Minimal thinness for jump processes

Panki Kim

Department of Mathematical Sciences
Seoul National University

Seoul National University
PDE and Functional Analysis
Research Center

German-Japanese conference on Stochastic Analysis and Applications
Katahira Campus, Tohoku University
August 31–September 4, 2015

References

This talk is based on joint works with

Renming Song (University of Illinois, USA) & Zoran Vondraček (University of Zagrab, Croatia)

References

This talk is based on joint works with

Renming Song (University of Illinois, USA) & Zoran Vondraček (University of Zagrab, Croatia)

KSV1 K, Song & Vondraček, Minimal thinness with respect to symmetric Lévy processes. To appear in *Trans. Amer. Math. Soc.* 2015.

KSV2 K, Song & Vondraček: Minimal thinness with respect to subordinate killed Brownian motions. Preprint 2015. arXiv:1503.03153

Table of Contents

Thin sets

Let $X = (X_t, \mathbb{P}_x)$ be a Markov process in $\mathbb{R}^d$, $E \subset \mathbb{R}^d$ and $T_E = \inf\{t > 0 : X_t \in E\}$. We say that E is thin at y with respect to X if $\mathbb{P}_y(T_E = 0) = 0$; X starting at y does not hit E immediately.

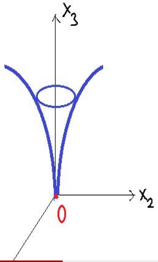
Thin sets

Let $X = (X_t, \mathbb{P}_x)$ be a Markov process in $\mathbb{R}^d$, $E \subset \mathbb{R}^d$ and $T_E = \inf\{t > 0 : X_t \in E\}$. We say that E is thin at y with respect to X if $\mathbb{P}_y(T_E = 0) = 0$; X starting at y does not hit E immediately.

Thinness is a concept which describes smallness of a set E at the point y with respect to the process X.

Classical example: the Lebesgue thorn set

Let $f : [0, \infty) \to [0, \infty)$ be increasing, $f(r) > f(0)$ for all $r > 0$, $f(r)/r$ non-decreasing for r small. Let $E = \{x = (\widetilde{x}, x_d) : |\widetilde{x}| < f(x_d)\}$.

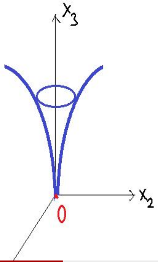


Classical example: the Lebesgue thorn set

Let $f : [0,\infty) \to [0,\infty)$ be increasing, $f(r) > f(0)$ for all $r > 0$, $f(r)/r$ non-decreasing for r small. Let $E = \{x = (\widetilde{x}, x_d) : |\widetilde{x}| < f(x_d)\}$. Then E is thin at 0 with respect to Brownian motion iff

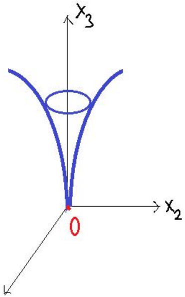
$$\int_0^1 \left(\frac{f(r)}{r}\right)^{d-3} \frac{dr}{r} < \infty\,, \qquad d \geq 4\,,$$

$$\int_0^1 \left|\log \frac{f(r)}{r}\right|^{-1} \frac{dr}{r} < \infty\,, \qquad d = 3\,.$$



The Lebesgue thorn for the isotropic α-stable process

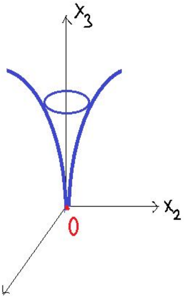
Let $f : [0, \infty) \to [0, \infty)$ be increasing, $f(r) > f(0)$ for all $r > 0$, $f(r)/r$ non-decreasing for r small. Let $E = \{x = (\widetilde{x}, x_d) : |\widetilde{x}| < f(x_d)\}$, $d \geq 3$.



The Lebesgue thorn for the isotropic α-stable process

Let $f : [0, \infty) \to [0, \infty)$ be increasing, $f(r) > f(0)$ for all $r > 0$, $f(r)/r$ non-decreasing for r small. Let $E = \{x = (\widetilde{x}, x_d) : |\widetilde{x}| < f(x_d)\}$, $d \geq 3$. Then E is thin at 0 with respect to the isotropic α-stable process if and only if

$$\int_0^1 \left(\frac{f(r)}{r}\right)^{d-\alpha-1} \frac{dr}{r} < \infty .$$



Martin boundary

Let $D \subset \mathbb{R}^d$ be an open connected Greenian set and $G_D^W(x, y)$ its Green's function for Brownian motion W on D.

Martin boundary

Let $D \subset \mathbb{R}^d$ be an open connected Greenian set and $G_D^W(x,y)$ its Green's function for Brownian motion W on D.
Fix $x_0 \in D$ and define

$$M_D^W(x,y) = \frac{G_D^W(x,y)}{G_D^W(x_0,y)}, \quad x,y \in D,$$

the Martin kernel. Then D has a Martin boundary $\partial_M D$ and $M_D^W(x,\cdot)$ extends continuously to $\partial_M D$. The function $M_D^W(x,z)$, $x \in D$, $z \in \partial_M D$ is the Martin kernel.

Martin boundary

Let $D \subset \mathbb{R}^d$ be an open connected Greenian set and $G_D^W(x, y)$ its Green's function for Brownian motion W on D.
Fix $x_0 \in D$ and define

$$M_D^W(x, y) = \frac{G_D^W(x, y)}{G_D^W(x_0, y)}, \quad x, y \in D,$$

the Martin kernel. Then D has a Martin boundary $\partial_M D$ and $M_D^W(x, \cdot)$ extends continuously to $\partial_M D$. The function $M_D^W(x, z)$, $x \in D$, $z \in \partial_M D$ is the Martin kernel.
A point $z \in \partial_M D$ is called a minimal Martin boundary point if $M_D^W(\cdot, z)$ is a minimal harmonic function in D. The minimal Martin boundary is denoted by $\partial_m D$.

Martin boundary

Let $D \subset \mathbb{R}^d$ be an open connected Greenian set and $G_D^W(x,y)$ its Green's function for Brownian motion W on D.
Fix $x_0 \in D$ and define

$$M_D^W(x,y) = \frac{G_D^W(x,y)}{G_D^W(x_0,y)}, \quad x,y \in D,$$

the Martin kernel. Then D has a Martin boundary $\partial_M D$ and $M_D^W(x,\cdot)$ extends continuously to $\partial_M D$. The function $M_D^W(x,z)$, $x \in D$, $z \in \partial_M D$ is the Martin kernel.
A point $z \in \partial_M D$ is called a minimal Martin boundary point if $M_D^W(\cdot,z)$ is a minimal harmonic function in D. The minimal Martin boundary is denoted by $\partial_m D$.

If D is a bounded Lipschitz domain, then the Martin and the minimal Martin boundary with respect to W are identified with ∂D.

Minimal thinness

A subset $E \subset D$ is minimally thin in D at $z \in \partial_m D$ with respect to killed Brownian motion W^D if

$$\mathbb{E}_\cdot[M_D^W(W_{T_E}^D, z)] \neq M_D^W(\cdot, z).$$

Minimal thinness

A subset $E \subset D$ is minimally thin in D at $z \in \partial_m D$ with respect to killed Brownian motion W^D if

$$\mathbb{E}_\cdot[M_D^W(W_{T_E}^D, z)] \neq M_D^W(\cdot, z)\,.$$

Probabilistic interpretation: Let $(W^{D,z}, \mathbb{P}_x^z)$ be the Brownian motion conditioned to exit D at z – Doob's h-transform with the harmonic function $h = M_D^W(\cdot, z)$. The lifetime of $W^{D,z}$ is ζ and $W_{\zeta-}^{D,z} = z$. Then A is minimally thin in D at $z \in \partial_m D$ if and only if $\mathbb{P}_x^z(T_A < \zeta) < 1$ for some $x \in D$.

Minimal thinness

A subset $E \subset D$ is minimally thin in D at $z \in \partial_m D$ with respect to killed Brownian motion W^D if

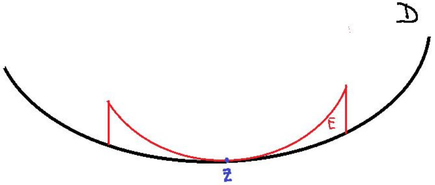
$$\mathbb{E}_\cdot[M_D^W(W^D_{T_E}, z)] \neq M_D^W(\cdot, z)\,.$$

Probabilistic interpretation: Let $(W^{D,z}, \mathbb{P}_x^z)$ be the Brownian motion conditioned to exit D at z – Doob's h-transform with the harmonic function $h = M_D^W(\cdot, z)$. The lifetime of $W^{D,z}$ is ζ and $W^{D,z}_{\zeta-} = z$. Then A is minimally thin in D at $z \in \partial_m D$ if and only if $\mathbb{P}_x^z(T_A < \zeta) < 1$ for some $x \in D$.

