The logarithmic Sobolev inequality and the convergence of a semigroup in the Zygmund space

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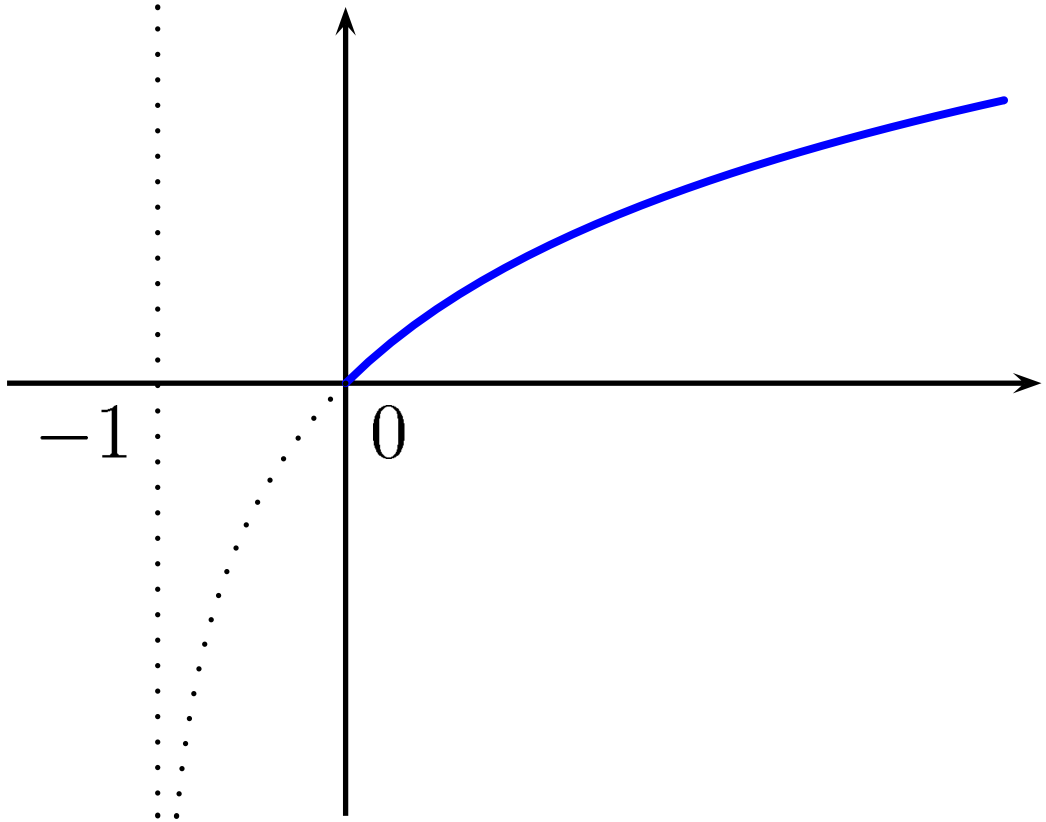
1. Entropy and the Zygmund space

Zygmund space

- $(M, \mathcal{B}, m)$: a measure space
- $m(M) = 1$
- $\langle f \rangle = E[f] = \int_M f\,dm$

Let $\boldsymbol{\phi}\colon [\mathbf{0}, \boldsymbol{\infty}) \to \mathbb{R}$ be defined by

