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Diffusion determines the recurrent graph

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Motivation

Kac famously posed the question, "Can one hear the shape of a drum?". Mathematically, this question asks whether two domains with unitarily equivalent Laplacians are necessarily congruent. It is well-known that the answer is "No" as long as a sufficiently large class of domains is allowed.
A closely related question by Arendt replaces the unitary intertwining operator by an order isomorphism and asks if the domains have to be congruent, that is:

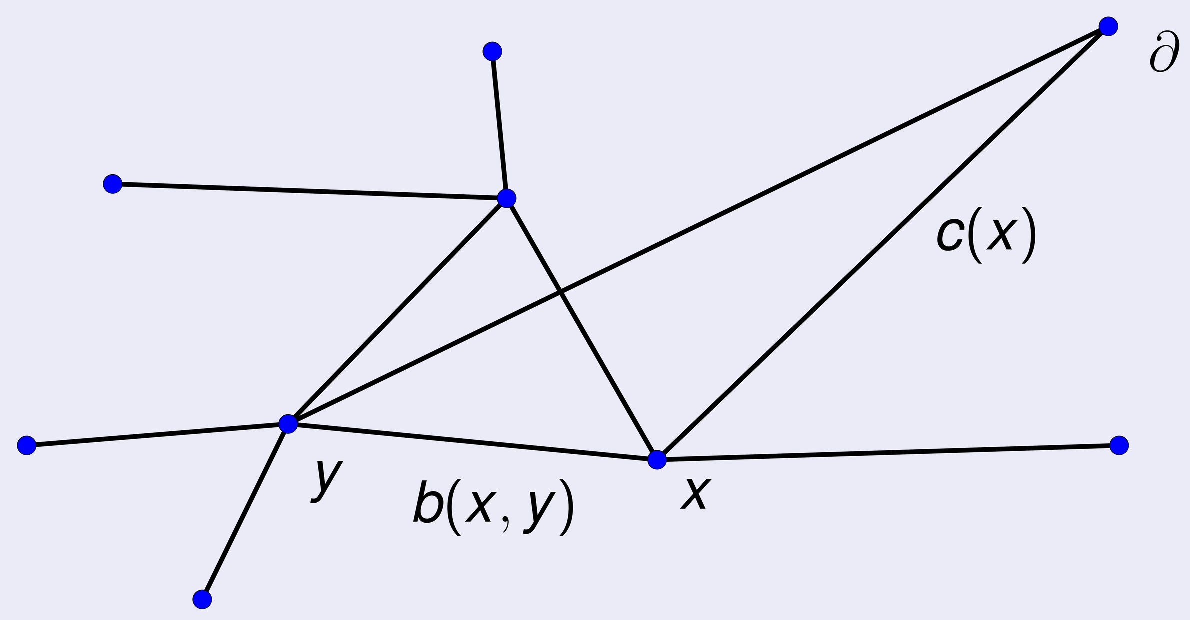
"Does diffusion determine the body?" [1] Contrary to the original question, the answer to this modified question is "Yes."
We study an analogous problem in a discrete setting, where the domains in Euclidean space and usual Laplace operators are replaced by weighted graphs and graph Laplacians. Similar to the above questions, the objective of our work could be paraphrased as "Does diffusion determine the graph structure?"

Graphs and discrete Dirichlet forms

A weighted graph is a quadruple (X, b, c, m) consisting of a vertex set X, a symmetric edge weight $b\colon X \times X \longrightarrow [0, \infty)$, a killing term $c\colon X \longrightarrow [0, \infty)$ and measure $m\colon X \longrightarrow (0, \infty)$ such that $b(x, x) = 0$ and $\sum_y b(x, y) < \infty$ for all $x \in X$.
The Dirichlet form with Dirichlet boundary conditions Q is defined as the closure of

$$\tilde{Q}(u, v) = \frac{1}{2}\sum_{x,y} b(x, y)(u(x) - u(y))(v(x) - v(y)) + \sum_x c(x)u(x)v(x)$$

on $C_c(X) \subset \ell^2(X, m)$. Its generator L in $\ell^2(X, m)$ is called Laplacian with Dirichlet boundary conditions.



Order isomorphisms of ℓ^p-spaces

A linear map $U\colon E \longrightarrow F$ between ordered vector spaces is called order isomorphism if it is bijective and $f \geq 0$ iff $Uf \geq 0$.
Let (X_i, m_i), $i \in \{1, 2\}$, be discrete measure spaces and $p \in [1, \infty)$. If $U\colon \ell^p(X_1, m_1) \longrightarrow \ell^p(X_2, m_2)$ is an order isomorphism, then there is an associated scaling $h\colon X_2 \longrightarrow (0, \infty)$ and a bijection $\tau\colon X_2 \longrightarrow X_1$ such that

$$Uf = h \cdot (f \circ \tau)$$

for all $f \in \ell^p(X_1, m_1)$.

