Lamplighter Random Walks on Fractals

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Motivation : Make clear the relation between geometrical properties of graphs and asymptotics of random walks.

Image of Lamplighter Random Walks

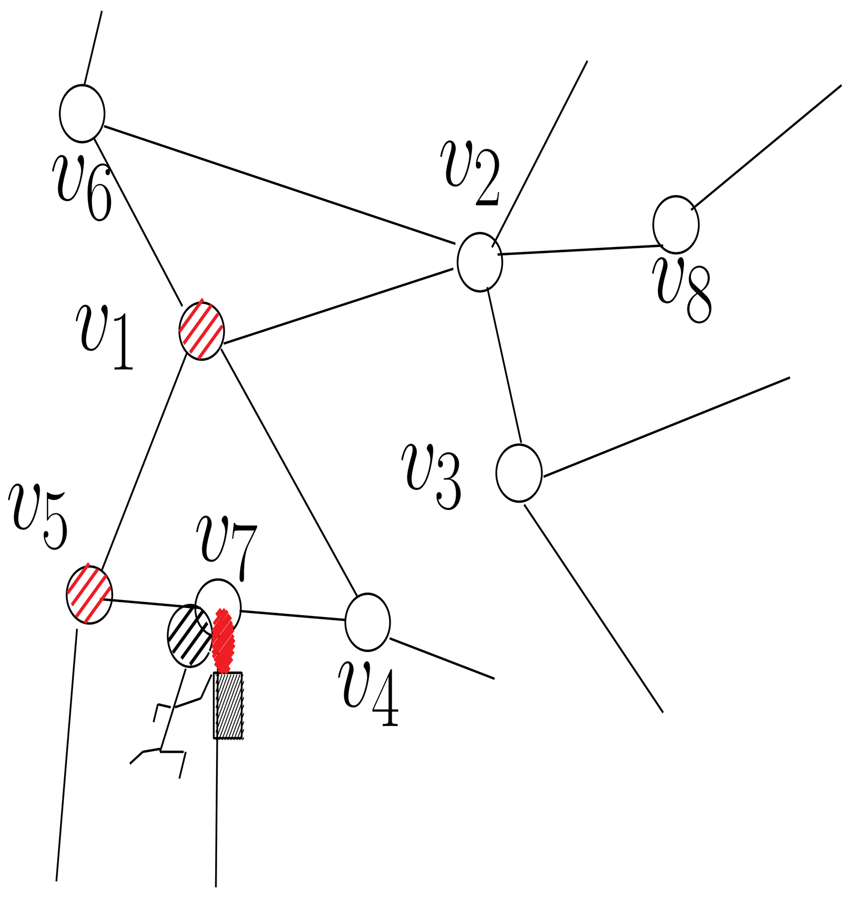


Figure: We denote this figure as (η, v_7), where $\eta : V(G) \to \{0,1\}$ where $\eta(u) = 1$ if $u = v_1, v_5$ and $\eta(u) = 0$ (otherwise)