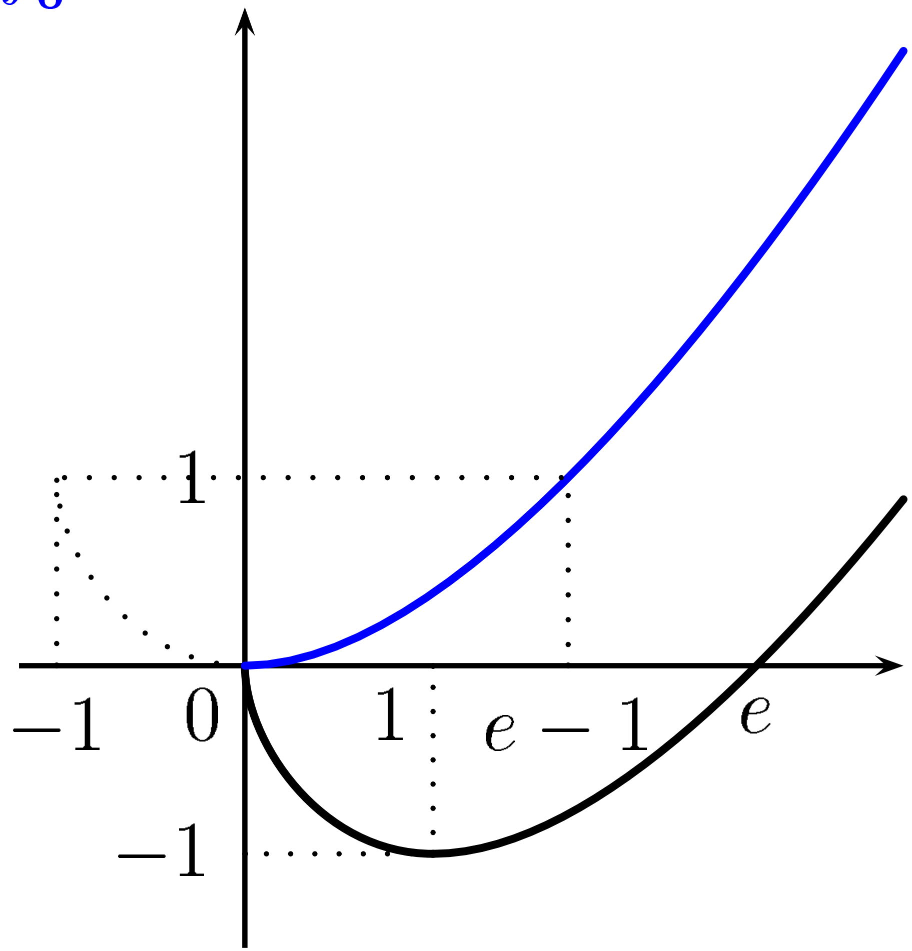
$$\boldsymbol{\phi(x) = \log(1 + x)} \tag{1}$$



graph of $\boldsymbol{\phi(x) = \log(1 + x)}$

Set

$$\Phi(x) = \int_0^x \phi(y)\,dy = (1+x)\log(1+x) - x. \tag{2}$$



Graphs of Φ and $x \log x - x$

The *Zygmund space* $Z = L \log L$ is defined by

$$Z = \{f; E[\Phi(|f|)] < \infty\}. \tag{3}$$

The norm N_Φ in Z is defined by

$$N_\Phi(f) = \inf\{\lambda; E[\Phi(|f|/\lambda)] \leq 1\}. \tag{4}$$

The dual space of Z can be defined as follows. Let ψ be the inverse function of ϕ:

$$\psi(x) = e^x - 1.$$

Set

$$\Psi(x) = \int_0^x \psi(y)\,dy = \int_0^x (e^y - 1)\,dy = e^x - x - 1.$$

The dual space of Z is the Orlicz space associated with Ψ. The following inequality is fundamental:

$$xy \leq \Phi(x) + \Psi(y). \tag{5}$$

By using this inequality we can show

$$\|f\|_1 \leq (e-1)N_\Phi(f). \tag{6}$$

So Z is smaller than L^1. Moreover we have

$$N_\Phi(f - \langle f \rangle) = 2N_\Phi(f).$$

Entropy

Define an entorpy of $f \geq 0$ by

$$\mathrm{Ent}(f) = E[f \log(f/\langle f \rangle)]. \tag{7}$$

We discuss the relation between the Zygmunt space and the entropy.

Proposition 1. For any non-negative function f, we have

$$\langle f \rangle E[\Phi(|(f - \langle f \rangle)/\langle f \rangle|)] \leq \mathrm{Ent}(f) \tag{8}$$

If $\langle f \rangle \geq 1$, we have another inequality.

