Pathwise Uniqueness for SDEs with Non-Regular Drift and Non-Constant Diffusion

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Abstract

A new approach to prove pathwise uniqueness for SDEs of the form

$$dX_t = b(t, X_t)dt + dW_t$$

was introduced by E. Fedrizzi and F. Flandoli in [1]. We generalize this method to SDEs with time and space dependent diffusion

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t,$$

where the matrix-valued function σ is non-degenerated, bounded, continuous and its weak derivative $\partial_x\sigma$ as well as σ is in $L^q_{p'}$. The proof is based on a transformation via solutions to PDEs of the form

$$\partial_t u + \frac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}\partial_{x_i}\partial_{x_j}u = -b.$$

1 Assumptions

Consider the SDE

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \qquad t \in [0, T], \qquad (1)$$

where W is an m-dimensional Brownian motion on a filtered probability space $(\Omega, (\mathcal{F}_t)_t, \mathbb{P})$, $x \in \mathbb{R}^d$ and $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are measurable functions with the following properties:

(1) $b \in L^q_p(T)$, for some $p, q > 2(d + 1)$,

(2) σ is continuous in (t, x),

(3) σ is non-degenerate, i.e. there exists a constant $c_\sigma > 0$ such that

$$\langle \sigma\sigma^*(t, x)\xi, \xi\rangle \geq c_\sigma\langle I\xi, \xi\rangle \quad \forall\, \xi \in \mathbb{R}^d \;\; \forall\, (t, x) \in [0, T] \times \mathbb{R}^d,$$

(4) σ is bounded by a constant $\tilde{c}_\sigma$,

(5) $\sigma \in W^{0,1}_{q,p}(T)$.

Here

$$L^q_p(T) = \{\, f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d\ (\mathbb{R}^{d\times m}) \mid \|f\|_{L^q_p(T)} < \infty \,\},$$

with

$$\|f\|_{L^q_p(T)} = \left(\int_0^T \left(\int_{\mathbb{R}^d} |f(t, x)|^p dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}},$$

$$W^{1,2}_{q,p}(T) = \{\, f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \mid f, \partial_t f, \partial_x f, \partial_x^2 f \in L^q_p(T) \,\},$$
$$W^{0,1}_{q,p}(T) = \{\, f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m} \mid f, \partial_x f \in L^q_p(T) \,\},$$

where we consider weak differentials.

In addition, assume that for all $f \in L^q_p(T)$ there exists a solution $u \in W^{1,2}_{q,p}(T)$ to the equation

$$\partial_t u + \frac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}\partial_{x_i}\partial_{x_j}u = f \quad \text{on } [0, T], \quad u|_{t=T} = 0$$

with

$$\|u\|_{W^{1,2}_{q,p}(T)} \leq C\|f\|_{L^q_p(T)},$$

where C is independent of f. This is e.g. the case if $q \geq p$ and σ is uniformly continuous (see [2]).

2 Result

Theorem 1. Under the above assumptions, we have pathwise uniqueness for equation (1).

3 Sketch of proof

The proof follows the ideas of [1].

3.1 Step by step to a more convenient drift term

Step 1
Let U_b be a solution to the PDE

$$\partial_t u + \frac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}\partial_{x_i}\partial_{x_j}u = -b \quad \text{on } [0, T], \quad u\big|_{t=T} = 0.$$

By a generalized Itô formula for weak differentiable functions, we get

$$U_b(t, X_t) = U_b(0, x) + \int_0^t \partial_x U_b(s, X_s)b(s, X_s)ds + \int_0^t \partial_x U_b(s, X_s)\sigma(s, X_s)dW_s + \underbrace{\int_0^t \partial_t U_b(s, X_s) + \frac{1}{2}\sum_{i,j}(\sigma\sigma^*(s, X_s))_{ij}\partial_{x_i}\partial_{x_j}U_b(s, X_s)ds}_{-\int_0^t b(s,X_s)ds}.$$

Replacing $\int_0^t b(s, X_s)ds$ in the integral equation (1) yields

$$X_t = x + U_b(0, x) - U_b(t, X_t) + \int_0^t \underbrace{\partial_x U_b(s, X_s)b(s, X_s)}_{=:\mathcal{T}(b)(s,X_s)}ds + \int_0^t (\partial_x U_b(s, X_s)\sigma(s, X_s) + \sigma(s, X_s))dW_s \qquad (2)$$

Step 2
Let $U_{\mathcal{T}(b)}$ be a solution to the PDE

$$\partial_t u + \frac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}\partial_{x_i}\partial_{x_j}u = -\mathcal{T}(b) \quad \text{on } [0, T], \quad u|_{t=T} = 0.$$

Now, applying Itô's formula to $U_{\mathcal{T}(b)}$ and, as before, replacing the drift term in (2), we obtain

$$X_t = x + U_b(0, x) + U_{\mathcal{T}(b)}(0, x) - U_b(t, X_t) - U_{\mathcal{T}(b)}(t, X_t) + \int_0^t \partial_x U_{\mathcal{T}(b)}(s, X_s)b(s, X_s)ds + \int_0^t (I + \partial_x U_b(s, X_s) + \partial_x U_{\mathcal{T}(b)}(s, X_s))\sigma(s, X_s)dW_s.$$

Step n+1 ($\mathcal{T}^0(b) := b$, $\mathcal{T}^{k+1}(b) := \partial_x U_{\mathcal{T}^k(b)} \cdot b$)

$$\underbrace{X_t + \sum_{k=0}^n U_{\mathcal{T}^k(b)}(t, X_t)}_{=:Y_t^{(n)}} = \underbrace{x + \sum_{k=0}^n U_{\mathcal{T}^k(b)}(0, x)}_{=Y_0^{(n)}} + \int_0^t \mathcal{T}^{n+1}(b)(s, X_s)ds + \int_0^t \underbrace{\left(\sum_{k=0}^n \partial_x U_{\mathcal{T}^k(b)}(s, X_s) + I \right) \sigma(s, X_s)}_{=:\sigma^n(s,X_s)} dW_s.$$

