A Central Limit Theorem for Stochastic Heat Equations in Random Environments

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Problem Formulation

It is well-known that a central limit theorem holds for finite-dimensional diffusions in random environments (see, e.g., [2]). We consider the infinite-dimensional extension of this problem. Let $H = L^2[0,1]$; $C_0[0,1]$ and μ_0 be the path space and distribution of a 1-d standard Brownian motion on $[0,1]$ starting from 0.

Suppose that $(\Sigma, \mathscr{A}, q)$, $(\Omega, \mathscr{F}, P)$ are two probability spaces, $\{(V(u), B(u))\}_{u\in H}$ is an $\mathbb{R}\times H$-valued random field on $(\Sigma, \mathscr{A}, q)$ and W_t is a cylindrical BM on $(\Omega, \mathscr{F}, P)$. Let $u(t,\cdot) : \Sigma \times \Omega \to H$ be the solution to the 1-dimensional stochastic heat equation

$$\begin{cases} \partial_t u(t,x) = \dfrac{\partial_x^2}{2} u(t,x) - DV(u(t)) - B(u(t)) + \dot{W}(dt,dx), \\ \partial_x u(t,0) = \partial_x u(t,1) = 0, \\ u(0,x) = v(x), \end{cases} \qquad (*)$$

where we assume $V \in C_b^1(H)$ and DV is its Fréchet derivative. Our aim is to investigate the limit distribution of $t^{-\frac{1}{2}}u(t)$.

Assumptions

Assume that the random mapping $DV + B : H \to H$ is bounded and Lipschitz continuous q-a.s., thus the mild solution $u(t,x)$ to $(*)$ is unique and continuous in (t,x) (see [1]). Furthermore, assume:

(A1) V and $\|DV + B\|_H$ are uniformly bounded;

(A2) The distribution of the random field $\{(V,B)\}$ is stationary and ergodic under $\{\iota_c; c\in\mathbb{R}\}$, where ι_c stands for the shift on the path space defined as $\iota_c(\{V,B\}) = \{V(\cdot + c), B(\cdot + c)\}$.

(A3) $\forall F \in C_b^1(H)$, $E_{\mu_0}\left[\langle DF, e^{-2V}\cdot B\rangle_H\right] = 0$, q-a.s.

Main Theorem

The solution to $(*)$ satisfies the central limit theorem in probability with respect to the environment, i.e.,

Theorem (Central Limit Theorem). *Let $\mathbf{1}$ be the constant function $\mathbf{1}(x) \equiv 1$ on $[0,1]$, Φ_σ be the probability density function of a 1-d centered Gaussian distribution with variance σ^2, then*

$$\lim_{t\to\infty} E_q \left| E_P\left[F\left(\frac{u(t)}{\sqrt{t}}\right)\right] - \int_{\mathbb{R}} F(\mathbf{1}\cdot y)\Phi_\sigma(y)dy \right| = 0,$$

for any $F \in C_b(C[0,1])$, where σ is a constant defined in $(**)$ below.

Environmental set and Smooth Functions

Thanks to (A2), w.l.o.g., we can assume that there exists a group of shift operators $\{\tau_c : \Sigma\to\Sigma; c\in\mathbb{R}\}$ as well as an $\mathbb{R}\times H$-valued function $(\mathbf{V},\mathbf{B})$ defined on the following set

$$\Xi = \{\xi : [0,1]\to\Sigma, \xi(\cdot) = \tau_{u(\cdot)}\sigma | u\in C[0,1], \sigma\in\Sigma\},$$

such that $\tau_{c_1}\circ\tau_{c_2} = \tau_{c_1+c_2}$, q is invariant and ergodic under $\{\tau_c\}$ and $\{(V(u), B(u))\} \overset{d.}{=} \{\mathbf{V}\circ\tau_{u(\cdot)}, \mathbf{B}\circ\tau_{u(\cdot)}\}$. The next Lemma is self-explanatory, but still plays an important role in our proof.

Lemma. *Suppose that $\tau_{v_1(\cdot)}\sigma_1 = \tau_{v_2(\cdot)}\sigma_2$, then $v_1(x) - v_2(x) \equiv c$. Therefore $\forall\xi\in\Xi$, $\exists$ a unique $v_\xi \in C_0[0,1]$ s.t. $\xi(\cdot) = \tau_{v_\xi(\cdot)}\xi(0)$.*

For a function $\mathbf{f}$ on the environment set Ξ, its derivative $\mathcal{D}\mathbf{f}$ is defined as an H-valued function (if exists) such that

$$\lim_{\|h\|_H\downarrow 0} \frac{1}{\|h\|_H}\left[\mathbf{f}(\tau_{h(\cdot)}\xi(\cdot)) - \mathbf{f}(\xi) - \langle\mathcal{D}\mathbf{f}(\xi), h\rangle_H\right] = 0.$$

Environmental process

Define the **environmental process** $\{\xi_t\}_{t\geq 0}$ as

$$\xi_t(\sigma,\omega) \triangleq \left(\tau_{u(\sigma,\omega,t,x)}\sigma\right)_{x\in[0,1]} \in \Xi.$$

The Markovian property of ξ_t follows from the lemma before.