Minimal thinness in the half-space $\mathbb{H} = \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$ introduced by J. Lelong-Ferrand in 1949. Minimal thinness in a general open set developed L. Naïm in 1957. Probabilistic interpretation by J. Doob in 1957.

Integral tests for minimal thinness

Let $D \subset \mathbb{R}^d$ have smooth boundary, $E \subset D$ and $z \in \partial D$.

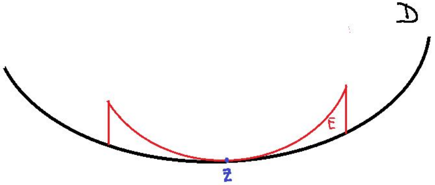


Integral tests for minimal thinness

Let $D \subset \mathbb{R}^d$ have smooth boundary, $E \subset D$ and $z \in \partial D$. If

$$\int_{E\cap B(z,1)} |x-z|^{-d}\, dx = \infty\,,$$

then E is not minimally thin in D at z (A. Beurling 1965, $d = 2$, B. Dahlberg 1976, $d \geq 3$).

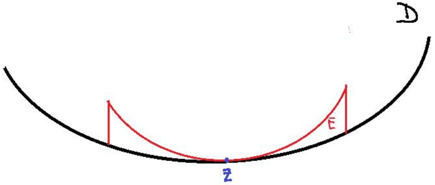


Integral tests for minimal thinness

Let $D \subset \mathbb{R}^d$ have smooth boundary, $E \subset D$ and $z \in \partial D$. If

$$\int_{E \cap B(z,1)} |x - z|^{-d}\, dx = \infty\,,$$

then E is not minimally thin in D at z (A. Beurling 1965, $d = 2$, B. Dahlberg 1976, $d \geq 3$). If E is the union of a subfamily of Whitney cubes of D, the converse is also true (H. Aikawa 1993).



Whitney decomposition

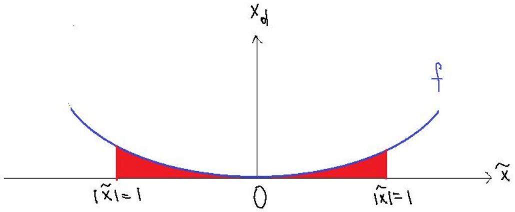
A Whitney decomposition of an open set D: a family $\{Q_j\}_{j\in\mathbb{N}}$ of closed cubes, with sides all parallel to the axes, satisfying the following properties:
(i) $D = \cup_j Q_j$,
(ii) $\text{int}(Q_j) \cap \text{int}(Q_k) = \emptyset$, $j \neq k$;
(iii) for any j, $\text{diam}(Q_j) \le \text{dist}(Q_j, \partial D) \le 4\text{diam}(Q_j)$,
where $\text{dist}(Q_j, \partial D)$ denotes the Euclidean distance between Q_j and ∂D.

Integral tests for minimal thinness, cont.

Let $f : \mathbb{R}^d \to [0, \infty)$ be a Lipschitz function. Then the set $E := \{x = (\widetilde{x}, x_d) \in \mathbb{H} : 0 < x_d \le f(\widetilde{x})\}$ is minimally thin in $\mathbb{H}$ at $z = 0$ if and only if

$$\int_{\{|\widetilde{x}|<1\}} f(\widetilde{x})|\widetilde{x}|^{-d}\, d\widetilde{x} < \infty\,;$$

K. Burdzy 1987 (probabilistic proof), S. J. Gardiner 1991.



Wiener-type criteria for minimal thinness (H. Aikawa 1993)

Let D be a smooth bounded domain; $E \subset D$ is minimally thin in D at $z \in \partial D$ if and only if

$$\sum_{j \geq 1} \mathrm{dist}(z, Q_j)^{-d} \mathrm{dist}(Q_j, \partial D)^2 \, \mathrm{Cap}(E \cap Q_j) < \infty \, .$$

Here $\{Q_j\}$ is a Whitney decomposition of D, Cap the Newtonian capacity.

Until very recently, no concrete criteria for minimal thinness with respect to jump processes (for subordinate Brownian motion in half space, see K, Song & Vondracek, 2012).

Until very recently, no concrete criteria for minimal thinness with respect to jump processes (for subordinate Brownian motion in half space, see K, Song & Vondracek, 2012).

Summary of the results in [KS1, KS2]

- We work with a broader class of purely discontinuous Markov processes and prove a version of Aikawa's Wiener-type criterion for minimal thinness at any finite (minimal Martin) boundary point.

Until very recently, no concrete criteria for minimal thinness with respect to jump processes (for subordinate Brownian motion in half space, see K, Song & Vondracek, 2012).

Summary of the results in [KS1, KS2]

- We work with a broader class of purely discontinuous Markov processes and prove a version of Aikawa's Wiener-type criterion for minimal thinness at any finite (minimal Martin) boundary point.
- We obtain criteria for minimal thinness of a subset of half-space-like open sets at infinity.

Until very recently, no concrete criteria for minimal thinness with respect to jump processes (for subordinate Brownian motion in half space, see K, Song & Vondracek, 2012).

Summary of the results in [KS1, KS2]

- We work with a broader class of purely discontinuous Markov processes and prove a version of Aikawa's Wiener-type criterion for minimal thinness at any finite (minimal Martin) boundary point.
- We obtain criteria for minimal thinness of a subset of half-space-like open sets at infinity.

Remark: In the classical case of the Laplacian, such results at infinity are direct consequences of the corresponding finite boundary point results by use of the inversion with respect to a sphere and the Kelvin transform, which is not available in our case.

Description of the processes

Let $X = (X_t, \mathbb{P}_x)$ be an isotropic unimodal Lévy process in $\mathbb{R}^d$: for each $t > 0$ there is a decreasing function $p_t : (0, \infty) \to (0, \infty)$ such that

$$\mathbb{P}_0(X_t \in A) = \int_A p_t(|x|)\, dx\,, \qquad A \subset \mathbb{R}^d \text{ Borel}\,.$$

Its Lévy measure ν of X has a *radial decreasing* density.

Description of the processes

Let $X = (X_t, \mathbb{P}_x)$ be an isotropic unimodal Lévy process in $\mathbb{R}^d$: for each $t > 0$ there is a decreasing function $p_t : (0, \infty) \to (0, \infty)$ such that

$$\mathbb{P}_0(X_t \in A) = \int_A p_t(|x|)\, dx\,, \qquad A \subset \mathbb{R}^d \text{ Borel}\,.$$

Its Lévy measure ν of X has a *radial decreasing* density.
The characteristic exponent Ψ of X defined by

$$\mathbb{E}_0\left[e^{i\langle x, X_t\rangle}\right] = e^{-t\Psi(x)}, \quad x \in \mathbb{R}^d.$$

Description of the processes

Let $X = (X_t, \mathbb{P}_x)$ be an isotropic unimodal Lévy process in $\mathbb{R}^d$: for each $t > 0$ there is a decreasing function $p_t : (0, \infty) \to (0, \infty)$ such that

$$\mathbb{P}_0(X_t \in A) = \int_A p_t(|x|)\, dx\,, \qquad A \subset \mathbb{R}^d \text{ Borel}\,.$$

Its Lévy measure ν of X has a *radial decreasing* density.
The characteristic exponent Ψ of X defined by

$$\mathbb{E}_0\left[e^{i\langle x, X_t\rangle}\right] = e^{-t\Psi(x)}\,, \quad x \in \mathbb{R}^d.$$

Ψ is a radial function; use interchangeably $\Psi(x)$ and $\Psi(|x|)$

Description of the processes, cont.

Scaling conditions:

(**H1**): There exist constants $0 < \delta_1 \le \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1 \lambda^{2\delta_1} \Psi(t) \le \Psi(\lambda t) \ \le a_2 \lambda^{2\delta_2} \Psi(t), \quad \lambda \ge 1, t \ge 1 \,.$$

Description of the processes, cont.

Scaling conditions:

(**H1**): There exist constants $0 < \delta_1 \le \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1 \lambda^{2\delta_1} \Psi(t) \le \Psi(\lambda t) \ \le a_2 \lambda^{2\delta_2} \Psi(t), \quad \lambda \ge 1, t \ge 1\,.$$

(**H2**): There exist constants $0 < \delta_3 \le \delta_4 < 1$ and $a_3, a_4 > 0$ such that

$$a_3 \lambda^{2\delta_3} \Psi(t) \le \Psi(\lambda t) \ \le a_4 \lambda^{2\delta_4} \Psi(t), \quad \lambda \le 1, t \le 1\,.$$

Description of the processes, cont.

