Infinite Dimensional Continuity and Fokker-Planck-Kolmogorov Equations

Michael Röckner (University of Bielefeld)

joint work with

Vladimir I. Bogachev, Giuseppe Da Prato and Stanislav V. Shaposhnikov

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Contents

1. Framework and Definitions

Let $T_0 > 0$ and $A = (a^{ij})_{i,j\in\mathbb{N}}$, $\quad B = (B^i)_{i\in\mathbb{N}}$, with

$$a^{ij} : \mathbb{R}^\infty \times [0, T_0] \to \mathbb{R}, \quad a^{ij} = a^{ji}, \quad A_N := \left(a^{ij}\right)_{1\le i,j\le N} \text{ nonnegative for each } N \in \mathbb{N},$$

$$B^i : \mathbb{R}^\infty \times [0, T_0] \to \mathbb{R}$$

Borel functions. Consider the corresponding Kolmogorov operator on $\mathbb{R}^\infty \times [0, T_0]$

$$L\varphi(x,t) = \sum_{i,j=1}^{\infty} a^{ij}(x,t)\partial_{e_i}\partial_{e_j}\varphi(x,t) + \sum_{i=1}^{\infty} B^i(x,t)\partial_{e_i}\varphi(x,t), \quad \varphi \in \mathrm{Dom}(L),$$

where $\mathrm{Dom}(L) =$ linear span of $\left\{\varphi = \tilde{\varphi}(P_N) \;\middle|\; N \in \mathbb{N},\ \tilde{\varphi} \in C_0^{2,1}\left(\mathbb{R}^N \times [0, T_0]\right)\right\}$ and $P_N : \mathbb{R}^\infty \to \mathbb{R}^N \times [0, T_0]$ denotes the canonical projection.
Furthermore, ∂_{e_i} denotes partial derivative in direction $e_i := (0, \dots 0, 1, 0, \dots)$.
Below we consider any function $f : \mathbb{R}^N \times [0, T_0] \to \mathbb{R}$ as a function on $\mathbb{R}^\infty \times [0, T_0]$ and hence do not distinguish between φ and $\tilde{\varphi}$ in the definition of $\mathrm{Dom}(L)$.

We say a measure $\mu = \mu_t \, \mathrm{d}t$ on $\mathbb{R}^\infty \times [0, T_0]$, where $\mu_t, \ t \in [0, T_0]$, are probability measures on $\mathbb{R}^\infty$ (measurable in t), is a solution to the **Fokker-Planck-Kolmogorov equation** associated to L, if for every $\varphi \in \mathrm{Dom}(L)$

$$\int_{\mathbb{R}^\infty} \varphi(x, t) \, \mu_t(\mathrm{d}x) = \int_{\mathbb{R}^\infty} \varphi(x, 0) \, \nu(\mathrm{d}x) + \int_0^t \int_{\mathbb{R}^\infty} [\partial_s \varphi + L\varphi] \, \mathrm{d}\mu_s \, \mathrm{d}s \quad \text{for } \mathrm{d}t - \text{a.e. } t \in [0, T_0]. \tag{FPKE}$$

Here ν is a probability measure on $\mathbb{R}^\infty$, given as initial condition. (FPKE) is also shortly written as a Cauchy problem for paths $\mu_t, \ t \in [0, T_0]$, of probability measures as

$$\partial_t \mu_t = L^* \mu_t$$
$$\mu_0 = \nu,$$

where L^* is the formal adjoint (in $x \in \mathbb{R}^\infty$) of L.
In (FPKE) we implicitly assume that $a^{ij}, \ B^i \in L^1 \left(\mathbb{R}^\infty \times [0, T_0], \mu \right)$.

1. Framework and Definitions

Aims:

- General existence and uniqueness results for (FPKE)
- Applications to obtain transition probabilities for SPDE

Remark.

- Both non-degenerate (cylindrical noise) and degenerate Kolmogorov operators are covered. The latter include the case $A \equiv 0$, i.e. where (FPKE) is just the **continuity equation**.
- Special case: $\nu := \delta_x (=$ Dirac measure in $x)$. Then the solution $\mu_t(\mathrm{d}y), t \in [s, T_0]$, of (FPKE) started at $s \in [0, T_0]$ in δ_x is usually denoted $p_{s,t}(x, \mathrm{d}y)$. If (FPKE) is well-posed (i.e. has a unique solution for all initial measures ν and any $s \in [0, T_0]$), then for all $r \leq s \leq t$, $x \in H$,

$$p_{s,t}(y, \mathrm{d}z)\, p_{r,s}(x, \mathrm{d}y) = p_{r,t}(x, \mathrm{d}z) \quad \text{"Chapman-Kolmogorov-Equations"}.$$

  And $p_{s,t}$, $s \leq t$, are the transition probabilities of the (time-inhomogeneous) Markov process generated by L, if it exists.

