Minimizing movement scheme for time dependent gradient flows in probability space

Bonn International Graduate School in Mathematics

universitätbonn

Kopfer, Eva

Advisor : Prof. Dr. Karl-Theodor Sturm

Let M be a smooth closed manifold equipped with a smooth family of Riemannian metrics g_t.

We suppose that g_t evolves under the **Ricci flow**

$$\partial_t g = -2Ric(g) \qquad t \in [0,T],$$

where $Ric(g)$ denotes the Ricci tensor of g.

We consider the **forward heat equation** acting on functions

$$\partial_t u = \Delta u,$$

where $\Delta = \Delta_{g(t)}$ denotes the Laplace-Beltrami operator.
By duality we obtain the **backward heat equation** on measures

$$\partial_t \nu = -\Delta \nu. \tag{1}$$

Aim
We are interested in **reinterpreting the diffusion equation (1) as a dynamic gradient flow** for the **relative entropy** with respect to the **Kantorovich distance**.
Method
At first we need to understand the meaning of gradient flows in a metric setting, where the metric is changing.
To establish existence of the trajectory, we apply the **time discrete approximation** of the gradient flow .

Let $\mathcal{P}(M)$ be the space of **Borel probability measures** on M. We consider the follwing two objects on this space:

- the **Kantorovich distance** at time t between two measures $\mu,\nu \in \mathcal{P}(M)$ is defined by
$$W_t(\mu,\nu) := \inf \left\{ \int_{M\times M} d_t^2(x,y)\,d\pi(x,y) \right\}^{1/2},$$
where the infimum is taken over all $\pi \in \mathcal{P}(M\times M)$ such that $\pi(A\times M) = \mu(A)$, $\pi(M\times A) = \nu(A)$;
- the **relative entropy** H_t with respect to the volume measure m_t, which is defined by
$$H_t(\mu) := \int_M \rho \log \rho \, dm_t,$$
provided that $\mu \in \mathcal{P}(M)$ is absolutely continuous with density ρ. Otherwise $H_t(\mu) = \infty$.

Motivation (dynamic gradient flow)
Let $V\colon [0,T]\times M \to \mathbb{R}$ be a smooth function, $x\colon [0,T] \to M$ a smooth curve. $G(a,b)$ denotes the Green function on $[0,1]$.

$$V \text{ is } \textbf{dynamically convex} \; :\Leftrightarrow \mathrm{Hess}_t V_t \geq \frac{1}{2}\partial_t g_t$$
$$\Leftrightarrow V_t(\gamma_a) \leq (1-a)V_t(\gamma_0) + aV_t(\gamma_1) - \frac{1}{2}\int_0^1 G(a,b)\partial_t g_t(\dot\gamma_b,\dot\gamma_b)db$$
$$\forall d_t\text{-geodesics } (\gamma_a).$$

Then the following are equivalent

$$\dot x_t = -\nabla_t V_t(x_t)$$
$$\Leftrightarrow \quad \partial_s \frac{1}{2} d_t^2(x_s,z)|_{s=t} \leq V_t(z) - V_t(x_t) - \frac{1}{2}\int_0^1 (1-b)\partial_t g_t(\dot\gamma_b,\dot\gamma_b)db \quad \forall d_t\text{-geodesics } \gamma \text{ from } x_t \text{ to } z.$$

For each x_0 there exists **at most one** dynamic gradient flow for V sarting in x_0.

Definition
A functional V on $\mathcal{P}(M)$ is **dynamically convex** if for every W_t geodesic (γ_a)

$$V_t(\gamma_a) \leq (1-a)V_t(\gamma_0) + aV_t(\gamma_1) - \frac{1}{2}\int_0^1 G(a,b)\partial_t(|\dot\gamma_b|_t^2)db,$$

where $|\dot\gamma_b|_t := \lim_{\epsilon\to 0} \frac{W_t(\gamma_{b+\epsilon},\gamma_b)}{\epsilon}$ is the metric speed with respect to W_t.

A **dynamic gradient flow** for V is a curve (μ_t) in $\mathcal{P}(M)$ satisfying the **dynamic Evolution Variation Inequality (EVI)**, i.e.

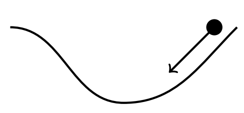
$$\partial_s \frac{1}{2} W_t^2(\mu_s,\sigma)|_{s=t} \leq V_t(\sigma) - V_t(\mu_t) - \frac{1}{2}\int_0^1 (1-b)\partial_t(|\dot\gamma_b|_t^2)db \quad \forall \sigma,$$

where (γ_b) is a W_t-geodesic from μ_t to σ.