Scaling conditions:

(**H1**): There exist constants $0 < \delta_1 \le \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1 \lambda^{2\delta_1} \Psi(t) \le \Psi(\lambda t) \ \le a_2 \lambda^{2\delta_2} \Psi(t), \quad \lambda \ge 1, t \ge 1\,.$$

(**H2**): There exist constants $0 < \delta_3 \le \delta_4 < 1$ and $a_3, a_4 > 0$ such that

$$a_3 \lambda^{2\delta_3} \Psi(t) \le \Psi(\lambda t) \ \le a_4 \lambda^{2\delta_4} \Psi(t), \quad \lambda \le 1, t \le 1\,.$$

(**H1**) governs small time – small space behavior of X, needed for local results; (**H2**) large time – large space, needed for behavior at ∞.

Description of the processes, cont.

Scaling conditions:

(**H1**): There exist constants $0 < \delta_1 \le \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1 \lambda^{2\delta_1} \Psi(t) \le \Psi(\lambda t) \le a_2 \lambda^{2\delta_2} \Psi(t), \quad \lambda \ge 1, t \ge 1 .$$

(**H2**): There exist constants $0 < \delta_3 \le \delta_4 < 1$ and $a_3, a_4 > 0$ such that

$$a_3 \lambda^{2\delta_3} \Psi(t) \le \Psi(\lambda t) \le a_4 \lambda^{2\delta_4} \Psi(t), \quad \lambda \le 1, t \le 1 .$$

(**H1**) governs small time – small space behavior of X, needed for local results; (**H2**) large time – large space, needed for behavior at ∞.

Description of the processes, cont.

Scaling conditions:

(**H1**): There exist constants $0 < \delta_1 \le \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1 \lambda^{2\delta_1} \Psi(t) \le \Psi(\lambda t) \le a_2 \lambda^{2\delta_2} \Psi(t), \quad \lambda \ge 1, t \ge 1 .$$

(**H2**): There exist constants $0 < \delta_3 \le \delta_4 < 1$ and $a_3, a_4 > 0$ such that

$$a_3 \lambda^{2\delta_3} \Psi(t) \le \Psi(\lambda t) \le a_4 \lambda^{2\delta_4} \Psi(t), \quad \lambda \le 1, t \le 1 .$$

(**H1**) governs small time – small space behavior of X, needed for local results; (**H2**) large time – large space, needed for behavior at ∞.
M. Zähle, P. Kim, R. Song, Z. Vondracek, T. Grzywny, K. Bogdan, M. Ryznar

Examples

(a) Isotropic α-stable process; $\Psi(x) = |x|^{\alpha}$, $\alpha \in (0, 2)$;

Examples

(a) Isotropic α-stable process; $\Psi(x) = |x|^\alpha$, $\alpha \in (0, 2)$;

(b) Independent sum of two isotropic stable processes;
$\Psi(x) = |x|^\alpha + |x|^\beta$, $\alpha, \beta \in (0, 2)$;

Examples

(a) Isotropic α-stable process; $\Psi(x) = |x|^\alpha$, $\alpha \in (0,2)$;

(b) Independent sum of two isotropic stable processes; $\Psi(x) = |x|^\alpha + |x|^\beta$, $\alpha, \beta \in (0,2)$;

(c) Subordinate Brownian motion via subordinator whose Laplace exponent ϕ satisfies scaling conditions similar to (**H1**) and (**H2**). For example, ϕ is comparable to a regularly varying function at zero and at infinity with (not necessarily same) indices from $(0,1)$: $\Psi(x) = \phi(|x|^2)$ where $\phi(x) \asymp |x|^\beta \ell_1(x)$, $|x| \to 0$, $\phi(x) \asymp |x|^\gamma \ell_2(x)$, $|x| \to \infty$;

Preliminary results

We assume that the process X is transient. Then it has the (radial) Green function (occupation density) $G(x) = G(|x|) = \int_0^\infty p_t(x)\, dt$.

Preliminary results

We assume that the process X is transient. Then it has the (radial) Green function (occupation density) $G(x) = G(|x|) = \int_0^\infty p_t(x)\,dt$.
Estimates for G: There exists $C > 1$ such that (KSV 2014, T. Grzywny 2014)

$$C^{-1}\frac{1}{|x|^d\Psi(|x|^{-1})} \le G(x) \le C\frac{1}{|x|^d\Psi(|x|^{-1})}, \qquad |x| \le 1, \ (x \in \mathbb{R}^d \setminus \{0\}).$$

Preliminary results

We assume that the process X is transient. Then it has the (radial) Green function (occupation density) $G(x) = G(|x|) = \int_0^\infty p_t(x)\, dt$.
Estimates for G: There exists $C > 1$ such that (KSV 2014, T. Grzywny 2014)

$$C^{-1} \frac{1}{|x|^d \Psi(|x|^{-1})} \le G(x) \le C \frac{1}{|x|^d \Psi(|x|^{-1})}, \qquad |x| \le 1, \ (x \in \mathbb{R}^d \setminus \{0\}).$$

The Lévy measure of X has the density $j(x) = j(|x|)$ satisfying

$$C^{-1} \frac{\Psi(|x|^{-1})}{|x|^d} \le j(x) \le C \frac{\Psi(|x|^{-1})}{|x|^d}, \qquad |x| \le 1, \ (x \in \mathbb{R}^d \setminus \{0\}).$$

Minimal thinness revisited

Let $D \subset \mathbb{R}^d$ be an open set, X^D the process X killed upon exiting D.

Minimal thinness revisited

Let $D \subset \mathbb{R}^d$ be an open set, X^D the process X killed upon exiting D.
Let $G_D(x, y)$ be the Green function of D with respect to the process X^D.

Minimal thinness revisited

Let $D \subset \mathbb{R}^d$ be an open set, X^D the process X killed upon exiting D.
Let $G_D(x, y)$ be the Green function of D with respect to the process X^D.

Fix $x_0 \in D$ and define $M_D(x, y) = \frac{G_D(x,y)}{G_D(x_0,y)}$, $x, y \in D$ – the Martin kernel.
Then D has a Martin boundary $\partial_M D$ and $M_D(x, \cdot)$ extends continuously to $\partial_M D$. The function $M_D(x, z)$, $x \in D$, $z \in \partial_M D$ is the Martin kernel.
A point $z \in \partial_M D$ is called a minimal Martin boundary point if $M_D(\cdot, z)$ is a minimal harmonic function for the process X^D. The minimal Martin boundary is denoted by $\partial_m D$.

Minimal thinness revisited, cont.

A subset $E \subset D$ is minimally thin in D at $z \in \partial_m D$ with respect to X^D if

$$\mathbb{E}_\cdot[M_D(X^D_{T_E}, z)] \neq M_D(\cdot, z)\,.$$

Probabilistic interpretation in terms of the process X^D conditioned to die at z. Abstract theory developed by H. Föllmer 1969.

Minimal thinness revisited, cont.

A subset $E \subset D$ is minimally thin in D at $z \in \partial_m D$ with respect to X^D if

$$\mathbb{E}_\cdot[M_D(X^D_{T_E}, z)] \neq M_D(\cdot, z)\,.$$

Probabilistic interpretation in terms of the process X^D conditioned to die at z. Abstract theory developed by H. Föllmer 1969.

We will consider the following two types of $D \subset \mathbb{R}^d$:

(1) D a $C^{1,1}$ open set: Then the finite part of $\partial_m D$ can be identified with the Euclidean ∂D;

Minimal thinness revisited, cont.

A subset $E \subset D$ is minimally thin in D at $z \in \partial_m D$ with respect to X^D if

$$\mathbb{E}_{\cdot}[M_D(X^D_{T_E}, z)] \neq M_D(\cdot, z)\,.$$

Probabilistic interpretation in terms of the process X^D conditioned to die at z. Abstract theory developed by H. Föllmer 1969.