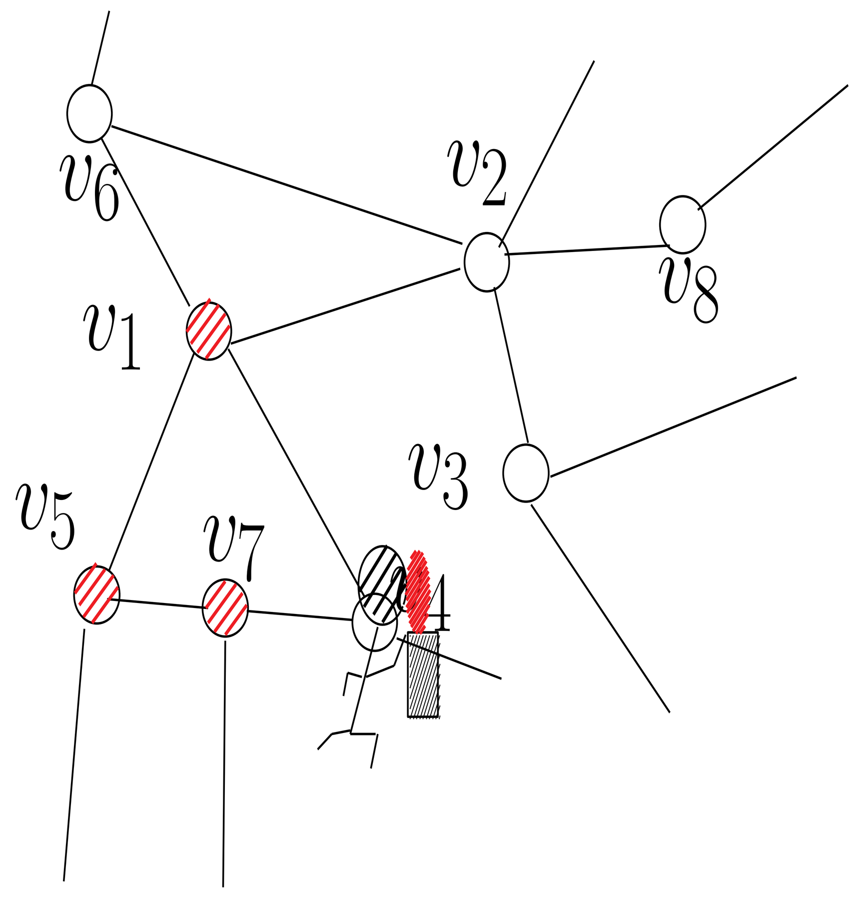


Figure: We denote this figure as (η, v_4), where $\eta : V(G) \to \{0,1\}$ where $\eta(u) = 1$ if $u = v_1, v_5, v_7$ and $\eta(u) = 0$ (otherwise)

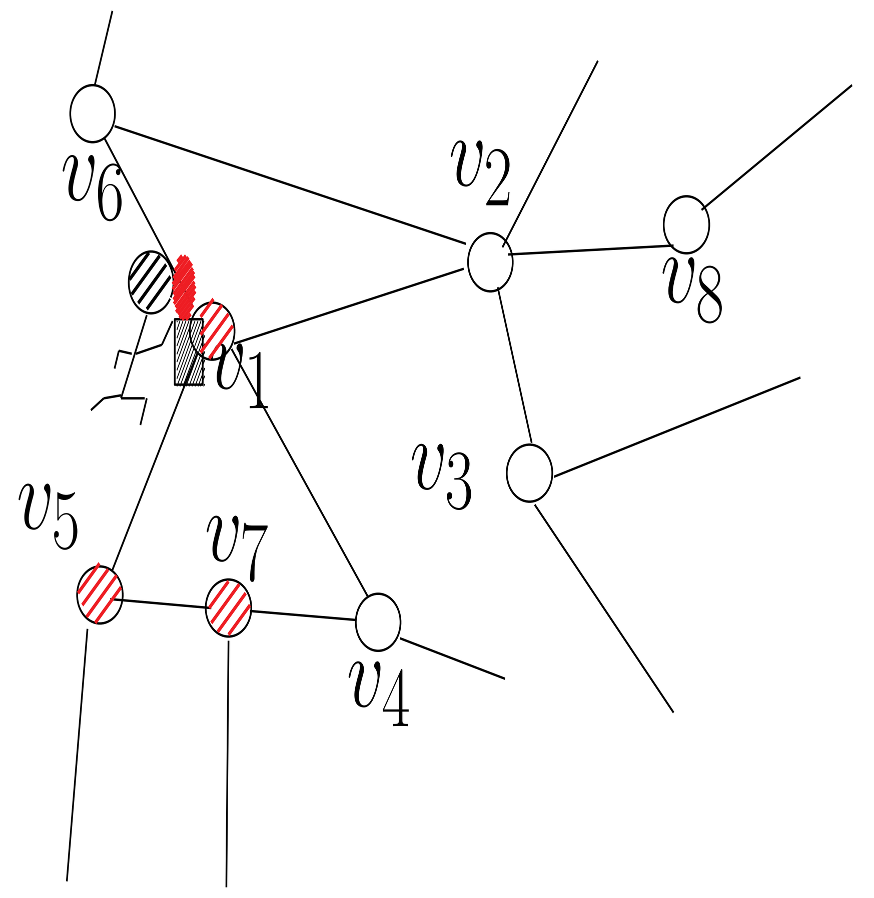


Figure: We denote this figure as (η, v_2), where $\eta : V(G) \to \{0,1\}$ where $\eta(u) = 1$ if $u = v_5, v_7$ and $\eta(u) = 0$ (otherwise)