2. Relation to S(P)DE

Consider the associated S(P)DE

$$\mathrm{d}X(t) = B(X(t), t)\,\mathrm{d}t + \sigma(X(t), t)\,\mathrm{d}W(t) \qquad \text{(S(P)DE)}$$
$$X(0) \sim \nu,$$

e.g. on $H := \ell^2 \subset \mathbb{R}^\infty$, with W being a cylindrical Wiener process on H and "your favourite" assumptions on B and $\sigma := \sqrt{A}$. Suppose (S(P)DE) has a solution $(\mathbb{P}_x)_{x \in H}$ in the sense of Stroock-Varadhan's martingale problem. Then $\mu := \mu_t\,\mathrm{d}t$ solves (FPKE) with

$$\mu_t := \mathbb{P}_\nu \bullet X(t)^{-1} \;(\text{"marginals"})\,, \quad \mathbb{P}_\nu := \int_H \mathbb{P}_x\,\nu(\,\mathrm{d}x).$$

Hence:

- weak (= martingale) existence for (S(P)DE) $\Longrightarrow$ existence for (FPKE)

BUT: $\not\Longleftarrow$

Remark: Under very broad assumptions (see [Stannat, Memoirs AMS 1999])

- "$\Longleftarrow$" holds $\Leftrightarrow$ generalized Dirichlet form given by L is **quasi regular**.
- "$\Longleftarrow$" holds in finite dimensions under very general conditions, see [Figalli, JFA 2008],[Trevisian, PhD-Thesis, SNS Pisa 2014].

2. Relation to S(P)DE

Furthermore (under some integrability conditions):

- uniqueness for (FPKE) $\Longrightarrow$ weak (= martingale) uniqueness for (S(P)DE)

Reason. Stroock-Varadhan's proof is "stable" under integrability conditions.

BUT: $\not\Leftarrow$

Reason. If coefficients A and B are too singular (so that "quasi-regularity" fails to hold), there might exist solutions to (FPKE) which are **not** the marginals of a martingale solution to (S(P)DE).

Conditions:

(A) Each a^{ij} only depends on $t, x_1, x_2, \ldots, x_{\max\{i,j\}}$, is continuous and for every $N \in \mathbb{N}$ and $A_N := (a^{ij})_{1\le i,j\le N}$ there exist $\gamma_N,\ \lambda_N \in (0,\infty)$ and $\beta_N \in (0,1]$ such that for all $x, y \in \mathbb{R}^N,\ t \in [0, T_0]$

$$\gamma_N |y|^2 \le \langle A_N(x,t)y,\ y\rangle_{\mathbb{R}^N} \le \gamma_N^{-1}|y|^2,\ \|A_N(x,t) - A_N(y,t)\| \le \lambda_N |x-y|^{\beta_N}.$$

Consider a **convex** set $\mathcal{P}_\nu$ of solutions $\mu = \mu_t\, \mathrm{d}t$ to (FPKE) such that:

(B) $|B^k| \in L^2\left(\mathbb{R}^\infty \times [0, T_0],\ \mu\right),\ k \in \mathbb{N}$, and $\exists N_\ell \in \mathbb{N},\ N_\ell \nearrow \infty$ as $\ell \to \infty$ and $C_b^{2,1}$-mappings $b_\ell : \mathbb{R}^{N_\ell} \times [0, T_0] \to \mathbb{R}^{N_\ell}$ such that for $B_\ell := \left(B^1, \ldots, B^{N_\ell}\right)$

$$\lim_{\ell\to\infty} \int_0^{T_0} \int_{\mathbb{R}^\infty} \left| A_{N_\ell}(x,t)^{-\frac{1}{2}} \left(B_\ell(x,t) - b_\ell\left(P_{N_\ell}x, t\right)\right)\right|^2 \mu_t(dx)\, \mathrm{d}t = 0.$$

Example

(i) ("triangular case") Each component B^k depends only on $x_1, \dots, x_k$ and is in $L^2(\mathbb{R}^\infty \times [0, T_0],\ \mu)$.

