Variational convergence on Riemannian manifolds
Stochastic Analysis and Applications
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Space: Weighted Riemannian manifold

- $(\Omega, g, d\mu)$: Complete weighted Riemannian manifold.
- $\Sigma \subset \Omega$: Compact submanifold of co-dimension 1 with measure $d\sigma$.
- $\Sigma_\epsilon = \epsilon$-neighborhood of Σ

Real world example:

- Σ: membrane (endothelial cell monolayer).
- Permeability of Σ:
$$P_D = \frac{J}{\Delta u}$$
where J is the flux at $\partial_{left}\Sigma_\epsilon$ and $\Delta u = u(x_0) - u(x_1)$ with $x_0 \in \partial_{left}\Sigma_\epsilon$ and $x_1 \in \partial_{right}\Sigma_\epsilon$.

Functionals (energies)

Let $\alpha, \beta \geq 0$, $\epsilon > 0$.

- F^ϵ: functionals representing the "real situations"

$$F^\epsilon[u] = \int_{\Omega \setminus \Sigma^\epsilon} g(\nabla u, \nabla u) d\mu + \int_{\Sigma^\epsilon} \epsilon^{-\alpha} g_\Sigma(\nabla u, \nabla u) + \epsilon^{-\beta} g_{\Sigma^\perp}(\nabla u, \nabla u)\, d\mu$$

- F: functional of the "ideal model":

$$F[u] = \int_\Omega g(\nabla u, \nabla u) d\mu + \int_\Sigma \delta_{\beta=1} g_\Sigma(\nabla u, \nabla u) d\sigma$$

Real world application: Redefining the "permeability" of endothelial cell monolayer.

Hille-Yosida semigroups

- A, A^ϵ: The associated generators; namely, self-adjoint operators such that

$$F(u,v) = (-Au, v)_{L^2} \text{ and } F^\epsilon(u,v) = (-A^\epsilon u, v)_{L^2}.$$

- T_t, T_t^ϵ with $t > 0$: the associated Hille-Yosida semigroups; namely,

$$T_t = \exp(tA) \text{ and } T_t^\epsilon = \exp(tA^\epsilon)$$

  T_t and T_t^ϵ solve the heat equations:

$$A(T_t u) = \frac{d}{dt} T_t u \text{ and } A^\epsilon(T_t^\epsilon u) = \frac{d}{dt} T_t^\epsilon u.$$

Probability spaces

$\mathcal{C}=$ the set of all continuous trajectories in Ω

$$\mathcal{C} = \{c : [0, \infty) \to \Omega : \ c \text{ is continuous with time } t > 0\}.$$

$\mathbb{P}_p$ and $\mathbb{P}_p^\epsilon$: the associated measures on $\mathcal{C}$ with $p \in \Omega$; namely, $\mathbb{P}_p$ and $\mathbb{P}_p^\epsilon$ are the probability measures on $\mathcal{C}$ satisfying:

$$(T_t \chi_O)(p) = \mathbb{P}_p(c(t) \in O), \quad O \subset \Omega$$

and

$$(T_t^\epsilon \chi_O)(p) = \mathbb{P}_p^\epsilon(c(t) \in O), \quad O \subset \Omega.$$

For an initial distribution ν_0 (a probability measure on Ω), set

$$\mathbb{P} = \int_\Omega \mathbb{P}_p \nu_0(dp) \text{ and } \mathbb{P}^\epsilon = \int_\Omega \mathbb{P}_p^\epsilon \nu_0(dp).$$

Goals

1. Study the convergence of the diffusion equations:

$$T_t^\epsilon \to T_t \quad (L^2)$$

2. Study the weak convergence of the measures:

$$\mathbb{P}^\epsilon \to \mathbb{P}.$$

Strategy

(1) Show the Mosco convergence of $F^\epsilon \to F$. This will imply the convergence:

$$T_t^\epsilon \to T_t \quad (L^2)$$

in particular, the finite-dimensional convergence of $\mathbb{P}^\epsilon$ to $\mathbb{P}$.

