The conservativeness of Girsanov transformed symmetric Markov processes

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1 Settings

- E: locally compact separable metric space
- m: Radon measure with $\mathrm{supp}[m] = E$
- $\mathbb{M} = (\Omega, X_t, \mathbb{P}_x, \zeta)$: m-symmetric Hunt process
- $(\mathcal{E}, \mathcal{F})$: Dirichlet form on $L^2(E;m)$ generated by $\mathbb{M}$

$$\mathcal{F}^{\dagger}_{\mathrm{loc}} := \left\{ u \in \mathcal{F}_{\mathrm{loc}} \,\middle|\, \int_{y\in E} (u(x)-u(y))^2 J(dx,dy) \text{ is a Radon measure} \right\}$$

Fukushima's decomposition [Kuwae ('10, '12)]

For $u \in \mathcal{F}^{\dagger}_{\mathrm{loc}}$, $u(X_t) - u(X_0)$ can be decomposed as

$$u(X_t) - u(X_0) = M^{[u]}_t + N^{[u]}_t,$$

where $M^{[u]}$: martingale AF, $N^{[u]}$: continuous AF.

2 Girsanov transformations

Take nonnegative $\rho \in \mathcal{F}^{\dagger}_{\mathrm{loc}}$.

- $E_\rho := \{x \in E : 0 < \rho(x) < \infty\}$
- L^ρ_t the solution to

$$L^\rho_t = 1 + \int_0^t L^\rho_{s-} \frac{1}{\rho(X_{s-})} dM^{[\rho]}_s, \quad t < \zeta \wedge \tau_{E_\rho}.$$

$\Longrightarrow$ L^ρ_t : supermartingale multiplicative functional

$d\widetilde{\mathbb{P}}_x := L^\rho_t d\mathbb{P}_x$.

$\rightsquigarrow$ $\widetilde{\mathbb{M}}^\rho = (\Omega, X_t, \widetilde{\mathbb{P}}_x)$ is a symmetric Markov process on E_ρ (*the Girsanov transformed process*)

$(\widetilde{\mathcal{E}}^\rho, \widetilde{\mathcal{F}}^\rho)$: Dirichlet form generated by $\widetilde{\mathbb{M}}^\rho$.

3 Main Result

Theorem 1

Let $\rho \in \mathcal{F}_e \cap \mathfrak{B}_b(E)$ with $\rho > 0$. Then

(i) $(\widetilde{\mathcal{E}}^\rho, \widetilde{\mathcal{F}}^\rho)$ is recurrent,

(ii) $\mathcal{F} \subset \widetilde{\mathcal{F}}^\rho$ and for $u \in \mathcal{F}$,

$$\widetilde{\mathcal{E}}^\rho(u,u) = \int_E \rho(x)^2 \mu^c_{\langle u\rangle}(dx) + \int_{E\times E} (u(x)-u(y))^2 \rho(x)\rho(y) J(dx,dy).$$

Remark.

The conservativeness of $\mathbb{M}$ is not assumed.
However, for $\rho \in \mathcal{F}_e$, $\widetilde{\mathbb{M}}^\rho$ is always conservative.

Theorem 2

Assume $\mathbb{M}$ is conservative and $\mu_{\langle\rho\rangle}(E) < \infty$.
Then $\widetilde{\mathbb{M}}^\rho$ is conservative and never hits $\{\rho(x) = 0\}$.
i.e., $\mathbb{P}_x(\zeta \wedge \tau_{E_\rho} = \infty) = 1$, $\rho^2 m$-a.e.

Key Lemma

If there exists a constant $c > 1$ s.t. $c^{-1} \le \rho \le c$,
then $\widetilde{\mathbb{M}}^\rho$ is conservative.

$\Updownarrow$

L^ρ_t is a martingale under $\mathbb{P}_x$ $\cdots (*)$

Idea of the proof.

We cannot apply Novikov's theorem to show $(*)$.
$\rightsquigarrow$ We check a condition due to Z.-Q. Chen ('12).

4 Application

Assumption. $\mathbb{M}$: transient, irreducible and strong Feller

μ: positive Green-tight measure ($^\forall\varepsilon > 0$, $^\exists K$: cpt, $^\exists\delta > 0$ s.t. $^\forall B \subset K$ with $\mu(B) < \delta$, $\|R(\mathbb{1}_{K^c \cup B}\,\mu)\|_\infty < \varepsilon$)

$$\xrightarrow{\text{Takeda('14)}} \ ^\exists h \in \mathcal{F}_e \text{ s.t. } h > 0, \ \int_E h^2 d\mu = 1, \ \mathcal{E}(h,h) = \lambda(\mu) \ \left(:= \inf\left\{ \mathcal{E}(u,u) : u \in \mathcal{F}_e, \int_E u^2 d\mu = 1 \right\} \right),$$

$$\text{and } h(X_t) - h(X_0) = M^{[h]}_t - \int_0^t h(X_s)\, dA^{\lambda(\mu)\cdot\mu}_s.$$

$\Downarrow$ Ito's formula applied to $\log h(X_t)$

$$\frac{h(X_t)}{h(X_0)} \exp A^{\lambda(\mu)\cdot\mu}_t = L^h_t$$

$\Downarrow$ Theorem 1

$$\mathcal{E}(hu,hu) - \lambda(\mu)\int_E (hu)^2 d\mu = \frac{1}{2}\int_E h(x)^2 \mu^c_{\langle u\rangle}(dx) + \int_{E\times E} (u(x)-u(y))^2 h(x)h(y) J(dx,dy), \quad u \in \mathcal{F}.$$