We will consider the following two types of $D \subset \mathbb{R}^d$:

(1) D a $C^{1,1}$ open set: Then the finite part of $\partial_m D$ can be identified with the Euclidean ∂D;

(2) D a half-space-like $C^{1,1}$ open set, $\mathbb{H}_1 \subset D \subset \mathbb{H}$: Then the infinite part of $\partial_m D$ consists of a single point, ∞;

Wiener-Aikawa-type criteria for minimal thinness

(1) Let D be a $C^{1,1}$ open set, $\{Q_j\}$ a Whitney decomposition for D and $E \subset D$. Then E is minimally thin at $z \in \partial D$ if and only if

$$\sum_{j:Q_j \subset B(z,1)} \mathrm{dist}(z, Q_j)^{-d} \Psi(\mathrm{dist}(Q_j, \partial D)^{-1})^{-1} \mathrm{Cap}(E \cap Q_j) < \infty$$

Wiener-Aikawa-type criteria for minimal thinness

(1) Let D be a $C^{1,1}$ open set, $\{Q_j\}$ a Whitney decomposition for D and $E \subset D$. Then E is minimally thin at $z \in \partial D$ if and only if

$$\sum_{j: Q_j \subset B(z,1)} \mathrm{dist}(z, Q_j)^{-d} \Psi(\mathrm{dist}(Q_j, \partial D)^{-1})^{-1} \mathrm{Cap}(E \cap Q_j) < \infty$$

(2) Let D be a half-space-like $C^{1,1}$ open set, $\mathbb{H}_1 \subset D \subset \mathbb{H}$, $\{Q_j\}$ a Whitney decomposition for D and $E \subset D$. Then E is minimally thin at infinity in D if and only if

$$\sum_{j: Q_j \subset B(0,1)^c} \mathrm{dist}(0, Q_j)^{-d} \Psi(\mathrm{dist}(0, Q_j)^{-1})^{-1} \mathrm{Cap}(E \cap Q_j) < \infty$$

Wiener-Aikawa-type criteria for minimal thinness

(1) Let D be a $C^{1,1}$ open set, $\{Q_j\}$ a Whitney decomposition for D and $E \subset D$. Then E is minimally thin at $z \in \partial D$ if and only if

$$\sum_{j: Q_j \subset B(z,1)} \mathrm{dist}(z, Q_j)^{-d} \Psi(\mathrm{dist}(Q_j, \partial D)^{-1})^{-1} \mathrm{Cap}(E \cap Q_j) < \infty$$

(2) Let D be a half-space-like $C^{1,1}$ open set, $\mathbb{H}_1 \subset D \subset \mathbb{H}$, $\{Q_j\}$ a Whitney decomposition for D and $E \subset D$. Then E is minimally thin at infinity in D if and only if

$$\sum_{j: Q_j \subset B(0,1)^c} \mathrm{dist}(0, Q_j)^{-d} \Psi(\mathrm{dist}(0, Q_j)^{-1})^{-1} \mathrm{Cap}(E \cap Q_j) < \infty$$

$\mathrm{Cap}(\overline{B(0,r)}) \asymp r^d \Psi(r^{-1})$

Integral tests for minimal thinness

The following integral tests are consequences of Wiener-Aikava-type criteria:

Integral tests for minimal thinness

The following integral tests are consequences of Wiener-Aikava-type criteria:

Theorem: [KSV1]
(1) Let D be a $C^{1,1}$ open set, $E \subset D$. If E is minimally thin at $z \in \partial D$, then

$$\int_{E\cap B(z,1)} |x-z|^{-d}\, dx < \infty\,.$$

If E is the union of a subfamily of $\{Q_j\}$, the converse also holds true.

Integral tests for minimal thinness

The following integral tests are consequences of Wiener-Aikava-type criteria:

Theorem: [KSV1]
(1) Let D be a $C^{1,1}$ open set, $E \subset D$. If E is minimally thin at $z \in \partial D$, then

$$\int_{E \cap B(z,1)} |x-z|^{-d}\, dx < \infty .$$

If E is the union of a subfamily of $\{Q_j\}$, the converse also holds true.
(2) Let D be a half-space-like $C^{1,1}$ open set, $\mathbb{H}_1 \subset D \subset \mathbb{H}$. If E is minimally thin at infinity, then

$$\int_{E \cap B(0,1)^c} |x|^{-d}\, dx < \infty .$$

If E is the union of a subfamily of $\{Q_j\}$, the converse also holds true.

Sets below graphs of Lipschitz functions

$f : \mathbb{R}^{d-1} \to [0, \infty)$ Lipschitz, $E = \{x = (\widetilde{x}, x_d) \in \mathbb{H} : 0 < x_d \le f(\widetilde{x})\}$.

Sets below graphs of Lipschitz functions

$f : \mathbb{R}^{d-1} \to [0, \infty)$ Lipschitz, $E = \{x = (\widetilde{x}, x_d) \in \mathbb{H} : \ 0 < x_d \le f(\widetilde{x})\}$.

Corollary:

(a) E is minimally thin in $\mathbb{H}$ at 0 if and only if $\int_{|\widetilde{x}|<1} f(\widetilde{x})|\widetilde{x}|^{-d}\, d\widetilde{x} < \infty$.

(b) E is minimally thin in $\mathbb{H}$ at ∞ if and only if $\int_{|\widetilde{x}|>1} f(\widetilde{x})|\widetilde{x}|^{-d}\, d\widetilde{x} < \infty$.

Sets below graphs of Lipschitz functions

$f : \mathbb{R}^{d-1} \to [0,\infty)$ Lipschitz, $E = \{x = (\widetilde{x}, x_d) \in \mathbb{H} : \ 0 < x_d \le f(\widetilde{x})\}$.

Corollary:

(a) E is minimally thin in $\mathbb{H}$ at 0 if and only if $\int_{|\widetilde{x}|<1} f(\widetilde{x})|\widetilde{x}|^{-d}\, d\widetilde{x} < \infty$.

(b) E is minimally thin in $\mathbb{H}$ at ∞ if and only if $\int_{|\widetilde{x}|>1} f(\widetilde{x})|\widetilde{x}|^{-d}\, d\widetilde{x} < \infty$.

Remark: the criteria in the theorem and corollary do not depend on the process X (or Ψ).

Sets below graphs of Lipschitz functions

$f : \mathbb{R}^{d-1} \to [0, \infty)$ Lipschitz, $E = \{x = (\widetilde{x}, x_d) \in \mathbb{H} : 0 < x_d \leq f(\widetilde{x})\}$.

Corollary:
(a) E is minimally thin in $\mathbb{H}$ at 0 if and only if $\int_{|\widetilde{x}|<1} f(\widetilde{x})|\widetilde{x}|^{-d}\, d\widetilde{x} < \infty$.
(b) E is minimally thin in $\mathbb{H}$ at ∞ if and only if $\int_{|\widetilde{x}|>1} f(\widetilde{x})|\widetilde{x}|^{-d}\, d\widetilde{x} < \infty$.

Remark: the criteria in the theorem and corollary do not depend on the process X (or Ψ). They are the same as in the classical potential theory (Brownian motion case).

Sets below graphs of Lipschitz functions

$f : \mathbb{R}^{d-1} \to [0,\infty)$ Lipschitz, $E = \{x = (\widetilde{x}, x_d) \in \mathbb{H} : \ 0 < x_d \le f(\widetilde{x})\}$.

Corollary:

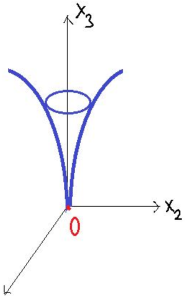
(a) E is minimally thin in $\mathbb{H}$ at 0 if and only if $\int_{|\widetilde{x}|<1} f(\widetilde{x})|\widetilde{x}|^{-d}\, d\widetilde{x} < \infty$.

(b) E is minimally thin in $\mathbb{H}$ at ∞ if and only if $\int_{|\widetilde{x}|>1} f(\widetilde{x})|\widetilde{x}|^{-d}\, d\widetilde{x} < \infty$.

Remark: the criteria in the theorem and corollary do not depend on the process X (or Ψ). They are the same as in the classical potential theory (Brownian motion case). Somewhat surprising! An explanation hinges on sharp two-sided estimates for $G_D(x,y)$ which imply that the singularity of the Martin kernel $M_D(x,z)$ near $z \in \partial D$ is of the order $|x-z|^{-d}$ for all such processes.

The Lebesgue thorn revisited

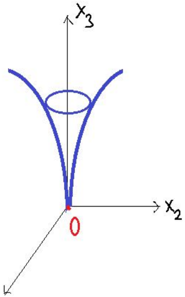
Let $f : [0,\infty) \to [0,\infty)$ be increasing, $f(r) > f(0)$ for all $r > 0$, $f(r)/r$ non-decreasing for r small. Let $E = \{x = (\widetilde{x}, x_d) : |\widetilde{x}| < f(x_d)\}$, $d \geq 3$.