(ii) Suppose (for simplicity) $a^{ij} = \delta^{ij}\alpha_i$ with $\alpha_1 > 0$.
If

$$\sum_{k=1}^{\infty} \alpha_k^{-1} \int_0^{T_0} \int_{\mathbb{R}^\infty} \left| B^k(x,t) \right|^2 \mu_t(\,\mathrm{d}x)\,\mathrm{d}t < \infty,$$

then (simple exercise) $B = (B^k)$ fulfills (B).

(iii) $B = G + F$, G as in (i), F as in (ii). In this case one can take $\mathcal{P}_\nu$ as the set of all solutions to (FPKE) such that G satisfies (i) and F satisfies (ii).

Theorem I. Suppose (A) holds and let $\mathcal{P}_\nu$ be as above. Then $\#\mathcal{P}_\nu \leq 1$.

Proof. Assume that $\sigma^1 = \sigma_t^1 dt$ and $\sigma^2 = \sigma_t^2 dt$ belong to $\mathcal{P}_\nu$. Then $\sigma := (\sigma^1 + \sigma^2)/2 \in \mathcal{P}_\nu$. Let $d \in \mathbb{N}$, $\psi \in C_0^\infty(\mathbb{R}^d)$ and $|\psi(x)| \leq 1$ for all $x \in \mathbb{R}^d$. By condition (B) for every $\varepsilon > 0$ there exist a natural number $N \geq d$ and a $C_b^{2,1}$-mapping $b = (b^k)_{k=1}^N : \mathbb{R}^N \times [0, T_0] \to \mathbb{R}^N$ such that

$$\int_0^{T_0} \int_{\mathbb{R}^\infty} |A_N^{-1/2}(x, s)(B_N(x, s) - b(x_1, \ldots, x_N, s))|^2 \, \sigma_s(dx) \, ds < \varepsilon. \qquad (*)$$

3. Uniqueness: Non-degenerate Case

Fix " $\mathrm{d}t$-a.e." $t \in [0, T_0]$. Let f be a solution to the finite-dimensional Cauchy problem

$$\begin{cases} \partial_t f + \sum_{i,j=1}^N a^{ij}\partial_{x_i}\partial_{x_j} f + \sum_{i=1}^N b^i \partial_{x_i} f = 0 \quad \text{on } \mathbb{R}^N \times (0,t), \\ f(t,x) = \psi(x). \end{cases} \tag{PDE$_N$}$$

By standard PDE-theory such a solution exists and belongs to the class $C_b(\mathbb{R}^N \times [0,t]) \bigcap C_b^{2,1}(\mathbb{R}^N \times (0,t))$. Moreover, according to the maximum principle $|f(x,s)| \leq 1$ for all $(x,s) \in \mathbb{R}^N \times [0,t]$.

Set $\mu = \sigma^1 - \sigma^2$. The measure μ solves (FPKE) with zero initial condition. Hence

$$\int_{\mathbb{R}^\infty} f(x,t)\,\mu_t(dx) = \int_0^t \int_{\mathbb{R}^\infty} \Big[\partial_s f + \sum_{i,j=1}^N a^{ij}\partial_{x_j}\partial_{x_i} f + \sum_{i=1}^N B^i \partial_{x_i} f\Big]\, d\mu_s\, ds.$$

Hence by (PDE$_N$)

$$\int_{\mathbb{R}^\infty} \psi\, d\mu_t = \int_0^t \int_{\mathbb{R}^\infty} \langle B - b, \nabla f\rangle\, d\mu_s\, ds. \qquad (**)$$

3. Uniqueness: Non-degenerate Case

Let us estimate:

$$\int_0^t \int_{\mathbb{R}^\infty} |\sqrt{A_N}\nabla f|^2 \, d\sigma_s \, ds.$$

Using (FPKE) for σ and $\varphi = f^2$, taking into account that $(\partial_s + L)(f^2) = 2f(\partial_s + L)f + 2|\sqrt{A_N}\nabla f|^2$, we obtain from (PDE$_N$) that

$$\int_{\mathbb{R}^\infty} \psi^2 \, d\sigma_t - \int_{\mathbb{R}^\infty} f^2(x,0)\,\nu(dx) = 2\int_0^t \int_{\mathbb{R}^\infty} \Big[|\sqrt{A_N}\nabla f|^2 + f \sum_{i=1}^N (B^i - b^i)\partial_{x_i} f \Big] \, d\sigma_s \, ds.$$

Therefore,

$$\int_0^t \int_{\mathbb{R}^\infty} |\sqrt{A_N}\nabla f|^2 \, d\sigma_s \, ds \le 2 + \int_0^{T_0} \int_{\mathbb{R}^\infty} |A_N^{-1/2}(x,s)(B_N(x,s) - b(x_1,\dots,x_N,s))|^2 \, \sigma_s(dx) \, ds,$$

hence by $(*)$

$$\int_0^t \int_{\mathbb{R}^\infty} |\sqrt{A_N}\nabla f|^2 \, d\sigma_s \, ds \le 2 + \varepsilon. \qquad (***)$$