Backward dynamic convexity and backward EVI
The Ricci flow ensures the **backward dynamic convexity of the relative entropy** H, i.e. $\tilde H$ is dynamically convex with respect to $\tilde W$, where $\tilde H_t = H_{T-t}$.
Since we are dealing with the backward heat equation, we have to deal with the EVI backwards in time, too, i.e. (μ_t) satisfies the **backward EVI** for H with respect to W if $(\tilde\mu_t)$ satisfies the EVI for $\tilde H$ with respect to $\tilde W$, where $\tilde\mu_t = \mu_{T-t}$.

Motivation (time discrete approximation)
Let $F\colon \mathbb{R}^d \to \mathbb{R}$ be a convex differentiable functional. Consider the gradient flow (x_t) starting from $\bar x \in \mathbb{R}^d$

$$\begin{cases} \dot x_t = -\nabla F(x_t), \\ x_0 = \bar x. \end{cases}$$



Fix a step size $h>0$, set $x_0 = \bar x$ and recursively determine x_n that minimizes

$$x \quad \mapsto \quad F(x) + \frac{|x-x_{n-1}|^2}{2h}. \tag{2}$$

The Euler-Lagrange equation of x_n is

$$\frac{x_n - x_{n-1}}{h} = -\nabla F(x_n),$$

which is a time discretization of $\dot x_t = -\nabla F(x_t)$.
Define the interpolation

$$x_t^h := x_n^h \text{ if } t \in ((n-1)h, nh].$$

Then as $h \to 0$

$$x_t^h \to x_t \text{ and } x_t \text{ is a gradient flow starting in } \bar x.$$

Fix a time step $h>0$ and an initial value $\bar\mu$.
Recursively define the **sequence of minimizers** $(\mu_n^h)_n$ according to the following discrete scheme

$$\mu_0^h := \bar\mu, \qquad \mu_n^h := \arg\min_{\nu\in\mathcal{P}(M)} \left\{ H_{nh}(\nu) + \frac{1}{2h} W_{nh}^2(\mu_{n-1}^h,\nu) \right\}.$$

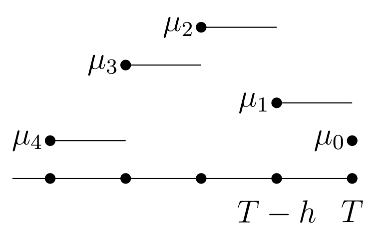
Existence of minimizers is provided by compactness and lower semicontinuity.
Uniqueness follows by usual strict convexity.

Euler-Lagrange equation for the minimizers
The minimizers μ_n^h satisfy the **discrete gradient flow equation**

$$-\int_M \mathrm{div}_{nh}\,\xi\, d\mu_n^h = \int_M \left\langle \frac{1}{h}\Psi_{\mu_n^h}^{\mu_{(n-1)}^h}, \xi \right\rangle_{nh} d\mu_n^h \qquad \xi \in \mathcal{C}_c^\infty(M,TM),$$

i.e. for $\mu_n^h = \rho_n^h m_{nh}$ we have $\nabla_{nh}\rho_n^h/\rho_n^h = \frac{1}{h}\Psi_{\mu_n^h}^{\mu_{(n-1)}^h}$, where $\Psi_{\mu_n^h}^{\mu_{(n-1)}^h}$ denotes the optimal transport vector field between μ_n^h and μ_{n-1}^h.

Limit trajectory
Define piecewise constant interpolations $\bar\mu_t^h$ backwards in time:



$$\bar\mu_T^h := \bar\mu,$$
$$\bar\mu_t^h := \mu_n^h, \text{ for } t \in [T-nh, T-(n-1)h).$$

With the help of Arzela-Ascoli one obtains a **limit curve** (μ_t) such that up to a subsequence

$$\bar\mu_t^{h_n} \to \mu_t \text{ weakly for all } t \text{ as } h_n \to 0.$$

Questions

- Is the limit curve (μ_t) the **solution to the backward heat equation (1)**?
- Does the limit curve (μ_t) satisfy the **backward EVI** for H with respect to W?

Possible Generalizations

- (M,g_t) is not compact but complete and evolves under a **super-Ricci flow**
$$\partial_t g \geq -2Ric(g).$$
- Consider instead of a manifold a familiy of metric measure space $(X,d_t,m_t)_t$ evolving under a super Ricci-flow, which means that the relative entropy H is backward dynamically convex.

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eva.kopfer@iam.uni-bonn.de