(2) Show that F and F^ϵ are regular Dirichlet forms. This together with (1) and the tightness of $\{\mathbb{P}^\epsilon\}_{\epsilon>0}$ implies the convergence:

$$\mathbb{P}^\epsilon \to \mathbb{P}.$$

1st Step: Semigroup convergence

Definition (Mosco convergence)

F^ϵ Mosco-converges to F in H iff

(M1) $\forall u \in L^2$, $\exists u_\epsilon \in L^2$ converging strongly to u s.t.

$$\limsup F^\epsilon[u_\epsilon] \leq F[u]$$

(M2) if v_ϵ weak converges to u in L^2, then

$$\liminf F^\epsilon[v_\epsilon] \geq F[u].$$

Theorem (Mosco)

$F^\epsilon \to F$ *Mosco-converges* $\Longleftrightarrow T_t^\epsilon u \to T_t u, \ \forall u \in L^2$.

Related Results

In 1970s, L. Carbone and C. Sbordone, H. Pham Huy and E. Sanchez-Palencia studied

$$\Omega = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$$
$$\Sigma_\epsilon = [-1, 1] \times [-\epsilon/2, \epsilon/2]$$
$$F^\epsilon[u] = \int_\Omega a_\epsilon |\nabla u|^2 d\mu$$
$$a_\epsilon(p) = \begin{cases} 1, & p \in \Omega \setminus \Sigma_\epsilon \\ 1/\epsilon, & p \in \Sigma_\epsilon \end{cases}$$
$$F[u] = \int_\Omega |\nabla u|^2 d\mu + \int_\Sigma u_x^2\, dx$$

and showed

$$F^\epsilon \to F \text{ in Mosco sense.}$$

Related Results

Very limited list of other main contributers in different settings:

In Euclidean spaces: U. Mosco, J.L. Lions, A. Bensoussan, De Girogi, I. Babuska, etc;
in Riemannian manifolds: Y. Ogura, etc;
in Measure Metric Spaces: T. Shioya, K. Kuwae, etc;
in Infinite dimension spaces: Kolesnikov, S. Albeverio, etc;
in Fractals: U. Mosco, M.R. Lancia, M.A. Vivaldi, Barlow-Bass, T. Kumagai, etc.

2nd Step: Tightness

Prove

$$\lim_{h\to 0} \sup_{\epsilon>0} \mathbb{P}^\epsilon(\mathcal{C}^\delta_{h,l}) = 0$$

where

$$\mathcal{C}^\delta_{h,l} = \{w \in \mathcal{C} \mid \sup_{|t-s|<h,\ 0\leq t<s\leq l} |w(s) - w(t)| > \delta\}$$

Application in Probability theory

Theorem (T. Uemura 1994)

Assume that a^ϵ *and* a *defined in* $\mathbb{R}^n$ *satisfies:*

1. *bounded from below and above uniformly in* $\epsilon > 0$
2. $a_\epsilon \to a$ *almost everywhere.*

Then

1. $\int a_\epsilon |\nabla u|^2 d\mu$ *Mosco converges to* $\int a|\nabla u|^2 d\mu$,
2. $\{\mathbb{P}^\epsilon\}_{\epsilon>0}$ *is tight, accordingly,* $\mathbb{P}^\epsilon$ *weak converges to* $\mathbb{P}$.

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Note. Uemura's result was generalized by K. Kuwae and T. Uemura (1995) to metric-measure spaces.

Changing Speed Measures

Definition (Kuwae and Shioya)

Let H_ϵ and H be Hilbert spaces.

- $u_\epsilon \in H_\epsilon \to u \in H$ *strongly* if $\exists \tilde{u}_\delta \in$ Core such that
  1. $\lim_{\delta \to 0} \|\tilde{u}_\delta - u\|_H = 0$
  2. $\lim_{\delta \to 0} \limsup_{\epsilon \to 0} \|\tilde{u}_\delta - u_\epsilon\|_{H_\epsilon} = 0.$
- $u \in H_\epsilon \to u \in u$ *weakly* iff $v_\epsilon \to v$ strongly, then

$$\lim_{\epsilon \to 0} (u_\epsilon, v_\epsilon)_{H_\epsilon} = (u, v)_H.$$

Changing Speed Measures (Con't)

Let E_ϵ and E (resp) be functionals defined in H_ϵ and H (resp).