3. Uniqueness: Non-degenerate Case

Applying $(**)$, $(***)$ and the fact that $|\mu| \leq \sigma^1 + \sigma^2 = 2\sigma$ we have

$$\int_{\mathbb{R}^\infty} \psi \, d\mu_t \leq 2\sqrt{\varepsilon(2+\varepsilon)},$$

hence (letting $\varepsilon \to 0$)

$$\int_{\mathbb{R}^\infty} \psi \, d\mu_t \leq 0.$$

Replacing ψ with $-\psi$ we obtain

$$\int_{\mathbb{R}^\infty} \psi \, d\mu_t = 0 \text{ for all } \psi \in C_0^\infty(\mathbb{R}^d),\ d \in \mathbb{N}.$$

Hence $\mu_t \equiv 0$ for $\mathrm{d}t$-a.e. $t \in [0, T_0]$, i.e.

$$\sigma_1 = \sigma_2.$$

□

Let $D \subset \mathbb{R}^d,\ d \in \mathbb{N},\ D$ open, bounded, and Δ_D the Dirichlet Laplacian on $L^2(D)$ with eigenbasis $e_k,\ k \in \mathbb{N}$, and eigenvalues $-\lambda_k^2,\ k \in \mathbb{N}$. Then $L^2(D) \cong \ell^2 \subset \mathbb{R}^\infty$.
As before consider

$$L\varphi = \sum_{i,j=1}^{\infty} a^{ij} \partial_{e_i} \partial_{e_j} \varphi + \sum_{i=1}^{\infty} B^i \partial_{e_i} \varphi, \quad \varphi \in \mathrm{Dom}(L),$$

or associated (heuristic) S(P)DE (with $\sigma = \sqrt{A}$)

$$\mathrm{d}X(t) = B(X(t), t)\,\mathrm{d}t + \sigma(X(t), t)\,\mathrm{d}W(t).$$

Assume $A = (a^{ij})$ satisfies condition (A) with $\gamma_N = \gamma > 0$. Let

$$B(u, t)(\xi) := \Delta_D u(\xi) + f(t, \xi, u(\xi)), \quad u \in L^2(D),$$

for some Borel $f : [0, T_0] \times D \times \mathbb{R} \to \mathbb{R}$.

More precisely, $B = (B^i) : \mathbb{R}^\infty \times [0, T_0] \to \mathbb{R}^\infty$ is defined by

$$B^i(u,t) := \begin{cases} -\lambda_i^2 u_i + \langle f(t,\cdot,u), e_i\rangle_{L^2(D)}, \text{ if } u = (u_i) = \big(\langle e_i, u\rangle_{L^2(D)}\big) \in \ell^2, \\ 0, \text{ if } u \in \mathbb{R}^\infty \backslash \ell^2. \end{cases}$$

Proposition. Suppose $\exists$ Borel $C : [0, T_0] \to [0,\infty)$ and $m \geq 1$ such that

$$|f(t,\xi,u)| \leq C(t)(1 + |u|^m), \text{ for all } (t,\xi,u) \in [0, T_0] \times D \times \mathbb{R}.$$

Then there exists **at most one** solution $\mu = \mu_t \,\mathrm{d}t$ to (FPKE) such that

$$\int_0^{T_0} C(t)^2 \int_{L^2(D)} \|u\|^{2m}_{L^{2m}(D)} \,\mu_t(\mathrm{d}u)\,\mathrm{d}t < \infty. \qquad (\mathrm{I}_{RD})$$

Proof. Consequence of Theorem I. □

Remark. Existence of a solution to (FPKE) satisfying (I_{RD}) is known, e.g. if $D = (0,1)$, $a^{ij} = \delta^{ij}\alpha$, $\alpha > 0$, and if

$$f(t,\xi,u) = f_1(t,\xi,u) + f_2(t,\xi,u),$$

where $f_i(\cdot,\xi,\cdot)$, $i = 1,2$, is continuous for all $\xi \in D$ and for some $c_1, c_3 \in L^2((0,T_0))$, $c_2 \in L^1((0,T_0))$

(i) $|f_1(t,\xi,u)| \le c_1(t)(1+|u|^m)$

(ii) $(f_1(t,\xi,u) - f_1(t,\xi,v))(u-v) \le c_2(t)|u-v|^2$

(iii) $|f_2(t,\xi,u)| \le c_3(t)(1+|u|)$,

provided

$$\int_{L^2(D)} \|u\|_{L^{2m}(D)}^{2m} \nu(\mathrm{d}u) < \infty.$$

(See e.g. Bogachev/DaPrato/R.: JDE 2010).

