Convergence of Brownian motions on $\mathrm{RCD}^*(K,N)$ spaces

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Abstract Suppose that a sequence of metric measure spaces (X_n, d_n, m_n) satisfies $\mathrm{RCD}^*(K,N)$ with $\mathrm{Diam}(X_n) \le D$ and $m_n(X_n) = 1$. Then we show that the following (A) and (B) are equivalent: (A) the measured Gromov–Hausdorff convergence of (X_n, d_n, m_n), (B) the weak convergence of the laws of Brownian motions on (X_n, d_n, m_n).

1 Motivation

(Q) Does convergence of Brownian motions follow only from convergence of the underlying spaces?

Let

- $\mathcal{X}_n = (X_n, d_n, m_n)$ "good" metric measure spaces;
- Ch_n Cheeger energies on $\mathcal{X}_n$;
- $\mathbb{B}_n = (\{B^n_t\}_{t\ge 0}, \{\mathbb{P}^n_x\}_{x\in X_n})$ Brownian motions on $\mathcal{X}_n$.

$$\begin{array}{ccc} \mathcal{X}_n & \xrightarrow{\textbf{(A)}\ \text{mGH}} & \mathcal{X}_\infty \\ {\scriptstyle \mathsf{Ch}_n}\downarrow & & {\scriptstyle \mathsf{Ch}_\infty}\downarrow \\ \mathbb{B}_n & \xrightarrow{\textbf{(B)}\ \text{in law}} & \mathbb{B}_\infty \end{array}$$

where mGH means *measured Gromov–Hausdorff convergence.*

The question (Q) means, more precisely,

(Q) *Does* **(A)** *imply* **(B)** *(or, vice versa)?*

2 $\mathrm{RCD}^*(K,N)$ spaces

- $(\mathcal{P}_2(X,d), W_2)$: L^2-Wasserstein space.
- $\mu \in \mathcal{P}_\infty(X,d,m) \overset{def}{\iff} \mu \in P_2(X,d)$ & bdd support & absol. cont. w.r.t. m.

Set, for $\theta \in [0,\infty)$,

$$\Theta_\kappa(\theta) = \begin{cases} \dfrac{\sin(\sqrt{\kappa}\theta)}{\sqrt{\kappa}} & \text{if} \quad \kappa > 0, \\ \theta & \text{if} \quad \kappa = 0, \\ \dfrac{\sinh(\sqrt{-\kappa}\theta)}{\sqrt{-\kappa}} & \text{if} \quad \kappa < 0, \end{cases}$$

and set for $t \in [0,1]$,

$$\sigma^{(t)}_\kappa(\theta) = \begin{cases} \dfrac{\Theta_\kappa(t\theta)}{\Theta_\kappa(\theta)} & \text{if} \quad \kappa\theta^2 \neq 0 \quad \text{and} \quad \kappa\theta^2 < \pi^2, \\ t & \text{if} \quad \kappa\theta^2 = 0, \\ +\infty & \text{if} \quad \kappa\theta^2 \ge \pi^2. \end{cases}$$

Definition 1 ([1, 2]). ($\mathrm{CD}^*(K,N)$ and $\mathrm{RCD}^*(K,N)$) Let $K \in \mathbb{R}$ and $1 < N < \infty$.

(i) (X,d,m) satisfies $\mathrm{CD}^*(K,N) \overset{def}{\iff} \forall \mu_0 = \rho_0 m, \mu_1 = \rho_1 m \in \mathcal{P}_\infty(X,d,m)$, $\exists$ opt. coupl. q of μ_0 and μ_1 $\exists$geod. $\mu_t = \rho_t m \in (\mathcal{P}_\infty(X,d,m), W_2)$ connect. μ_0 and μ_1 s.t. $\forall t \in [0,1]$ and $\forall N' \ge N$,

$$\int \rho_t^{-\frac{1}{N'}} d\mu_t \ge \int_{X\times X} \Big[\sigma^{(1-t)}_{K/N'}(d(x_0,x_1))\rho_0^{-1/N'}(x_0) + \sigma^{(t)}_{K/N'}(d(x_0,x_1))\rho_1^{-1/N'}(x_1)\Big] dq(x_0,x_1).$$

(ii) (X,d,m) satisfies $\mathrm{RCD}^*(K,N) \overset{def}{\iff}$

- $\mathrm{CD}^*(K,N)$
- the Cheeger energy Ch is linear:

$$2\mathsf{Ch}(u) + 2\mathsf{Ch}(v) = \mathsf{Ch}(u+v) + \mathsf{Ch}(u-v), \quad u,v \in W^{1,2}(X,d,m),$$

where

$$\mathsf{Ch}(u) = \frac{1}{2}\inf\Big\{\liminf_{n\to\infty} \int |\nabla u_n|^2 dm : u_n \in \mathrm{Lip}(X), \int_X |u_n - u|^2 dm \to 0\Big\}$$

$$W^{1,2}(X,d,m) = \{u \in L^2(X,m) : \mathsf{Ch}(u) < \infty\}.$$

3 Main result

Assumption 1. Let $1 < N < \infty$, $K \in \mathbb{R}$ and $0 < D < \infty$. For $n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, let $\mathcal{X}_n = (X_n, d_n, m_n)$ be a metric measure space satisfying

$$\begin{cases} \mathrm{RCD}^*(K,N) \\ \mathrm{Diam}(X_n) \le D \\ m_n(X_n) = 1. \end{cases}$$

Theorem 1. Suppose that Assumption 1 holds. Then the following **(A)** and **(B)** are equivalent:

(A) $\mathcal{X}_n \overset{mGH}{\to} \mathcal{X}_\infty$;

(B) There exist

$$\begin{cases} \text{a compact metric space } (X,d) \\ \text{isometric embeddings } \iota_n : X_n \to X \ (n \in \overline{\mathbb{N}}) \\ x_n \in X_n \ (n \in \overline{\mathbb{N}}) \end{cases}$$

such that

$$\iota_n(B^n_\cdot)_\# \mathbb{P}^{x_n}_n \to \iota_\infty(B^\infty_\cdot)_\# \mathbb{P}^{x_\infty}_\infty \quad \text{weakly}$$

in $\mathcal{P}(C([0,\infty);X))$. Here $\#$ means *push-forward.*

References

[1] K. Bacher and K.-T. Sturm. *J. Funct. Anal.*, **259**(1):28–56, 2010.

[2] M. Erbar, K. Kuwada, and K.-T. Sturm, *to appear, Invent. math.*.