The Lebesgue thorn revisited

Let $f : [0, \infty) \to [0, \infty)$ be increasing, $f(r) > f(0)$ for all $r > 0$, $f(r)/r$ non-decreasing for r small. Let $E = \{x = (\widetilde{x}, x_d) : |\widetilde{x}| < f(x_d)\}$, $d \geq 3$. Then E is thin at 0 with respect to the isotropic α-stable process if and only if

$$\int_0^1 \left(\frac{f(r)}{r}\right)^{d-\alpha-1} \frac{dr}{r} < \infty .$$



Description of processes and D

Let D be either a bounded $C^{1,1}$ domain, or a $C^{1,1}$ domain with compact complement, or a domain above the graph of a bounded $C^{1,1}$ function.

Description of processes and D

Let D be either a bounded $C^{1,1}$ domain, or a $C^{1,1}$ domain with compact complement, or a domain above the graph of a bounded $C^{1,1}$ function.

$W = (W_t)$ Brownian motion in $\mathbb{R}^d$, $W^D = (W^D_t)$ Brownian motion killed upon exiting D, $S = (S_t)$ an independent subordinator with Laplace exponent ϕ: $\mathbb{E}[\exp(-\lambda S_t)] = \exp(-t\phi(\lambda))$.

Description of processes and D

Let D be either a bounded $C^{1,1}$ domain, or a $C^{1,1}$ domain with compact complement, or a domain above the graph of a bounded $C^{1,1}$ function.

$W = (W_t)$ Brownian motion in $\mathbb{R}^d$, $W^D = (W_t^D)$ Brownian motion killed upon exiting D, $S = (S_t)$ an independent subordinator with Laplace exponent ϕ: $\mathbb{E}[\exp(-\lambda S_t)] = \exp(-t\phi(\lambda))$.
$X_t := W(S_t)$ subordinate Brownian motion; a Lévy process with the characteristic exponent $\Psi(x) = \phi(|x|^2)$.

Description of processes and D

Let D be either a bounded $C^{1,1}$ domain, or a $C^{1,1}$ domain with compact complement, or a domain above the graph of a bounded $C^{1,1}$ function.

$W = (W_t)$ Brownian motion in $\mathbb{R}^d$, $W^D = (W_t^D)$ Brownian motion killed upon exiting D, $S = (S_t)$ an independent subordinator with Laplace exponent ϕ: $\mathbb{E}[\exp(-\lambda S_t)] = \exp(-t\phi(\lambda))$.
$X_t := W(S_t)$ subordinate Brownian motion; a Lévy process with the characteristic exponent $\Psi(x) = \phi(|x|^2)$.
X^D the process X killed upon exiting D, killed subordinate Brownian motion (KSBM).

Description of processes and D

Let D be either a bounded $C^{1,1}$ domain, or a $C^{1,1}$ domain with compact complement, or a domain above the graph of a bounded $C^{1,1}$ function.

$W = (W_t)$ Brownian motion in $\mathbb{R}^d$, $W^D = (W^D_t)$ Brownian motion killed upon exiting D, $S = (S_t)$ an independent subordinator with Laplace exponent ϕ: $\mathbb{E}[\exp(-\lambda S_t)] = \exp(-t\phi(\lambda))$.
$X_t := W(S_t)$ subordinate Brownian motion; a Lévy process with the characteristic exponent $\Psi(x) = \phi(|x|^2)$.
X^D the process X killed upon exiting D, killed subordinate Brownian motion (KSBM).
$Y^D_t : W^D(S_t)$ subordinate killed Brownian motion (SKBM); a strong Markov process in D with the infinitesimal generator $\phi(-\Delta_{|D})$.

Description of processes and D

Let D be either a bounded $C^{1,1}$ domain, or a $C^{1,1}$ domain with compact complement, or a domain above the graph of a bounded $C^{1,1}$ function.

$W = (W_t)$ Brownian motion in $\mathbb{R}^d$, $W^D = (W_t^D)$ Brownian motion killed upon exiting D, $S = (S_t)$ an independent subordinator with Laplace exponent ϕ: $\mathbb{E}[\exp(-\lambda S_t)] = \exp(-t\phi(\lambda))$.
$X_t := W(S_t)$ subordinate Brownian motion; a Lévy process with the characteristic exponent $\Psi(x) = \phi(|x|^2)$.
X^D the process X killed upon exiting D, killed subordinate Brownian motion (KSBM).
Y_t^D : $W^D(S_t)$ subordinate killed Brownian motion (SKBM); a strong Markov process in D with the infinitesimal generator $\phi(-\Delta_{|D})$.
The semigroup of Y^D subordinate to the semigroup of X^D in the sense that for $f : D \to [0,\infty)$, $\mathbb{E}_x[f(Y_t^D)] \le \mathbb{E}_x[f(X_t^D)]$ – Y^D is a "smaller" process.

Scaling conditions for ϕ

Scaling conditions for ϕ:

(**A1**): There exist constants $0 < \delta_1 \leq \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1 \lambda^{\delta_1} \phi(t) \leq \phi(\lambda t) \leq a_2 \lambda^{\delta_2} \phi(t), \quad \lambda \geq 1, t \geq 1 .$$

Scaling conditions for ϕ

Scaling conditions for ϕ:

(**A1**): There exist constants $0 < \delta_1 \le \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1 \lambda^{\delta_1} \phi(t) \le \phi(\lambda t) \le a_2 \lambda^{\delta_2} \phi(t), \quad \lambda \ge 1, t \ge 1 .$$

(**A2**): There exist constants $0 < \delta_3 \le \delta_4 < 1$ and $a_3, a_4 > 0$ such that

$$a_3 \lambda^{\delta_3} \phi(t) \le \phi(\lambda t) \le a_4 \lambda^{\delta_4} \phi(t), \quad \lambda \le 1, t \le 1 .$$

Scaling conditions for ϕ

Scaling conditions for ϕ:

(**A1**): There exist constants $0 < \delta_1 \le \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1 \lambda^{\delta_1} \phi(t) \le \phi(\lambda t) \le a_2 \lambda^{\delta_2} \phi(t), \quad \lambda \ge 1, t \ge 1 .$$

(**A2**): There exist constants $0 < \delta_3 \le \delta_4 < 1$ and $a_3, a_4 > 0$ such that

$$a_3 \lambda^{\delta_3} \phi(t) \le \phi(\lambda t) \le a_4 \lambda^{\delta_4} \phi(t), \quad \lambda \le 1, t \le 1 .$$

(**A1**) always assumed, (**A2**) for unbounded D.

Scaling conditions for ϕ

Scaling conditions for ϕ:
(**A1**): There exist constants $0 < \delta_1 \le \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1 \lambda^{\delta_1} \phi(t) \le \phi(\lambda t) \le a_2 \lambda^{\delta_2} \phi(t), \quad \lambda \ge 1, t \ge 1 .$$

(**A2**): There exist constants $0 < \delta_3 \le \delta_4 < 1$ and $a_3, a_4 > 0$ such that

$$a_3 \lambda^{\delta_3} \phi(t) \le \phi(\lambda t) \le a_4 \lambda^{\delta_4} \phi(t), \quad \lambda \le 1, t \le 1 .$$

(**A1**) always assumed, (**A2**) for unbounded D.

If ϕ satisfies (**A1**), respectively (**A2**), then $\Psi(x) = \phi(|x|^2)$ satisfies (**H1**), respectively (**H2**).

Scaling conditions for ϕ

Scaling conditions for ϕ:
(**A1**): There exist constants $0 < \delta_1 \le \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1 \lambda^{\delta_1} \phi(t) \le \phi(\lambda t) \le a_2 \lambda^{\delta_2} \phi(t), \quad \lambda \ge 1, t \ge 1 .$$

(**A2**): There exist constants $0 < \delta_3 \le \delta_4 < 1$ and $a_3, a_4 > 0$ such that

$$a_3 \lambda^{\delta_3} \phi(t) \le \phi(\lambda t) \le a_4 \lambda^{\delta_4} \phi(t), \quad \lambda \le 1, t \le 1 .$$

(**A1**) always assumed, (**A2**) for unbounded D.

If ϕ satisfies (**A1**), respectively (**A2**), then $\Psi(x) = \phi(|x|^2)$ satisfies (**H1**), respectively (**H2**).
Scaling conditions on ϕ can be weakened to include subordinators which scale with order zero: $\phi(\lambda) = \log(1 + \lambda^{\alpha/2})$ $(0 < \alpha \le 2)$ – geometric stable subordinators and Γ-subordinators.

More on Laplace exponent ϕ

The Laplace exponent ϕ is a Bernstein function:

$$\phi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\, \mu(dx)\,.$$

More on Laplace exponent ϕ

The Laplace exponent ϕ is a Bernstein function:

$$\phi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\, \mu(dx)\,.$$

Assume $a = 0$ (no killing), $b = 0$ (zero drift) and μ infinite.