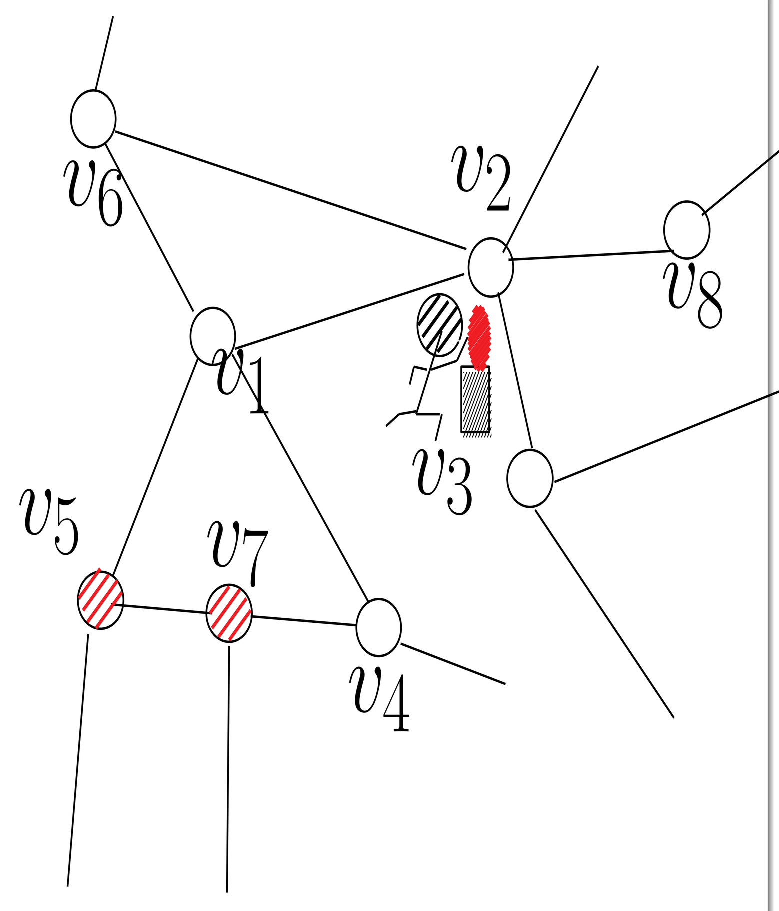
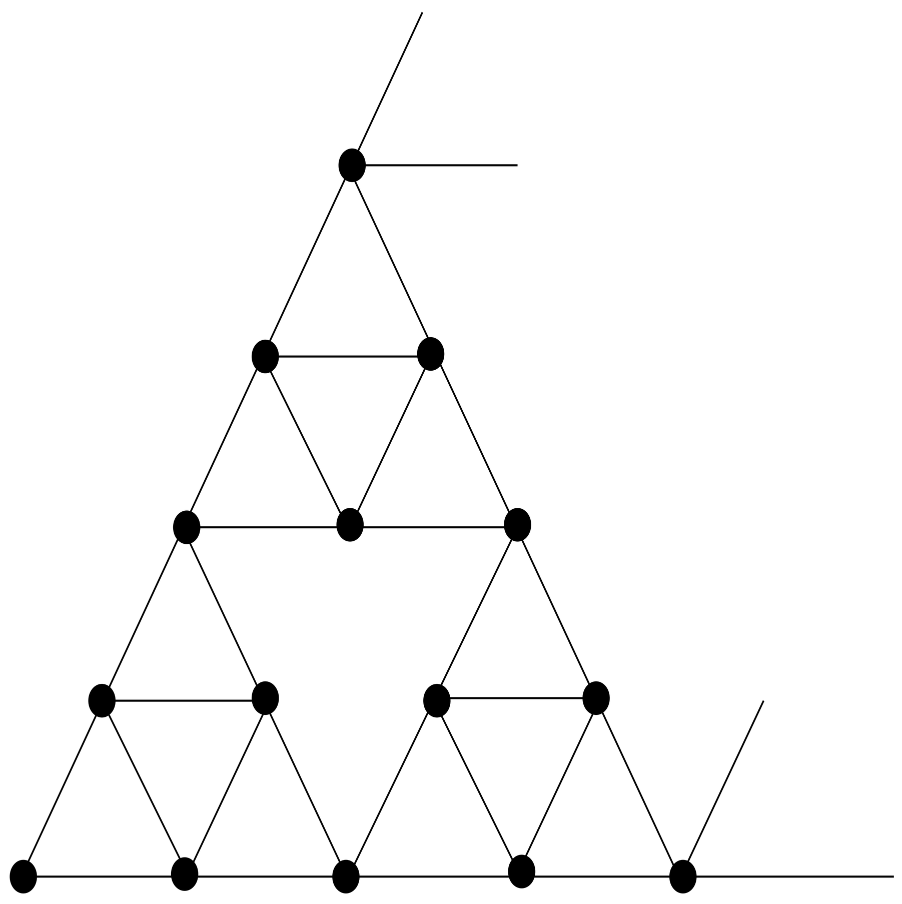