Proposition 2. For any nonnegative function $\boldsymbol{f}$ with $\langle \boldsymbol{f} \rangle \geq \mathbf{1}$, we have

$$\boldsymbol{E}[\boldsymbol{\Phi}(|\boldsymbol{f} - \langle \boldsymbol{f} \rangle|)] \leq \langle \boldsymbol{f} \rangle\, \mathbf{Ent}(\boldsymbol{f}). \tag{9}$$

Now we have

Proposition 3. For any non-negative function $\boldsymbol{f}$, we have

$$\boldsymbol{N}_{\boldsymbol{\Phi}}(\boldsymbol{f} - \langle \boldsymbol{f} \rangle) \leq \mathbf{max}\{\sqrt{\langle \boldsymbol{f} \rangle}, \sqrt{\mathbf{Ent}(\boldsymbol{f})}\}\sqrt{\mathbf{Ent}(\boldsymbol{f})}. \tag{10}$$

Now we will prove the reversed inequality. Recall

$$\mathbf{Ent}(f) = E[f \log(f/\langle f\rangle)] \tag{11}$$

Proposition 4. For any non-negative function f, we have

$$\mathbf{Ent}(f) \leq \frac{\langle f\rangle}{\log(4/e)} E[\Phi(|(f - \langle f\rangle)/\langle f\rangle|)]. \tag{12}$$

If f satisfy $\langle f\rangle \leq 1$, we have the following.

Proposition 5. For any non-negative function f with $\langle f \rangle \leq 1$, we have

$$\mathbf{Ent}(f) \leq E[\Phi(|f - \langle f \rangle|)] + 2. \tag{13}$$

Proposition 6. For any non-negative function f, we have

$$\mathbf{Ent}(f) \leq 3N_\Phi(f - \langle f \rangle). \tag{14}$$

The logarithmic Sobolev inequality

Let us recall the logarithmic Sobolev inequality.

- $\mathcal{E}$: a Dirichlet form
- $\{T_t\}$: a Markovian semigroup in $L^2(m)$
- $\mathfrak{A}$: the generator of $\{T_t\}$

The following inequality is called a logarithmic Sobolev inequality:

$$\int_M f^2(x) \log(f(x)^2/\|f\|_2^2)\, dm \leq \frac{2}{\gamma_{\mathbf{LS}}} \mathcal{E}(f, f). \tag{15}$$

If we assume the logarithmic Sobolev inequality (15), it is known that for any non-negative function f, we have

$$\mathbf{Ent}(T_t f) \leq e^{-2\gamma_{\mathrm{LS}} t}\,\mathbf{Ent}(f). \tag{16}$$

We set

$$\gamma_{Z\to Z} = -\overline{\lim}\,\frac{1}{t}\log \|T_t - m\|_{Z\to Z} \tag{17}$$

Combining the previous results, we have

Theorem 7. We have the following inequality:

$$\gamma_{\mathbf{LS}} \leq \gamma_{Z \to Z} \tag{18}$$

Under the assumption of the logarithmic Sobole inequality, we can show that tha independence of the spectrum.

(Kusuoka - S [2015])

Assume $\mathfrak{A}$ is normal. Then $\sigma(\mathfrak{A}_p)$ is independent of p $(1 < p < \infty)$.

Here $\mathfrak{A}_p$ is the generator in L^p.

Question: What happens in the Zygmund space?

2. Operators in Zygmund space

We define an Orlitz norm $\| \ \|_\Phi$ as follows:

$$\|f\|_\Phi = \sup\{E[g|f|]; E[\Psi(g)] \leq 1\}. \tag{19}$$

Here non-negative functions g run over all fundtions with $E[\Psi(g)] \leq 1$. We also have

$$\|f\|_\Phi = \sup\{E[g|f|]; E[e^g - g] \leq 2\}. \tag{20}$$

Tow norms N_Φ and $\| \ \|_\Phi$ are equivalent:

$$N_\Phi(f) \leq \|f\|_\Phi \leq 2N_\Phi(f). \tag{21}$$

Proposition 8. A linear operator$T$ in Z is bounded if and only if there exist positive constants A, B such that

$$\|Tf\|_{\Phi} \leq AE[\Phi(|f|)] + B. \tag{22}$$

Corollary 9. A linear operator T in Z is bounded if and only if there exist positive constants A, B such that for all non-negative function g with $E[e^g] \leq 4$, we have

$$E[g|Tf|] \leq AE[|f| \log |f|] + B. \tag{23}$$

3. Spectrum of the Kummer operator

In this section, we consider the Kummer operator

- $M = [0, \infty)$
- $m(dx) = \frac{1}{\Gamma(\alpha+1)} x^{\alpha} e^{-x} dx$
- $H = L^2([0, \infty), m)$
- $\mathfrak{A} = x \frac{d}{dx^2} + (1 + \alpha - x) \frac{d}{dx}$

We assume that $\alpha > 0$.

We give a representation of the resolvent by using the confluent hypergeometric functions.