More on Laplace exponent ϕ

The Laplace exponent ϕ is a Bernstein function:

$$\phi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\, \mu(dx)\,.$$

Assume $a = 0$ (no killing), $b = 0$ (zero drift) and μ infinite.

Assume also that ϕ is a special Bernstein function, i.e., $\phi^*(\lambda) = \frac{\lambda}{\phi(\lambda)}$ is again a Bernstein function.

More on Laplace exponent ϕ

The Laplace exponent ϕ is a Bernstein function:

$$\phi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\, \mu(dx)\,.$$

Assume $a = 0$ (no killing), $b = 0$ (zero drift) and μ infinite.

Assume also that ϕ is a special Bernstein function, i.e., $\phi^*(\lambda) = \frac{\lambda}{\phi(\lambda)}$ is again a Bernstein function.
Important consequence: Let V^* be the potential operator of the subordinate killed Brownian motion $W^D(S_t^*)$ where S^* is the independent subordinator with Laplace exponent ϕ^*:
If $U_D(x, y)$ is the Green function of Y^D and $G_D^W(x, y)$ the Green function of W^D, then

$$U_D(x, y) = V^*(G_D^W(\cdot, y))(x)$$

Identification of (minimal) Martin boundary

Identification of (minimal) Martin boundary is non-trivial and, in many case it depends on Harnack inequality and boundary Harnack principle.

Identification of (minimal) Martin boundary

Identification of (minimal) Martin boundary is non-trivial and, in many case it depends on Harnack inequality and boundary Harnack principle.

The key step is to establish that for every $z \in \partial D$, there exists the limit

$$\lim_{y \to z} M_D(x, y) = \lim_{y \to z} \frac{G_D(x, y)}{G_D(x_0, y)}$$

through some oscillation reduction technique based on the boundary Harnack principle.

Identification of (minimal) Martin boundary

Identification of (minimal) Martin boundary is non-trivial and, in many case it depends on Harnack inequality and boundary Harnack principle.

The key step is to establish that for every $z \in \partial D$, there exists the limit

$$\lim_{y\to z} M_D(x,y) = \lim_{y\to z} \frac{G_D(x,y)}{G_D(x_0,y)}$$

through some oscillation reduction technique based on the boundary Harnack principle.

But the boundary Harnack principle does not hold for SKBM, partially because the jumping kernel J^D of Y^D has the estimates

$$J^D(x,y) \asymp \left(\frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \wedge 1\right) \frac{\phi(|x-y|^{-2})}{|x-y|^d}, \quad |x-y| \le M\,.$$

Martin kernel for SKBM

The existence of the limit

$$\lim_{y \to z \in \partial D} \frac{U_D(x,y)}{U_D(x_0,y)} =: M_D(x,z)$$

is based on two facts:

Martin kernel for SKBM

The existence of the limit

$$\lim_{y\to z\in\partial D}\frac{U_D(x,y)}{U_D(x_0,y)}=:M_D(x,z)$$

is based on two facts:

$$\lim_{y\to z} P_t^{W,D}\left(\frac{G_D^W(\cdot,y)}{G_D^W(x_0,y)}\right)=P_t^{W,D}M_D^W(\cdot,z)\,,\quad P_t^{W,D} \text{ semigroup of } W_t^D\,,$$

Martin kernel for SKBM

The existence of the limit

$$\lim_{y \to z \in \partial D} \frac{U_D(x,y)}{U_D(x_0,y)} =: M_D(x,z)$$

is based on two facts:

$$\lim_{y \to z} P_t^{W,D} \left(\frac{G_D^W(\cdot,y)}{G_D^W(x_0,y)} \right) = P_t^{W,D} M_D^W(\cdot,z)\,, \quad P_t^{W,D} \text{ semigroup of } W_t^D\,,$$

$$\lim_{y \to z} V^* \left(\frac{G_D^W(\cdot,y)}{G_D^W(x_0,y)} \right) = V^* M_D^W(\cdot,z)\,.$$

Martin kernel for SKBM

The existence of the limit

$$\lim_{y\to z\in\partial D} \frac{U_D(x,y)}{U_D(x_0,y)} =: M_D(x,z)$$

is based on two facts:

$$\lim_{y\to z} P_t^{W,D}\left(\frac{G_D^W(\cdot,y)}{G_D^W(x_0,y)}\right) = P_t^{W,D} M_D^W(\cdot,z)\,, \quad P_t^{W,D} \text{ semigroup of } W_t^D\,,$$

$$\lim_{y\to z} V^*\left(\frac{G_D^W(\cdot,y)}{G_D^W(x_0,y)}\right) = V^* M_D^W(\cdot,z)\,.$$

$$M_D(x,z) = \frac{V^* M_D^W(x,z)}{V^* M_D^W(x_0,z)}.$$

Green and Martin kernel estimates for SKBM

$$U_D(x,y) \asymp \left(\frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \wedge 1 \right) |x-y|^{-d} \phi(|x-y|^{-2})^{-1}, \quad |x-y| \le 1$$

Based on sharp two sided estimates of the heat kernel of W^D.

Green and Martin kernel estimates for SKBM

$$U_D(x,y) \asymp \left(\frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \wedge 1 \right) |x-y|^{-d} \phi(|x-y|^{-2})^{-1}, \quad |x-y| \le 1$$

Based on sharp two sided estimates of the heat kernel of W^D.

For $|x-z| \le 1$,

$$M_D(x,z) \asymp \frac{\delta_D(x)}{|x-z|^{d+2}\phi(|x-z|^{-2})} \asymp \frac{U_D(x,x_0) \wedge 1}{|x-z|^{d+2}\phi(|x-z|^{-2})},$$

Poisson kernel estimates for SKBM

For any open subset B of D, let $U^{D,B}(x,y)$ be the Green function of Y^D killed upon exiting B. We define the Poisson kernel

$$K^{D,B}(x,y) := \int_B U^{D,B}(x,z)J^D(z,y)dz, \qquad (x,y) \in B \times (D \setminus \overline{B}).$$

Poisson kernel estimates for SKBM

For any open subset B of D, let $U^{D,B}(x,y)$ be the Green function of Y^D killed upon exiting B. We define the Poisson kernel

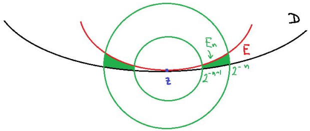
$$K^{D,B}(x,y) := \int_B U^{D,B}(x,z)J^D(z,y)dz, \qquad (x,y) \in B \times (D \setminus \overline{B}).$$

For every $M > 0$, there exists $c = c(M) > 0$ such that for any ball $B(x_0, r) \subset D$ of radius $r \in (0,1]$, we have for all $(x,y) \in B(x_0,r) \times (D \setminus \overline{B(x_0,r)})$ with $|x-y| \le M$,

$$K^{D,B(x_0,r)}(x,y) \le c\, \delta_D(y) \frac{\phi(((|y-x_0|-r)^{-2})}{(|y-x_0|-r)^{d+1}} \phi(r^{-2})^{-1}.$$

The first criterion for minimal thinness

$S_A = \inf\{t \geq 0 : \ Y_t^D \in A\}$, $R_u^A(x) = \mathbb{E}_x[u(Y_{S_A}^D)]$, $\widehat{R}_u^A(x) = \mathbb{E}_x[u(Y_{T_A}^D)]$.

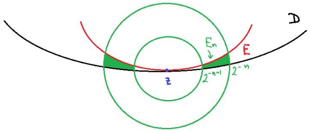


Let $E \subset D$. Fix $z \in \partial D$ and let
$E_n = E \cap \{x \in \mathbb{R}^d : 2^{-n-1} \leq |x - z| < 2^{-n}\}$.

The first criterion for minimal thinness

$$S_A = \inf\{t \ge 0 : \ Y_t^D \in A\}, \ R_u^A(x) = \mathbb{E}_x[u(Y_{S_A}^D)], \quad \widehat{R}_u^A(x) = \mathbb{E}_x[u(Y_{T_A}^D)] .$$

Let $E \subset D$. Fix $z \in \partial D$ and let
$E_n = E \cap \{x \in \mathbb{R}^d : 2^{-n-1} \le |x - z| < 2^{-n}\}$.