Definition (Kuwae and Shioya)

$(E_\epsilon, H_\epsilon) \to (E, H)$ Mosco iff

- $\forall u \in H$, $\exists u_\epsilon \in H_\epsilon$ converging strongly to u s.t.

$$\limsup E_\epsilon[u_\epsilon] \le E[u]$$

- $v_\epsilon \in H_\epsilon$ weak converges to u in $H \implies \liminf E_\epsilon[v_\epsilon] \ge E[u]$.

Theorem (Kuwae-Shioya)

$(E_\epsilon, H_\epsilon) \to (E, H)$ Mosco iff

$$T_t^\epsilon u_\epsilon \in H_\epsilon \to T_t u \in H \text{ strongly}$$

whenever $u_\epsilon \in H_\epsilon \to u \in H$ strongly (in Kuwae-Shioya sense).

Main Results.

Theorem (Regular Dirichlet forms)

- *Assume* $\partial\Omega = \emptyset$. A^ϵ *and* A *restricted to* $C_c^\infty(\Omega)$ *are essentially selfadjoint. In particular,*

$$Af = 0,\ f \in L^2 \implies f \equiv const$$

- F^ϵ *and* F *are strongly local regular Dirichlet forms. Namely, they generate Markov processes whose transition probabilities are given by* T_t^ϵ *and* T_t *(resp).*

Theorem (Semigroup convergence)

Suppose $\alpha \geq 0$ *and* $0 \leq \beta \leq 1$. *Then,* F^ϵ *Mosco-converges to* F *in* $L^2(\Omega; d\mu)$. *Therefore,*

$$T_t^\epsilon \to T_t$$

and

$$E_\epsilon((\lambda, \mu]) \to E((\lambda, \mu]),$$

where E_ϵ *and* E *are the spectral measures of the generators of* F^ϵ *and* F, *respectively,* $\lambda < \mu$ *are not in the point spectrum.*

Theorem (Semigroup convergence)

Suppose $\alpha \geq 0$ and $0 \leq \beta \leq 1$. Then, F^ϵ Mosco-converges to F in $L^2(\Omega; d\mu)$. Therefore,

$$T_t^\epsilon \to T_t$$

and

$$E_\epsilon((\lambda, \mu]) \to E((\lambda, \mu]),$$

where E_ϵ and E are the spectral measures of the generators of F^ϵ and F, respectively, $\lambda < \mu$ are not in the point spectrum.

Let $0 \leq \nu \leq 1$. The same results hold true for F^ϵ and F with changing speed measures $d\mu^\epsilon \to d\mu'$, where

$$d\mu^\epsilon = \chi_{\Omega \setminus \Sigma^\epsilon} d\mu + \epsilon^{-\nu} \chi_{\Sigma^\epsilon} d\sigma \quad d\mu' = d\mu + \delta_{\nu=1} d\sigma.$$

Theorem (Weak convergence of measures)

Suppose $0 \leq \beta \leq 1$ *and* $\alpha \geq \beta$*. Then the Wiener measures* $\{\mathbb{P}^\epsilon\}_{\epsilon>0}$ *and* $\{\mathbb{P}'^\epsilon\}_{\epsilon>0}$ *associated to* F^ϵ *in* $L^2(\Omega; d\mu)$ *and* $L^2(\Omega; d\mu_\epsilon)$ *respectively, are tight. Therefore,*

$$\mathbb{P}^\epsilon \to \mathbb{P}, \ \epsilon \to 0,$$
$$\mathbb{P}'^\epsilon \to \mathbb{P}', \ \epsilon \to 0.$$

where $\mathbb{P}$ *and* $\mathbb{P}'$ *are the measures associated to* F *in* $L^2(\Omega; d\mu)$ *and* $L^2(\Omega; d\mu')$*, respectively.*

Related projects

- Microfluidic. Joint with K. Funamoto (Tohoku) and I. Zervantonakis (Harvard)
- Markov properties of non-symmetric operators on a complete non-compact Riemannian manifold. Joint with M. Bordoni (Rome) and S. Gallot (Grenoble)
- Markov unique extensions of non-symmetric Schrodinger operators on a singular manifold.