One can prove that

$$\mathbb{E}\left[\int_0^T \left|\mathcal{T}^{n+1}(b)(t, X_t)\right|^2 dt \right] \to 0 \quad \text{for } n \to \infty.$$

3.2 Pathwise Uniqueness

Let $X_t^{(1)}$, $X_t^{(2)}$ be two solutions to equation (1) with the same initial values. We have

$$\left|X_t^{(1)} - X_t^{(2)}\right| \leq 2\left|Y_t^{(1,n)} - Y_t^{(2,n)}\right| \leq 3\left|X_t^{(1)} - X_t^{(2)}\right|,$$

which follows from the mean-value inequality and the fact that $\sum \|\partial_x U_{\mathcal{T}^k(b)}\|_{L^\infty(T)} \leq \frac{1}{2}$. Furthermore

$$\mathbb{E}\left[e^{A^{(n)}_{T\wedge\tau_R\wedge\tau_\varepsilon}} \right] \leq C \;\; \text{uniformly in } n,$$

where

$$A_t^{(n)} := \int_0^t \frac{\left|\sigma^{(n)}(s, X_s^{(1)}) - \sigma^{(n)}(s, X_s^{(2)})\right|^2}{\left|Y_s^{(1,n)} - Y_s^{(2,n)}\right|^2} \mathbb{1}_{\{Y_s^{(1,n)} \neq Y_s^{(2,n)}\}} ds.$$

To prove this is the hardest part! Then we have

$$\begin{aligned}
&\mathbb{E}\left[\left|X^{(1)}_{t\wedge\tau_R\wedge\tau_\varepsilon} - X^{(2)}_{t\wedge\tau_R\wedge\tau_\varepsilon}\right| \right] \\
&\leq 2\mathbb{E}\left[e^{\frac{1}{2}A^{(n)}_{t\wedge\tau_R\wedge\tau_\varepsilon}} e^{-\frac{1}{2}A^{(n)}_{t\wedge\tau_R\wedge\tau_\varepsilon}} \left|Y^{(1,n)}_{t\wedge\tau_R\wedge\tau_\varepsilon} - Y^{(2,n)}_{t\wedge\tau_R\wedge\tau_\varepsilon}\right| \right] \\
&\leq C\mathbb{E}\left[e^{-A^{(n)}_{t\wedge\tau_R\wedge\tau_\varepsilon}} \left|Y^{(1,n)}_{t\wedge\tau_R\wedge\tau_\varepsilon} - Y^{(2,n)}_{t\wedge\tau_R\wedge\tau_\varepsilon}\right|^2 \right]^{\frac{1}{2}},
\end{aligned}$$

where the term under the expectation can be estimated as follows:

$$\begin{aligned}
&d\left(e^{-A_t^{(n)}} \left|Y_t^{(1,n)} - Y_t^{(2,n)}\right|^2 \right) \\
&= e^{-A_t^{(n)}} d\left|Y_t^{(1,n)} - Y_t^{(2,n)}\right|^2 - \left|Y_t^{(1,n)} - Y_t^{(2,n)}\right|^2 e^{-A_t^{(n)}} dA_t^{(n)} \\
&\leq 2e^{-A_t^{(n)}} \left|Y_t^{(1,n)} - Y_t^{(2,n)}\right| \left|\mathcal{T}^{n+1}(b)(t, X_t^{(1)}) - \mathcal{T}^{n+1}(b)(t, X_t^{(2)})\right| dt \\
&\quad + 2e^{-A_t^{(n)}} \left\langle Y_t^{(1,n)} - Y_t^{(2,n)}, \left(\sigma^{(n)}(t, X_t^{(1)}) - \sigma^{(n)}(t, X_t^{(2)})dW_t\right) \right\rangle,
\end{aligned}$$

here we used Itô's formula and the Cauchy–Schwarz inequality in the last step.
Summarizing, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left|X^{(1)}_{t\wedge\tau_R\wedge\tau_\varepsilon} - X^{(2)}_{t\wedge\tau_R\wedge\tau_\varepsilon}\right| \right] \\
&\leq C\mathbb{E}\left[\int_0^T \left|X_s^{(1)} - X_s^{(2)}\right| \left|\mathcal{T}^{n+1}(b)(s, X_s^{(1)}) - \mathcal{T}^{n+1}(b)(s, X_s^{(2)})\right| ds \right]^{\frac{1}{2}} \\
&\quad + C\mathbb{E}\left[\int_0^{t\wedge\tau_R\wedge\tau_\varepsilon} e^{-A_s^{(n)}} \left\langle Y_s^{(1,n)} - Y_s^{(2,n)}, \left(\sigma^{(n)}(s, X_s^{(1)}) - \sigma^{(n)}(s, X_s^{(2)})\right) dW_s \right\rangle \right]^{\frac{1}{2}}
\end{aligned}$$

for all $n \in \mathbb{N}$.
The first term converges to 0 for $n \to \infty$ and the second integral is a martingale. Therefore, we have pathwise uniqueness.

4 Remark: What is the problem if $\sigma \neq I$?

By a change of measure (Girsanov) the solution of the equation

$$X_t = x + \int_0^t b(s, X_s)ds + W_t$$

is a Brownian motion. This fact was essential in the proof of [1]. The Girsanov transformation for non-constant diffusion is here no longer useful, since the Brownian motion, after change of measure, is not of the form (1). A priori it is even not clear that the process is Markovian.

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The infinite dimentional case has been done in

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