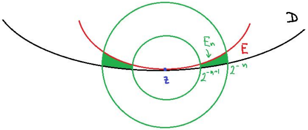
(1) In case D is a $C^{1,1}$ open set, E is minimally thin at D at $z \in \partial D$ iff

$$\sum_{n \ge 1} R_{M_D(\cdot,z)}^{E_n}(x_0) < \infty .$$

The first criterion for minimal thinness

$$S_A = \inf\{t \geq 0 : \; Y^D_t \in A\}, \; R^A_u(x) = \mathbb{E}_x[u(Y^D_{S_A})], \quad \widehat{R}^A_u(x) = \mathbb{E}_x[u(Y^D_{T_A})] .$$

Let $E \subset D$. Fix $z \in \partial D$ and let
$E_n = E \cap \{x \in \mathbb{R}^d : 2^{-n-1} \leq |x - z| < 2^{-n}\}$.



(1) In case D is a $C^{1,1}$ open set, E is minimally thin at D at $z \in \partial D$ iff

$$\sum_{n \geq 1} R^{E_n}_{M_D(\cdot, z)}(x_0) < \infty .$$

(2) If D is a half-space-like $C^{1,1}$ open set, E is minimally thin at D at ∞ iff $\sum_{n \geq 1} R^{E^n}_{M_D(\cdot, \infty)}(x_0) < \infty$, where $E^n = E \cap \{x \in \mathbb{R}^d : 2^n \leq |x| < 2^{n+1}\}$.

Let $g(x) = U_D(x, x_0) \wedge 1$. Then $M_D(x,z) \asymp \frac{g(x)}{|x-z|^{d+2}\phi(|x-z|^{-2})}$ which gives that $\sum_{n\geq 1} R^{E_n}_{M_D(\cdot,z)}(x_0) < \infty$ iff $\sum_{n=1} \frac{2^{n(d+2)}}{\phi(2^{2n})} R^{E_n}_g(E_n) < \infty$.

Let $g(x) = U_D(x, x_0) \wedge 1$. Then $M_D(x, z) \asymp \frac{g(x)}{|x-z|^{d+2}\phi(|x-z|^{-2})}$ which gives that $\sum_{n\geq 1} R^{E_n}_{M_D(\cdot,z)}(x_0) < \infty$ iff $\sum_{n=1} \frac{2^{n(d+2)}}{\phi(2^{2n})} R^{E_n}_g(E_n) < \infty$.

For every $A \subset Q_j$, we have $R^A_g(x_0) \asymp g(x_j)^2 \mathrm{Cap}_D(A)$,

where

$$\mathrm{Cap}_D(A) := \inf\{\mu(D) : \ U_D\mu \geq 1 \text{ on } A\}.$$

Quasi-additivity of capacity

Capacity is always subadditive: $\text{Cap}_D(\cup_{j=1}^{\infty} A_j) \le \sum_{j=1}^{\infty} \text{Cap}_D(A_j)$

Quasi-additivity of capacity

Capacity is always subadditive: $\mathrm{Cap}_D(\cup_{j=1}^{\infty} A_j) \le \sum_{j=1}^{\infty} \mathrm{Cap}_D(A_j)$

Capacity is (locally) quasi-additive (at z) with respect to $\{Q_j\}$: $\exists\ C > 0$

$$\sum_{j=1}^{\infty} \mathrm{Cap}_D(A \cap Q_j) \le C\, \mathrm{Cap}_D(A)\,, \quad A \subset D \cap B(z,r) \quad (A \subset D)$$

Comparable measure and Hardy's inequality

The proof of quasi-additivity of capacity relies on the existence of a comparable measure: the measure σ is comparable to the capacity Cap_D if

$$\sigma(Q_j) \asymp \mathrm{Cap}_D(Q_j) \text{ and } \sigma(A) \le c\mathrm{Cap}_D(A), \text{ all } A \subset D\,.$$

Comparable measure and Hardy's inequality

The proof of quasi-additivity of capacity relies on the existence of a comparable measure: the measure σ is comparable to the capacity Cap_D if

$$\sigma(Q_j) \asymp \mathrm{Cap}_D(Q_j) \text{ and } \sigma(A) \le c\mathrm{Cap}_D(A), \text{ all } A \subset D\,.$$

Define $\sigma(A) := \int_A \phi(\delta_D(x)^{-2})\,dx$, $A \subset D$. Then σ is comparable to Cap_D.

Comparable measure and Hardy's inequality

The proof of quasi-additivity of capacity relies on the existence of a comparable measure: the measure σ is comparable to the capacity Cap_D if

$$\sigma(Q_j) \asymp \mathrm{Cap}_D(Q_j) \text{ and } \sigma(A) \le c\mathrm{Cap}_D(A), \text{ all } A \subset D\,.$$

Define $\sigma(A) := \int_A \phi(\delta_D(x)^{-2})\,dx$, $A \subset D$. Then σ is comparable to Cap_D.

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ of Y^D satisfies a local Hardy's inequality at $z \in \partial D$ if there exist $c > 0$ and (the localization radius) $R_0 > 0$ such that

$$\mathcal{E}(v,v) \ge c \int_{D\cap B(z,r_0)} v^2(x)\phi(\delta_D(x)^{-2})\,dx\,, \qquad v \in \mathcal{F}\,.$$

Wiener-Aikawa-type criterion for minimal thinness for SKBM

Let $\{Q_j\}$ be a Whitney decomposition of D, $E \subset D$ and $z \in \partial D$. Then E is minimally thin in D at z with respect to Y^D if and only if

$$\sum_{j:Q_j \subset B(z,1)} \frac{\text{dist}(Q_j, \partial D)^2}{\text{dist}(z, Q_j)^{d+2}\phi(\text{dist}(z, Q_j)^{-2})} \text{Cap}(E \cap Q_j) < \infty$$

Wiener-Aikawa-type criterion for minimal thinness for SKBM

Let $\{Q_j\}$ be a Whitney decomposition of D, $E \subset D$ and $z \in \partial D$. Then E is minimally thin in D at z with respect to Y^D if and only if

$$\sum_{j:Q_j \subset B(z,1)} \frac{\mathrm{dist}(Q_j, \partial D)^2}{\mathrm{dist}(z, Q_j)^{d+2} \phi(\mathrm{dist}(z, Q_j)^{-2})} \mathrm{Cap}(E \cap Q_j) < \infty$$

Compare with the criterion for minimal thinness with respect to X^D:

$$\sum_{j:Q_j \subset B(z,1)} \frac{1}{\mathrm{dist}(z, Q_j)^{d} \phi(\mathrm{dist}(Q_j, \partial D)^{-2})} \mathrm{Cap}(E \cap Q_j) < \infty$$

Integral criterion for minimal thinness

Theorem: [KSV2] Let D be either a bounded $C^{1,1}$ domain, or a $C^{1,1}$ domain with compact complement, or a domain above the graph of a bounded $C^{1,1}$ function, let $E \subset D$ and $z \in \partial D$. If E is minimally thin in D at z with respect to Y^D, then

$$\int_{E\cap B(z,1)} \frac{\delta_D(x)^2 \phi(\delta_D(x)^{-2})}{|x-z|^{d+2}\phi(|x-z|^{-2})} dx < \infty \,.$$

Conversely, if E is the union of a subfamily of Whitney cubes for D, then the converse also holds true.

Integral criterion for minimal thinness

Theorem: [KSV2] Let D be either a bounded $C^{1,1}$ domain, or a $C^{1,1}$ domain with compact complement, or a domain above the graph of a bounded $C^{1,1}$ function, let $E \subset D$ and $z \in \partial D$. If E is minimally thin in D at z with respect to Y^D, then

$$\int_{E\cap B(z,1)} \frac{\delta_D(x)^2\phi(\delta_D(x)^{-2})}{|x-z|^{d+2}\phi(|x-z|^{-2})}dx < \infty\,.$$

Conversely, if E is the union of a subfamily of Whitney cubes for D, then the converse also holds true.

$f : \mathbb{R}^d \to [0,\infty)$ Lipschitz, $E = \{x = (\widetilde{x}, x_d \in \mathbb{H} : 0 < x_d \le f(\widetilde{x})\}$.
Corollary: E is minimally thin in $\mathbb{H}$ at 0 if and only if

$$\int_{\{|\widetilde{x}|<1\}} \frac{f(\widetilde{x})^3\phi(f(\widetilde{x})^{-2})}{|\widetilde{x}|^{d+2}\phi(|\widetilde{x}|^{-2})}\, d\widetilde{x} < \infty\,.$$

Censored stable process

Let D be either a bounded $C^{1,1}$ domain or a half-space. Let X be an isotropic α-stable process in $\mathbb{R}^d$, $\Psi(x) = |x|^\alpha$ $(0 < \alpha < 2)$.