Conditions:

(A′) Each a^{ij} only depends on $t, x_1, x_2, \dots, x_{\max\{i,j\}}$, is bounded and for each $N \in \mathbb{N}$ the matrix $A_N := (a^{ij})_{1 \le i,j \le N}$ is nonnegative such that $\sigma_N = \sqrt{A_N}$ has components σ_N^{ij} in $C^\infty(\mathbb{R}^N \times [0, T_0])$.

Consider a **convex** set $\mathcal{P}_\nu$ of solutions $\mu = \mu_t \,\mathrm{d}t$ to (FPKE) such that:

(B′) $|B^k| \in L^1(\mathbb{R}^\infty \times [0, T_0],\ \mu)$, $k \in \mathbb{N}$, and $\exists\ N_\ell \in \mathbb{N}$, $N_\ell \nearrow \infty$ as $\ell \to \infty$ and C^∞-mappings $b_\ell = (b_\ell^i)_{1 \le i \le N_\ell} : \mathbb{R}^{N_\ell} \times [0, T_0] \to \mathbb{R}^{N_\ell}$, a Borel $\Theta_\ell : \mathbb{R}^{N_\ell} \to \mathbb{R}$, $V_\ell \in C^2(\mathbb{R}^{N_\ell})$ with $V_\ell \ge 1$ and $C_\ell \in [0, \infty)$, $\delta_\ell \in (0, \infty)$ such that for $B_\ell := (B^1, \dots, B^{N_\ell})$:

(i) $(V_\ell \circ P_{N_\ell})^{\frac{1}{2}} \in L^1(\mathbb{R}^\infty \times [0, T_0],\ \mu)$ and

$$\lim_{\ell \to \infty} \int_0^{T_0} \int_{\mathbb{R}^\infty} \left| B_\ell(x, t) - b_\ell\left(P_{N_\ell} x, t\right) \right| \left(V_\ell\left(P_{N_\ell} x\right) \right)^{\frac{1}{2}} e^{C_\ell (T_0 - t)/2} \mu_t(\,\mathrm{d}x)\,\mathrm{d}t = 0,$$

(ii) the matrix $\mathcal{B}_\ell = \left(\partial_{x_j} b_\ell^i\right)$ and the operator

$$L_\ell \varphi = \sum_{i,j \le N_\ell} a^{ij} \partial_{x_i} \partial_{x_j} \varphi + \sum_{i \le N_\ell} b_\ell^i \partial_{x_i} \varphi$$

satisfy for all $(x,t) \in \mathbb{R}^{N_\ell} \times [0, T_0]$, $h \in \mathbb{R}^{N_\ell}$,

$$\langle \mathcal{B}_\ell(x,t)h,\ h\rangle_{\mathbb{R}^{N_\ell}} \le \Theta_\ell(x)|h|^2,$$ ("weighted monotonicity")

$$L_\ell V_\ell(x,t) \le \left(C_\ell - \Lambda_\ell(x,t)\right) V_\ell(x),$$ ("strong Lyapunov function")

where

$$\Lambda_\ell(x,t) := 4 \sum_{i,j,k \le N} \left|\partial_{x_k} \sigma_N^{ij}(x,t)\right|^2 + 2\Theta_\ell(x) + \delta_\ell \frac{|b_\ell(x,t)|^2}{1+|x|^2}.$$

Theorem II Suppose (A′) holds and let $\mathcal{P}_\nu$ be as above. Then $\#\mathcal{P}_\nu \le 1$.

Proof. Again reduction to finite dimensions and PDE-theory. See also [R./Sobol, Ann. Probab. 2006]. □