Confluent hypergeometric functions

A confluent hypergeometric functions is defined by

$$ {}_1F_1(a;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} x^n. \tag{24} $$

Here $(a)_n$ stands for the Pochhammer symbol:

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\cdots(a+n-1) & n \geq 1 \\ 1 & n = 0 \end{cases} \tag{25} $$

${}_1F_1(a;c;x)$ satisfies the following Kummer differential equation:

$$ xu'' + (c-x)u' = au. \tag{26} $$

This means that ${}_1F_1(a;c;x)$ is an “eigen-function” of the Kummer operator in the case of $1+\alpha=c$.

If ${}_1F_1(a;c;x) \in L^2$, then ${}_1F_1(a;c;x)$ is really an eigen-value. We set

$$M(a,1+\alpha;x) = {}_1F_1(a;1+\alpha;x). \tag{27}$$

This function is called the Kummer function. Another independent solution is

$$U(a,1+\alpha;x) = \frac{\Gamma(-\alpha)}{\Gamma(a-\alpha)}M(a,1+\alpha;x)+\frac{\Gamma(\alpha)}{\Gamma(a)}x^{-\alpha}M(a-\alpha,1-\alpha;x) \tag{28}$$

which is called the Kummer function of the second kind. Their

Wronskian is

$$W(M(a,1+\alpha;\,\cdot\,),U(a,1+\alpha;\,\cdot\,))(x) = -\frac{\Gamma(1+\alpha)}{\Gamma(a)}x^{-\alpha-1}e^{x}.$$

It is known that Laguerre polynomials are eigen-functions. In fact, we have

$$L_n^{\alpha}(x) = \frac{(\alpha+1)_n}{n!}M(-n,\alpha+1;x). \tag{29}$$

Thus the spectrum of $\mathfrak{A}$ is $\{0,-1,-2,\cdots\}$.

The asymptotic behavior is crucial in the computation of the resolvent.

When $x \to 0$, we have

$$M(a, 1+\alpha; x) \to 1, \tag{30}$$

$$U(a, 1+\alpha; x) \sim \frac{\Gamma(\alpha)}{\Gamma(a)} x^{-\alpha}. \tag{31}$$

When $\alpha = 0$, $x^{-\alpha}$ should be $\log x$.

When $x \to \infty$, we have

$$M(a, 1+\alpha; x) \sim \frac{\Gamma(1+\alpha)}{\Gamma(a)} e^x x^{a-1-\alpha}, \tag{32}$$

$$U(a, 1+\alpha; x) \sim x^{-a} \tag{33}$$

Here $a, 1+\alpha \neq 0, -1, -2, \ldots$.

Recall that $\alpha > 0$. We also assume that $a \neq 0, -1, -2, \ldots$. Then the resolvent $G_a = (a - \mathfrak{A})^{-1}$ has the following kernel expression.

$$G_a f(x) = \int_0^\infty G_a(x,y) f(y)\, dy \tag{34}$$

where

$$G_a(x,y) = \begin{cases} -M(a, 1+\alpha; y) U(a, 1+\alpha, x) \dfrac{1}{yW(y)} & y < x, \\ -M(a, 1+\alpha; x) U(a, 1+\alpha, y) \dfrac{1}{yW(y)} & y > x. \end{cases}$$

W is the Wronskian. Hence

$$G_a(x,y) = \begin{cases} \dfrac{\Gamma(a)}{\Gamma(1+\alpha)} M(a, 1+\alpha; y) U(a, 1+\alpha, x) e^{-y} y^\alpha & y < x, \\ \dfrac{\Gamma(a)}{\Gamma(1+\alpha)} M(a, 1+\alpha; x) U(a, 1+\alpha, y) e^{-y} y^\alpha & y > x. \end{cases}$$