Figure: We denote this figure as (η, v_1), where $\eta : V(G) \to \{0,1$ where $\eta(u) = 1$ if $u = v_1, v_5, v_7$; $\eta(u) = 0$ (otherwise)

Known Results

- Pittet and Saloff-Coste (1996) obtained the on-diagonal heat kernel estimates for the case of the lamplighter graph of $G = \mathbb{Z}^d$ (denoted by $\mathbb{Z}_2 \wr \mathbb{Z}^d$).
$$p_{2n}(\mathbf{g}, \mathbf{g}) \approx \exp\left[-n^{d/(d+2)}\right].$$
- Revelle (2003) obtained the LIL for the case of the lamplighter graph of $\mathbb{Z}$ (denoted by $\mathbb{Z}_2 \wr \mathbb{Z}$).
$$c_1 \le \limsup_{n\to\infty} \frac{d_{\mathbb{Z}_2\wr\mathbb{Z}}(Y_0, Y_n)}{(n \log\log n)^{\frac{1}{2}}} \le c_2, \qquad P\text{-a.s.},$$
$$c_3 \le \liminf_{n\to\infty} \frac{d_{\mathbb{Z}_2\wr\mathbb{Z}}(Y_0, Y_n)}{(n/\log\log n)^{\frac{1}{2}}} \le c_4, \qquad P\text{-a.s.}.$$
- Question ; How do the exponents change when we replace $\mathbb{Z}^d$ to general graph G?

Fractal



- Volume $V(x,r) = \sharp B(x,r) \sim r^{d_f}$
- $E[d(X_0, X_n)] \sim n^{1/d_w}$ (d_w: Walk dimension ≥ 2)

Main Results

Theorem (On-diagonal Heat Kernel Estimates)

Suppose that Assumption $(*)$ holds. Then for the random walks $\{Y_n\}$ on the lamplighter graph, we have
$$p_{2n}^{\mathbb{Z}_2\wr G}(\mathbf{x}, \mathbf{x}) \approx \exp[-n^{\frac{d_f}{d_f+d_w}}]$$
for all $\mathbf{x} = (\eta, x) \in \mathbb{Z}_2 \wr G$ (lamplighter graph of G).

Theorem (The LIL for the case of $d_f/d_w < 1$)

Assume that Assumption $(*)$ and $d_f/d_w < 1$ hold. Then there exist (non-random) constants $c_{11}, c_{12}, c_{13}, c_{14} > 0$ such that the following hold for all $\mathbf{x} \in V(\mathbb{Z}_2 \wr G)$:
$$c_{11} \le \limsup_{n\to\infty} \frac{d_{\mathbb{Z}_2\wr G}(Y_0, Y_n)}{n^{d_f/d_w}(\log\log n)^{1-d_f/d_w}} \le c_{12}, \qquad P_{\mathbf{x}}\text{-a.s.}$$
$$c_{13} \le \liminf_{n\to\infty} \frac{d_{\mathbb{Z}_2\wr G}(Y_0, Y_n)}{n^{d_f/d_w}(\log\log n)^{-d_f/d_w}} \le c_{14}, \qquad P_{\mathbf{x}}\text{-a.s.}$$

Theorem (The LIL for the case of $d_f/d_w > 1$)

Assume that Assumption $(*)$ and $d_f/d_w > 1$ hold. Then there exist (non-random) positive constants $c_{21}, c_{22}, c_{23}, c_{24}$ such that the following hold for all $\mathbf{x} \in V(\mathbb{Z}_2 \wr G)$:
$$c_{21} \le \limsup_{n\to\infty} \frac{d_{\mathbb{Z}_2\wr G}(Y_0, Y_n)}{n} \le c_{22}, \qquad P_{\mathbf{x}}\text{-a.s.}$$
$$c_{23} \le \liminf_{n\to\infty} \frac{d_{\mathbb{Z}_2\wr G}(Y_0, Y_n)}{n} \le c_{24}, \qquad P_{\mathbf{x}}\text{-a.s}$$