. \end{aligned}$$

W-entropy formula for deformation of flows

In case $m = n$, $f = 0$, let $(\rho(t), \phi(t))$ be solution to

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \nabla \phi) &= 0, \\ c^2\left(\partial_t \phi + \frac{1}{2}|\nabla \phi|^2\right) &= -\phi + \log \rho + 1. \end{aligned}$$

Define

$$H_n(\rho(t)) = \mathrm{Ent}(\rho(t)) - \mathrm{Ent}(\rho_n(t)).$$

Theorem (S. Li-Li 2015)

$$\begin{aligned} & \frac{d^2}{dt^2} H_n(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right) \frac{d}{dt} H_n(\rho(t)) \\ = & \int_M \left[|\mathrm{Hess}\phi - \alpha(t) g|^2 + Ric(\nabla \phi, \nabla \phi) \right] \rho dv + \frac{1}{c^2} \int_M \frac{|\nabla \rho|^2}{\rho} dv. \end{aligned}$$

W-entropy formula for deformation of flows

Note that, on $M = \mathbb{R}^n$, the model (ρ_n, ϕ_n) on $(P_2(\mathbb{R}^n), dx)$

$$\begin{aligned}\phi_n(x,t) &= \frac{\alpha(t)}{2}\|x\|^2 + \beta(t),\\ \rho_n(x,t) &= \frac{1}{(4\pi u^2(t))^{n/2}} e^{-\frac{\|x\|^2}{4u^2(t)}}.\end{aligned}$$

satisfies

$$\mathrm{Hess}\phi_n = \alpha(t)g.$$

Moreover

$$\frac{d^2}{dt^2}H_n(\rho_n(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right)\frac{d}{dt}H_n(\rho_n(t)) = \frac{1}{c^2}\int_M \frac{|\nabla\rho_n|^2}{\rho_n}dv.$$

Thus, (ρ_n, ϕ_n) gives the rigidity model for the entropy inequality.

W-entropy formula for deformation of flows

Let us introduce the W-entropy be such that

$$\frac{dW}{dt}(\rho(t)) = \frac{d^2}{dt^2}H(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right)\frac{d}{dt}H(\rho(t)) - \frac{1}{c^2}\int_M \frac{|\nabla\rho|^2}{\rho}dv.$$

By calculation, we have

$$\frac{d}{dt}W(\rho_n(t)) = -n\alpha^2(t).$$

In view of this, the above theorem is equivalent to the following comparison theorem

$$\frac{d}{dt}(W(\rho(t)) - W(\rho_n(t))) = \int_M \left[|\mathrm{Hess}\phi - \alpha(t)g|^2 + Ric(\nabla\phi, \nabla\phi)\right]\rho dv.$$

Thus, if $Ric \geq 0$, then

$$\frac{d}{dt}(W(\rho(t)) \geq \frac{d}{dt}W(\rho_n(t))), \quad \forall t > 0.$$

Moreover, if one can extend this to complete Riemannian manifolds, then the equality holds at some $t = t_0 > 0$ if and only if

$$M = \mathbb{R}^n, \ \ \rho = \rho_n, \ \ \phi = \phi_n.$$

Thank you !