- **(Stochastic) Generalized Burgers Equation**

Let $D := (0,1) \subset \mathbb{R}^1$ and Δ_D the Dirichlet Laplacian on $L^2(D)$ with eigenbasis $e_k,\ k \in \mathbb{N}$, and eigenvalues $-\lambda_k^2,\ k \in \mathbb{N}$. Then $L^2(D) \cong \ell^2 \subset \mathbb{R}^\infty$.
As before consider

$$L\varphi = \sum_{i,j=1}^{\infty} a^{ij} \partial_{e_i} \partial_{e_j} \varphi + \sum_{i=1}^{\infty} B^i \partial_{e_i} \varphi,\ \varphi \in \mathrm{Dom}(L),$$

or associated (heuristic) S(P)DE (with $\sigma = \sqrt{A}$)

$$\mathrm{d}X(t) = B(X(t),t)\,\mathrm{d}t + \sigma(X(t),t)\,\mathrm{d}W(t).$$

Assume that $a^{ij} = \langle Se_i, e_j \rangle_{L^2}$ for some symmetric nonnegative operator S on $L^2(0,1)$, $\mathrm{Tr}\ S < \infty$. For $m,\ \ell \in \mathbb{N},\ 2 \le m \le \ell + 2$, define

$$B(u)(\xi) := \Delta_D u(\xi) + \partial_\xi (u^m(\xi)) - u^{2\ell+1}(\xi),\ u \in L^2(D),$$

more precisely, $B = (B^i) : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ is defined by

$$B^i(u) := \begin{cases} -\lambda_i^2 u_i - \langle u^m,\ \partial_\xi e_i\rangle_{L^2(D)} - \langle u^{2\ell+1},\ e_i\rangle_{L^2(D)}, & \text{if } u = (u_i) = \Big(\langle e_i, u\rangle_{L^2(D)}\Big) \in \ell^2, \\ 0, & \text{if } u \in \mathbb{R}^\infty \setminus \ell^2. \end{cases}$$

Proposition There exists **at most one** solution $\mu = \mu_t\,\mathrm{d}t$ to (FPKE) such that for some $\kappa' > 0$

$$\int_0^{T_0} \int_{L^2(D)} \left[\|u\|_{L^{4\ell+2}(D)}^{2\ell+1} + \|u^m\|_{H_0^1(D)}\right] \exp\left(\kappa' \|u\|_{L^{2m-2}(D)}^{2m-2}\right) \mu_t(\,\mathrm{d}u)\,\mathrm{d}t < \infty. \qquad (\mathrm{I}_{GB})$$

Proof. Consequence of Theorem II. □

Remark

(i) Existence of solution to (FPKE) satisfying (I_{GB}) follows from a general result in Section 7 below (see Section 8 for more details).

(ii) Above Proposition partly generalizes uniqueness in [Gyongy/Rovira: SPA 2000]. Their results applies if $\ell = 0$.

- **2D Navier-Stokes Equation**

Let us first look at the dD-case: Let $D \subset \mathbb{R}^d$, open, bounded and

$$V_2 := \left\{ u = (u^j)_{1 \le j \le d} \,\middle|\, u^j \in H_0^1(D) \text{ for } j = 1, \dots, d \text{ and div } u = 0 \right\}$$

with norm

$$\|u\|_{V_2} := \left(\sum_{j=1}^{d} \|\nabla_\xi u^j\|_{L^2(D)}^2 \right)^{\frac{1}{2}}$$

Let $\quad H :=$ closure of V_2 in $L^2(D;\ \mathbb{R}^d)$

and $P_H := L^2(D;\ \mathbb{R}^d) \to H$ orthogonal ("Leray-Helmholtz") projection.
Let $\eta_k \in V_2,\ k \in \mathbb{N}$, be the eigenbasis of the Stokes-Laplacian Δ on $H \cong \ell^2 \subset \mathbb{R}^\infty$ with eigenvalues $-\lambda_k^2,\ k \in \mathbb{N}$. As before consider

$$L\varphi = \sum_{i,j=1}^{\infty} a^{ij} \partial_{\eta_i} \partial_{\eta_j} \varphi + \sum_{i=1}^{\infty} B^i \partial_{\eta_i} \varphi, \quad \varphi \in \mathrm{Dom}(L),$$

resp. (with $\sigma = \sqrt{A}$)

$$\mathrm{d}X(t) = B(X(t), t)\, \mathrm{d}t + \sigma(X(t), t)\, \mathrm{d}W(t).$$

6 Applications to (Stochastic) Generalized Burgers and 2D Navier-Stokes Equations

Assume $a^{ij} = \langle S\eta_i, \eta_j\rangle_{L^2(D)}$ with $\sum\limits_i a^{ii}\lambda_i^2 < \infty$ where S is symmetric nonnegative bounded operator on H, and define

$$B(u)(\xi) := P_H\Delta u(\xi) - P_H\left(\langle u, \nabla\rangle u\right)(\xi),$$

more precisely, $B = (B^i) : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is defined by

$$B^i(u) := \begin{cases} \langle u, \Delta\eta_i\rangle_{L^2} + \langle u, \langle u, \nabla\rangle\eta_i\rangle_{L^2(D)}, \text{ if } u = (\langle\eta_i,\ u\rangle_H) \in \ell^2, \\ 0, \text{ if } u \in \mathbb{R}^\infty \setminus \ell^2. \end{cases}$$