Proposition 1. *ξ_t is a continuous Markov process. Its invariant and ergodic probability measure is*

$$\pi(d\xi) = Z^{-1}e^{-2\mathbf{V}(\xi)}\cdot\mu_0(dv_\xi)\otimes q(d\xi(0)).$$

Let $(Dom(\mathcal{K}),\mathcal{K})$ be its infinitesimal generator on $\mathscr{H} = L^2(\Xi,\pi)$, then a sector condition holds for $\mathcal{K}$, i.e., there exists some constant $C = C(\mathbf{V},\mathbf{B})$ such that for all $\mathbf{f}$, $\mathbf{g}\in Dom(\mathcal{K})$,

$$\langle\mathcal{K}\mathbf{f},\mathbf{g}\rangle_\pi^2 \leq C(\mathbf{V},\mathbf{B})\langle-\mathcal{K}\mathbf{f},\mathbf{f}\rangle_\pi\langle-\mathcal{K}\mathbf{g},\mathbf{g}\rangle_\pi.$$

Given a test function φ on $[0,1]$, by Itô's formula we have

$$\langle u(t),\varphi\rangle_H = \langle u(0),\varphi\rangle_H + \int_0^t \mathbf{U}_\varphi(\xi_r)dr + \langle W_t,\varphi\rangle_H,$$

where $\mathbf{U}_\varphi(\xi) = \frac{1}{2}\langle v_\xi, \partial_x^2\varphi\rangle_H - \langle D\mathbf{V}(\xi) + \mathbf{B}(\xi),\varphi\rangle_H$. Since the second term is an additive functional of ξ_t, the central limit theorem just follows from Proposition 1, and the diffusion constant σ is

$$\sigma^2 = \lim_{\lambda\downarrow 0} E_\pi\|\mathcal{D}\mathbf{f}_{\lambda,\mathbf{1}} + \mathbf{1}\|_H^2, \qquad (**)$$

where $\mathcal{D}$ is in the weak sense and $\mathbf{f}_{\lambda,\mathbf{1}} = (\lambda - \mathcal{K})^{-1}\mathbf{U}_\mathbf{1}$.

A variational principle

The limit distribution of $t^{-\frac{1}{2}}u(t)$ decays in every direction which is orthogonal to constant; in the direction of constant functions, it does not decay, as stated in the following proposition.

Proposition 2. *If $B\equiv 0$, the diffusion constant σ in $(**)$ satisfies*

$$\sigma^2 = \inf_{\mathbf{f}\in C_b^1(\Xi)} E_\pi\left[\|\mathcal{D}\mathbf{f} + \mathbf{1}\|_H^2\right],$$

and there exists some strictly positive constant $C = C(\mathbf{V}) \leq 1$ such that $\sigma^2\in[C,1]$. If $B\not\equiv 0$, the lower bound still holds.

Two Examples of non-random coefficients

Example 1. *Let $\Sigma = [0,1]$ and q be the Lebesgue measure. Suppose that for each $x\in[0,1]$, $V(x,\cdot)\in C^1(\mathbb{R})$ is periodic: $V(x,y) = V(x,y+1)$. For $\sigma\in\Sigma$ and $u\in H$, $V(\sigma,u)\triangleq\int_0^1 V(x,u(x)+\sigma)dx$ and $B(\sigma,u)\triangleq 0$. It gives the equation with periodic coefficients.*

Example 2. *For $d\geq 2$ take $\Sigma = [0,1]^d$ and q to be the Lebesgue measure. Suppose that for each $x\in[0,1]$, $V(x,\cdot)\in C^1(\mathbb{R}^d)$ is periodic: $V(x,\mathbf{y}) = V(x,\mathbf{y}+\mathbf{e}_i)$, $\forall 1\leq i\leq d$. For $\boldsymbol{\lambda}\in\mathbb{R}^d$ with rationally independent coordinates, let $V(\sigma,u)\triangleq\int_0^1 V(x,u(x)\boldsymbol{\lambda}+\sigma)dx$ and $B(\sigma,u)\triangleq 0$. It gives the equation with quasi-periodic coefficients.*

References

[1] Funaki, T. 1983. Random motion of strings and related stochastic evolution equations. *Nagoya Math. J.* 89: 129-193.

[2] Komorowski, T., Landim, C., and Olla, S. 2012. *Fluctuations in Markov processes*. Springer-Verlag Berlin Heidelberg.

[3] Xu, L. *Central limit theorem for stochastic heat equations in random environments*, in preparation.