Censored stable process

Let D be either a bounded $C^{1,1}$ domain or a half-space. Let X be an isotropic α-stable process in $\mathbb{R}^d$, $\Psi(x) = |x|^\alpha$ $(0 < \alpha < 2)$.
Whenever X makes a jump outside D, this jump is suppressed and the process starts afresh at the position $X_{\tau_D -}$ (Ikeda-Nagasawa-Watanabe piecing together procedure). This new process Z^D is called a censored α-stable process in D (Bogdan, Burdzy & Chen, 2003). It is a stable process not allowed to jump outside D.

Censored stable process

Let D be either a bounded $C^{1,1}$ domain or a half-space. Let X be an isotropic α-stable process in $\mathbb{R}^d$, $\Psi(x) = |x|^\alpha$ $(0 < \alpha < 2)$.
Whenever X makes a jump outside D, this jump is suppressed and the process starts afresh at the position $X_{\tau_D -}$ (Ikeda-Nagasawa-Watanabe piecing together procedure). This new process Z^D is called a censored α-stable process in D (Bogdan, Burdzy & Chen, 2003). It is a stable process not allowed to jump outside D.
Alternatively, let

$$\mathcal{E}(u,u) = \int_D \int_D (u(x) - u(y))^2 |x-y|^{-d-\alpha}\, dx\, dy \qquad u \in C_c^\infty(D)\,.$$

Then $\mathcal{E}$ extends to a regular Dirichlet form on $L^2(D, dx)$ and Z^D is the corresponding Markov (Hunt) process.

Wiener-Aikawa-type criterion

For $0 < \alpha \le 1$, Z^D is conservative and never approaches ∂D. For $1 < \alpha < 2$, Z^D has finite lifetime ζ and $Z^D_{\zeta-} \in \partial D$.

Wiener-Aikawa-type criterion

For $0 < \alpha \leq 1$, Z^D is conservative and never approaches ∂D. For $1 < \alpha < 2$, Z^D has finite lifetime ζ and $Z^D_{\zeta-} \in \partial D$.

Let $\alpha \in (1,2)$, D be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ or a half-space ($d \geq 2$), $z \in \partial D$ and $E \subset D$. Then E is minimally thin at z with respect to the censored α-stable process if and only if

$$\sum_{j:Q_j\cap B(z,1)\neq\emptyset} \frac{\mathrm{dist}(Q_j,\partial D)^{2(\alpha-1)}}{\mathrm{dist}(z,Q_j)^{d+\alpha-2}}\,\mathrm{Cap}_D(E\cap Q_j) < \infty\,,$$

(Mimica & Vondracek 2014)

Processes related to the stable process

D a bounded $C^{1,1}$ domain or a half-space.

(a) X^D - α-stable process killed upon exiting D;

(b) Y^D - subordinate killed Brownian motion in D via $\alpha/2$-stable subordinator;

(c) Z^D - censored α-stable process, $1 < \alpha < 2$.

Processes related to the stable process

D a bounded $C^{1,1}$ domain or a half-space.

(a) X^D - α-stable process killed upon exiting D;

(b) Y^D - subordinate killed Brownian motion in D via $\alpha/2$-stable subordinator;

(c) Z^D - censored α-stable process, $1 < \alpha < 2$.

Criteria for minimal thinness of $E \subset D$ at $z \in \partial D$:

(a) For X^D: $\int_{E\cap B(z,1)} \frac{1}{|x-z|^d}\,dx < \infty$;

(b) For Y^D: $\int_{E\cap B(z,1)} \frac{\delta_D(x)^{\alpha-2}}{|x-z|^{d+\alpha-2}}\,dx < \infty$;

(c) For Z^D: $\int_{E\cap B(z,1)} \frac{\delta_D(x)^{-\alpha+2}}{|x-z|^{d-\alpha+2}}\,dx < \infty$;

Processes related to the stable process

D a bounded $C^{1,1}$ domain or a half-space.

(a) X^D - α-stable process killed upon exiting D;

(b) Y^D - subordinate killed Brownian motion in D via $\alpha/2$-stable subordinator;

(c) Z^D - censored α-stable process, $1 < \alpha < 2$.

Criteria for minimal thinness of $E \subset D$ at $z \in \partial D$:

(a) For X^D: $\int_{E\cap B(z,1)} \frac{1}{|x-z|^d}\,dx < \infty$;

(b) For Y^D: $\int_{E\cap B(z,1)} \frac{\delta_D(x)^{\alpha-2}}{|x-z|^{d+\alpha-2}}\,dx < \infty$;

(c) For Z^D: $\int_{E\cap B(z,1)} \frac{\delta_D(x)^{-\alpha+2}}{|x-z|^{d-\alpha+2}}\,dx < \infty$;

Minimal thinness for $Y^D \Longrightarrow$ minimal thinness for $X^D \Longrightarrow$ minimal thinness for Z^D.

Minimal thinness under the graph of a function

$f:\mathbb{R}^d \to [0,\infty)$ Lipschitz, $E=\{x=(\widetilde{x},x_d)\in \mathbb{H}:\ 0<x_d\le f(\widetilde{x})\}$.

Minimal thinness under the graph of a function

$f : \mathbb{R}^d \to [0, \infty)$ Lipschitz, $E = \{x = (\widetilde{x}, x_d) \in \mathbb{H} : 0 < x_d \le f(\widetilde{x})\}$.
Then E is minimally thin in $\mathbb{H}$ at $z = 0$

(a) for X^D iff $\int_{\{|\widetilde{x}|<1\}} \frac{f(\widetilde{x})}{|\widetilde{x}|^d} d\widetilde{x} < \infty$;

(b) for Y^D iff $\int_{\{|\widetilde{x}|<1\}} \frac{f(\widetilde{x})^{3-\alpha}}{|\widetilde{x}|^{d+2-\alpha}} d\widetilde{x} < \infty$;

(c) for Z^D iff $\int_{\{|\widetilde{x}|<1\}} \frac{f(\widetilde{x})^{\alpha-1}}{|\widetilde{x}|^{d+\alpha-2}} d\widetilde{x} < \infty$.

Minimal thinness under the graph of a function

$f : \mathbb{R}^d \to [0, \infty)$ Lipschitz, $E = \{x = (\widetilde{x}, x_d) \in \mathbb{H} : 0 < x_d \le f(\widetilde{x})\}$.
Then E is minimally thin in $\mathbb{H}$ at $z = 0$

(a) for X^D iff $\int_{\{|\widetilde{x}|<1\}} \frac{f(\widetilde{x})}{|\widetilde{x}|^d} d\widetilde{x} < \infty$;

(b) for Y^D iff $\int_{\{|\widetilde{x}|<1\}} \frac{f(\widetilde{x})^{3-\alpha}}{|\widetilde{x}|^{d+2-\alpha}} d\widetilde{x} < \infty$;

(c) for Z^D iff $\int_{\{|\widetilde{x}|<1\}} \frac{f(\widetilde{x})^{\alpha-1}}{|\widetilde{x}|^{d+\alpha-2}} d\widetilde{x} < \infty$.

Example: If $f(\widetilde{x}) = |\widetilde{x}|^\gamma$, $\gamma \ge 1$, all three integrals are finite iff $\gamma > 1$.

Minimal thinness under the graph of a function

$f : \mathbb{R}^d \to [0, \infty)$ Lipschitz, $E = \{x = (\widetilde{x}, x_d) \in \mathbb{H} : 0 < x_d \le f(\widetilde{x})\}$.
Then E is minimally thin in $\mathbb{H}$ at $z = 0$

(a) for X^D iff $\int_{\{|\widetilde{x}|<1\}} \frac{f(\widetilde{x})}{|\widetilde{x}|^d} d\widetilde{x} < \infty$;

(b) for Y^D iff $\int_{\{|\widetilde{x}|<1\}} \frac{f(\widetilde{x})^{3-\alpha}}{|\widetilde{x}|^{d+2-\alpha}} d\widetilde{x} < \infty$;

(c) for Z^D iff $\int_{\{|\widetilde{x}|<1\}} \frac{f(\widetilde{x})^{\alpha-1}}{|\widetilde{x}|^{d+\alpha-2}} d\widetilde{x} < \infty$.

Example: If $f(\widetilde{x}) = |\widetilde{x}|^\gamma$, $\gamma \ge 1$, all three integrals are finite iff $\gamma > 1$.
Let $f(\widetilde{x}) = |\widetilde{x}|\big(\log(1/|\widetilde{x}|)\big)^{-\beta}$, $\beta \ge 0$. Then E is minimally thin at $z = 0$

(a) for X^D iff $\beta > 1$;

(b) for Y^D iff $\beta > 1/(3 - \alpha)$;

(c) for Z^D iff $\beta > 1/(\alpha - 1)$.

Thank you !