Proposition. Suppose $d = 2$. Then there exists **at most one** solution $\mu = \mu_t\,\mathrm{d}t$ to (FPKE) such that for some $\delta > 0$

$$\int_0^{T_0}\int_H \left(1 + \|\Delta u\|_{L^2}^2\right) e^{\delta\|u\|_{V_2}^2}\mu_t(\,\mathrm{d}u)\,\mathrm{d}t < \infty, \qquad (\mathrm{I}_{NV})$$

where $\|\Delta u\|_{L^2}^2 := \infty$ if $u \notin H^{2,2}$.

Proof. Theorem II. □

Remark Existence of solution to (FPKE) satisfying (I_{NV}) follows from a general result in Section 7 below (see Section 8 for more details).

7. Existence: General Results

Consider the general situation described in Section 1. We have the following two existence results for solution of (FPKE) depending on the existence of Lyapunov functions $V : \mathbb{R}^\infty \to [1, +\infty]$ of either polynomial type (Theorem III below) or exponential type (Theorem IV below).
For $n \in \mathbb{N}$ we define

$$H_n := \text{ linear span of } \left\{ e_i \;\middle|\; i \in \mathbb{N} \right\} \quad (\subset \mathbb{R}^\infty),$$

where $e_i = (\delta_{ij})_{j\in\mathbb{N}} = (0, \dots, 0, 1, 0, \dots)$.

Theorem III. Suppose that there exists a compact function $\Theta : \mathbb{R}^\infty \to [0, +\infty]$, with compact sublevel sets, finite on each H_n and such that the functions a^{ij} and B^i are continuous in x on all sublevel sets $\{\Theta \leq R\}$,and there exist numbers $M_0, C_0 \geq 0$ and a Borel function $V\colon \mathbb{R}^\infty \to [1, +\infty]$ whose sublevel sets $\{V \leq R\}$ are compact and whose restrictions to H_n are of class C^2 and such that for all $x \in H_n$, $n \geq 1$, one has

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\partial_{e_i}V(x)\partial_{e_j}V(x) \leq M_0 V(x)^2, \quad LV(x,t) \leq C_0 V(x) - \Theta(x). \tag{1}$$

Assume also that there exist constants $C_i \geq 0$ and $k_i \geq 0$ such that for all i and $j \leq i$ one has

$$|a^{ij}(x,t)| + |B^i(x,t)| \leq C_i V(x)^{k_i}(1 + \delta(\Theta(x))\Theta(x)), \ (x,t) \in \mathbb{R}^\infty \times [0, T_0], \tag{2}$$

where δ is a bounded nonnegative Borel function on $[0, +\infty)$ with $\lim\limits_{s\to\infty} \delta(s) = 0$.

Then, if for all $k \in \mathbb{N}$, $W_k := \sup\limits_n \int_{\mathbb{R}^\infty} V^k \circ P_n \, \mathrm{d}\nu < \infty$ for the initial distribution ν, (FPKE) has a solution $\mu = \mu_t \, dt$ such that for all $t \in [0, T_0]$

$$\int_{\mathbb{R}^\infty} V^k \, d\mu_t + k \int_0^t \int_{\mathbb{R}^\infty} V^{k-1}\Theta \ \mathrm{d}\mu_s \ \mathrm{d}s \leq N_k W_k \quad \forall\, k \in \mathbb{N},$$

where $N_k := M_k e^{M_k} + 1, \quad M_k := k(C_0 + (k-1)M_0)$. In particular, $\mu_t(V < \infty) = 1$ for all t and $\mu_t(\Theta < \infty) = 1$ for $\mathrm{d}t$-a.e. $t \in [0, T_0]$.