Intertwining order isomorphisms

Let (X_i, b_i, c_i, m_i), $i \in \{1, 2\}$, be weighted graphs and $U\colon \ell^2(X_1, m_1) \longrightarrow \ell^2(X_2, m_2)$ an order isomorphism. Then U is said to intertwine L_1 and L_2 if $UD(L_1) = D(L_2)$ and $UL_1 f = L_2 Uf$ for all $f \in D(L_1)$. This is equivalent to $Ue^{tL_1} = e^{-tL_2}U$ for all $t \geq 0$.
It turns out that our setting is closely related to the original question concerning unitary intertwiners:
For every order isomorphism U intertwining the graph Laplacians L_1 and L_2 there is a constant $\beta > 0$ such that βU is unitary.

Metrics on graphs

In a connected graph, the combinatorial graph distance $d(x, y)$ is the minimal number of edges of a path connecting x and y.
Another useful (intrinsic) pseudometric is given by

$$\rho(x, y) = \inf_\gamma \sum_{k=1}^{n} \mathrm{Deg}(x_{k-1})^{-\frac{1}{2}} \wedge \mathrm{Deg}(x_k)^{-\frac{1}{2}},$$

where the infimum is taken over all paths $\gamma = (x_0, \ldots, x_n)$ connecting x and y.

The map τ as isometry

If $U\colon \ell^2(X_1, m_1) \longrightarrow \ell^2(X_2, m_2)$ is an order isomorphism intertwining the graph Laplacians, then the associated bijection τ is a graph isomorphism in the sense that $b_2(x, y) > 0$ if and only if $b_1(\tau(x), \tau(y)) > 0$. Moreover, the weighted vertex degree $\mathrm{Deg}(x) = \frac{1}{m(x)} \sum_y b(x, y)$ is preserved by τ.
Consequently, the bijection τ associated with U is an isometry with respect to both the combinatorial graph metric d and the pseudometric ρ.

A generalized ground state transform and recurrent graphs

If $(X, b, 0, m)$ is a graph, $\tau\colon X \longrightarrow Y$ a bijection and $H\colon X \longrightarrow (0, \infty)$ a harmonic function, then the graph $(Y, b_H, 0, m_H)$ defined by $b_H(\tau(x), \tau(y)) = H(x)H(y)b(x, y)$ and $m_H(\tau(x)) = H(x)^2 m(x)$ for $x, y \in X$ is called generalized ground state transform of $(X, b, 0, m)$.

If U is an order isomorphism intertwining the Laplacians on $(X_1, b_1, 0, m_1)$ and $(X_2, b_2, 0, m_2)$, then $(X_1, b_1, 0, m_1)$ is a generalized ground state transform of $(X_2, b_2, 0, m_2)$ and the harmonic function H in the definition differs from the scaling h only by a constant.

A connected graph is called recurrent if the semigroup generated by the Laplacian with Dirichlet boundary conditions L is recurrent, that is,

$$G(x, y) = \int_0^\infty e^{-tL}\delta_x(y)\, dt = \infty$$

for some (equivalently all) $x, y \in X$. Any generalized ground state transform of a recurrent graph $(X, b, 0, m)$ differs only by constant and a bijection from $(X, b, 0, m)$.

Main theorem [2]

Let $(X_i, b_i, 0, m_i)$, $i \in \{1, 2\}$, be recurrent graphs and let $U\colon \ell^2(X_1, m_1) \longrightarrow \ell^2(X_2, m_2)$ be an order isomorphism intertwining L_1 and L_2. Then there is a constant $\alpha > 0$ such that

$$b_2(x, y) = \alpha b_1(\tau(x), \tau(y)),$$
$$m_2(x) = \alpha m_1(\tau(x))$$

hold for all $x, y \in X_2$

Two special situations

In two important cases, full information of the graphs can be recovered: Let U be an order isomorphism intertwining the Laplacians on (X_i, b_i, c_i, m_i), $i \in \{1, 2\}$. If

- $m_i \equiv 1$

or

- $b_i\colon X_i \longrightarrow \{0, 1\}$ and $m_i(x) = \sum_y b(x, y)$,

then $b_1(\tau(x), \tau(y)) = b_2(x, y)$ and $c_1(\tau(x)) = c_2(x)$ for all $x, y \in X_2$.

References

[1] Arendt, W. *Does diffusion determine the body?* J. Reine Angew. Math. 550 (2002), 97–123.
[2] Keller, M; Lenz, D; Schmidt, M; Wirth, M. *Diffusion determines the recurrent graph.* Adv. Math. 269 (2015), 364–398.