Geometric structures of favorite sites of random walk

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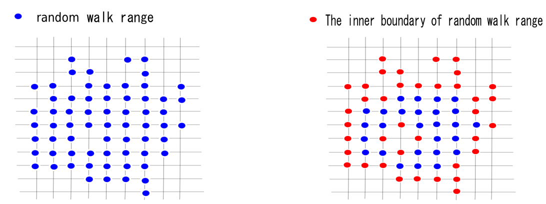
Definition

- Let $X_1, X_2, \ldots$ be i.i.d. $\mathbb{Z}^d$-valued random variables, $S_0 := 0$, $S_k := \Sigma_{i=1}^k X_i$ random walk taking values in $\mathbb{Z}^d$ with $d \geq 1$.
- $T_a := \inf\{m \geq 1 : S_m = a\}$
- $K(n; x) := \sum_{i=0}^n 1\{S_i = x\}$
- $R_n := \{x : K(n; x) \geq 1\} = \{S_0, S_1, \ldots, S_n\}$
- $R_n^{(p)} := \{x : K(n; x) = p\}$
- $\mathcal{N}(\mathrm{a})$:={z : dist(a,z)=1}
- $\partial R_n := \{x \in R_n : R_n \not\supset \mathcal{N}(x)\}$
- $\tilde{R}_n^{(p)} := \{x \in \partial R_n : k(n; x) = p\}$

The inner boundary points in $\mathbb{Z}^2$



Number of the eitire range and inner boundary points

Consider random walk in $\mathbb{Z}^d$ with $d \geq 1$.

Theorem 1 (Spitzer, 1976)**.**

$$\lim_{n\to\infty} \frac{|R_n|}{n} = P(0 \notin \{S_m\}_{m=1}^{\infty}) \quad a.s.$$

Theorem 2 (Jain and Pruitt, 1970)**.** *For simple random walk in $\mathbb{Z}^2$ it holds that*

$$E|R_n^{(p)}| = \pi^2 \frac{n}{(\log n)^2} + o. \quad \left(o = o\left(\frac{n}{(\log n)^2}\right)\right)$$

Theorem 3 (O., 2014)**.**

$$\lim_{n\to\infty} \frac{|\partial R_n|}{n} = P(\{S_m\}_{m=0}^{\infty} \cup \{S'_m\}_{m=0}^{\infty} \not\supset \mathcal{N}(0), \; 0 \notin \{S_m\}_{m=1}^{\infty}) \quad a.s.$$

For simple random walk in $\mathbb{Z}^2$ and $p \geq 1$,

$$\frac{c_1^{p-1}\pi^2 n}{4(\log n)^2} + o \leq E|\tilde{R}_n^{(p)}| \leq \frac{c_1^{p-1}\pi^2 n}{(\log n)^2} + o,$$

where $c_1 = P(T_0 < T_b)$ and $b \in \mathcal{N}(0)$.

Question

How often does random walk visit the sites in the inner boundary?
In particular, does the favorite site exist in the inner boundary for large time?

Set $M(n) := \max_{x\in\mathbb{Z}^d} K(n; x)$.

Theorem 4 (Erdös and Taylor,1960)**.** *It holds that for simple random walk in $\mathbb{Z}^d$ $(d \geq 3)$,*

$$\lim_{n\to\infty} \frac{M(n)}{\log n} = -\frac{1}{\log P(T_0 < \infty)} \quad a.s.,$$

and for simple random walk in $\mathbb{Z}^2$

$$\frac{1}{4\pi} \leq \liminf_{n\to\infty} \frac{M(n)}{(\log n)^2} \leq \limsup_{n\to\infty} \frac{M(n)}{(\log n)^2} \leq \frac{1}{\pi} \quad a.s.$$

Theorem 5 (Dembo, Peres, Rosen and Zeitouni, 2001)**.** *It holds that for simple random walk in $\mathbb{Z}^2$*

$$\lim_{n\to\infty} \frac{M(n)}{(\log n)^2} = \frac{1}{\pi} \quad a.s.$$

Set $\tilde{M}(n) := \max_{x\in\partial R_n} K(n; x)$.

Theorem 6 (O., 2014)**.** *It holds that for simple random walk in $\mathbb{Z}^d$ $(d \geq 2)$,*

$$\lim_{n\to\infty} \frac{\tilde{M}(n)}{\log n} = -\frac{1}{\log P(T_0 < T_b)} \quad a.s.$$

Conclusion

$$P\left(\lim_{n\to\infty} \frac{M(n)}{(\log n)^2} \neq \lim_{n\to\infty} \frac{\tilde{M}(n)}{(\log n)^2} \right) = 1.$$

In other words, the favorite site does not appear in the inner boundary for all but finitely many n a.s.