Theorem IV. Suppose that in Theorem III condition (1) is replaced by

$$LV(x,t) \leq V(x) - V(x)\Theta(x) \tag{1'}$$

and (2) is replaced by

$$|a^{ij}(x,t)| + |B^i(x,t)| \leq C_i(1 + \delta(V(x)\Theta(x))V(x)\Theta(x)), \ (x,t) \in \mathbb{R}^\infty \times [0, T_0]. \tag{2'}$$

Then, if $W_1 := \sup_n \int_{\mathbb{R}^\infty} V \circ P_n \, \mathrm{d}\nu < \infty$ for the initial distribution ν, (FPKE) has a solution $\mu = \mu_t \, dt$ such that for all $t \in [0, T_0]$

$$\int_{\mathbb{R}^\infty} V \, d\mu_t + \int_0^t \int_{\mathbb{R}^\infty} V\Theta \, d\mu_s \, ds \leq 4W_1.$$

- **(Stochastic) Generalized Burgers Equation**

Consider the situation in Section 6.1 and additionally assume that

$$a^{ij} = \langle Se_i, e_j \rangle_{L^2(0,1)}$$

with S a symmetric, nonnegative operator on $L^2(0,1)$, $Tr\ S < \infty$. Take

$$V(u) := \left(1 + \|u\|^2_{L^2(0,1)} + \|u\|^{2\ell+2}_{L^{2\ell+2}(0,1)}\right) \exp\left(\delta \|u\|^{2m-2}_{L^{2m-2}(0,1)}\right)$$

for small enough $\delta > 0$ and for a suitably small constant C_δ

$$\Theta(u) := C_\delta \left(1 + \|u\|^{4\ell+2}_{L^{4\ell+2}(0,1)} + \|u\|^2_{H^1_0(0,1)} + \|u^{m-1}\|^2_{H^1_0(0,1)}\right).$$

Then Theorem IV applies to give existence of a solution $\mu = \mu_t \, \mathrm{d}t$ to (FPKE), if $\sup_n \int V \circ P_n \, \mathrm{d}\nu < \infty$, such that $V \cdot \Theta \in L^1(\mathbb{R}^\infty \times [0, T_0], \ \mu)$. In particular, since by Sobolev embedding $H_0^1(0,1) \subset L^\infty(0,1)$ continuously and hence $\|u^m\|_{H_0^1(0,1)} \leq \|u\|^2_{H_0^1(0,1)} + \|u^{m-1}\|^2_{H_0^1(0,1)}$, also (I_{GB}) holds. So, this μ is the unique solution of (FPKE) satisfying (I_{GB}).

- **(Stochastic) Perturbed dD Navier-Stokes Equation**

Consider the situation in Section 6.2 and additionally assume that

$$a^{ij} = \delta^{ij}\alpha_i \text{ with } \alpha_i \in [0, \infty) \text{ such that } \sum_{i=1}^{\infty} \alpha_i < \infty.$$

Take for some suitable $C \in (0, \infty)$

$$V(u) := \|u\|_H^2 + 1 \text{ and } \Theta(u) := C\|u\|_{V_2}^2$$

Then Theorem III applies to give existence of a solution $\mu = \mu_t \, \mathrm{d}t$ to (FPKE), if $\sup \int V^k \circ P_n \, \mathrm{d}\nu < \infty$ for all $k \in \mathbb{N}$, such that $V^k\Theta \in L^1(\mathbb{R}^\infty \times [0, T_0], \mu)$ and $V^k \in L^1(\mathbb{R}^\infty, \mu_t)$ for all $t \in [0, T_0]$ and all $k \in \mathbb{N}$. In particular, $\mu_t(H) = 1$ for all $t \in [0, T_0]$ and $\mu_t(V_2) = 1$ for $\mathrm{d}t$-a.e. $t \in [0, T_0]$.

Remark The same result also holds if we replace B in Section 6.2 by $B + F$, where $F(u,t)(\xi) = f(u(\xi), t)$, $\xi \in D$, with $f : \mathbb{R}^d \times [0, T_0] \to \mathbb{R}$ bounded and continuous.

If $d = 2$, assume

$$\sum_{n=1}^{\infty} \alpha_n \lambda_n^2 < \infty$$

and take for some $\delta > 0$

$$V(u) := \exp\left(\delta \|u\|_{V_2}^2\right), \Theta(u) := 1 - \delta \sum_{n=1}^{\infty} \alpha_n \lambda_n^2 + \delta \|\Delta u\|_{L^2(D)}^2,$$

where again we set $V(u) = \infty$ and $\Theta(u) = \infty$ if $u \in \mathbb{R}^\infty \setminus V_2$ (using that $V_2 \subset H \cong \ell^2 \subset \mathbb{R}^\infty$). Then Theorem IV applies to give existence of a solution $\mu = \mu_t \, \mathrm{d}t$ to (FPKE), if $\sup_n \int V \circ P_n \, \mathrm{d}\nu < \infty$, such that $V\Theta \in L^1(\mathbb{R}^\infty \times [0, T_0], \mu)$. In particular, also (I_{NV}) holds. So, this μ is the unique solution of (FPKE) satisfying (I_{NV}).