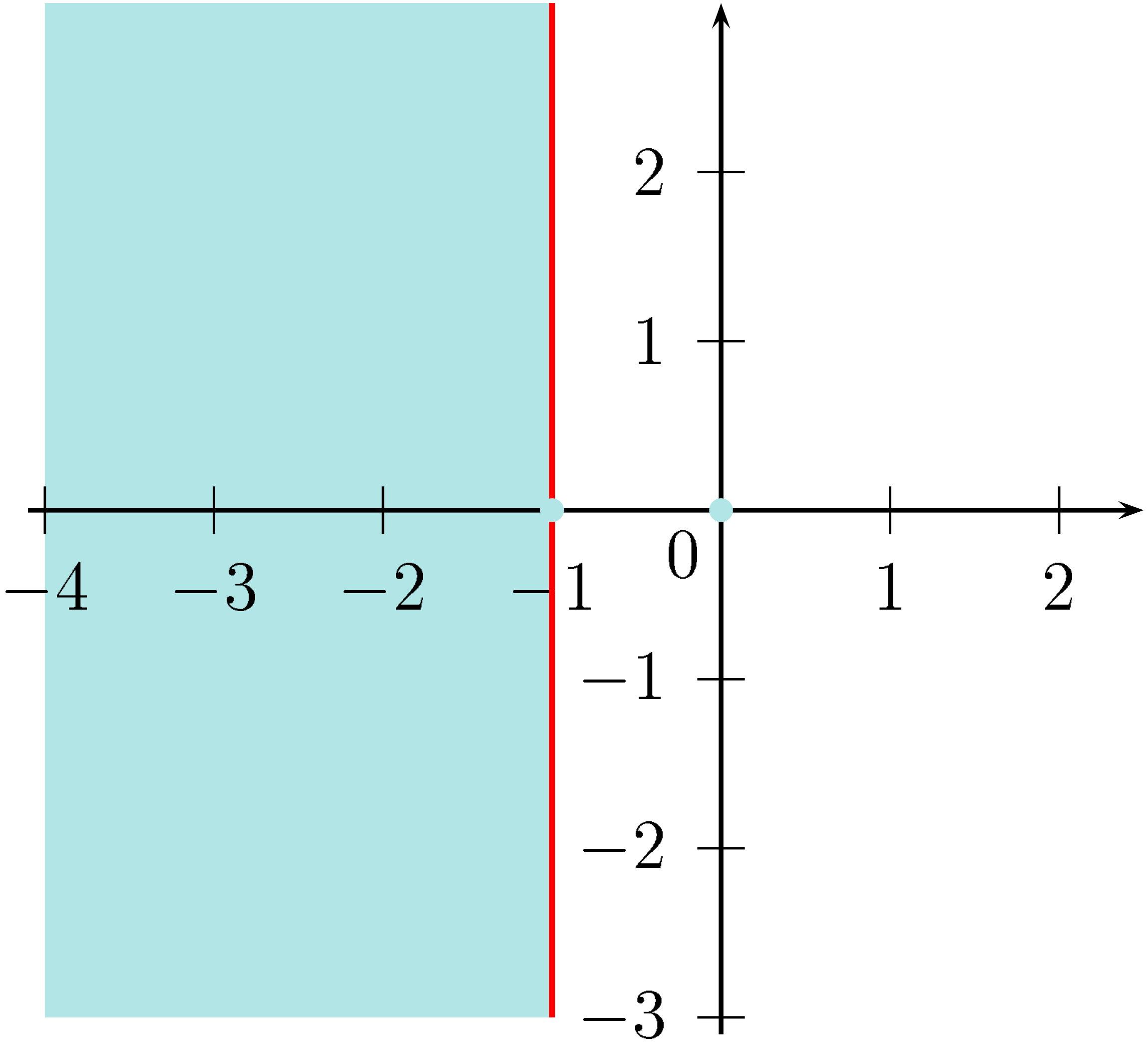
$\boldsymbol{G_a}$ is a bounded operator in $\boldsymbol{L^2}$. What happens in the case of Zygmund space?

4. The spectrum of the Kummer operaotr in Z

Now we can compute the spectrum of $\mathfrak{A}$ in Z. Since we have the kenel expression of the resolvent,we can compute the spectrum.

Theorem 10. The set of point spectums of $\mathfrak{A}$ is $\{z;\ \Re z < -1\} \cup \{-1\} \cap \{0\}$.

Theorem 11. When $\Re a > -1$, a belongs to the resolvent set.



The spectrum in Z.

In Theorem 7, we have shown $\gamma_{\mathbf{LS}} \leq \gamma_{Z\to Z}$. In this example $\gamma_{\mathbf{LS}} = \frac{\mathbf{1}}{\mathbf{2}}$ and $\gamma_{Z\to Z} = \mathbf{1}$, which means that $\gamma_{\mathbf{LS}} \neq \gamma_{Z\to Z}$.

Thanks !