Finite particle approximation of interacting Brownian motion

Yosuke Kawamoto (Kyushu University) E-mail : y-kawamoto@math.kyushu-u.ac.jp

Abstract

Consider interacting Brownian motion as infinite particle system. This system is described by infinite-dimensional SDE (ISDE). We construct a general theory of finite particle approximation. This general theory can be applied to many examples, for instance, ISDE related to random matrices, ISDE interacted Ruelle class potential.

1. General theory of finite particle approximation

1.1 Random point field and logarithmic derivative

Let $S(=R^d)$ be a state space and S be a configuration space defined by

$$\mathsf{S}=\{\mathsf{s}=\sum\delta_{s_i}\,;s_i\in S,\quad \mathsf{s}(K)<\infty \text{ for any compact set } K\}.$$

A probability measure μ on S is called random point field (RPF).

Let $\mu^{[1]}$ be 1-Campbell measure on $\mathsf{S}\times\mathsf{S}$ by

$$\mu^{[1]}(A\times B)=\int_A \mu_x(B)\rho^1(x)dx,$$

where μ_x is (reduced) Palm measure conditioned at x and ρ^1 is 1-correlation function.

Definition ('12 Osada)

$d_\mu\in L^p_{loc}(\mu^{[1]})$ is called logarithmic derivative of μ if for any $f\in C_0^\infty(S)\otimes D_0$,

$$\int_{S\times\mathsf{S}}\nabla_x f\, d\mu^{[1]}=-\int_{S\times\mathsf{S}} d_\mu\, f\, d\mu^{[1]}.$$

Remark

Logarithmic derivative is a key to connect RPF and SDE as we will show later. However, if μ has infinitely many particles, it is sometimes difficult to calculate the logarithmic derivative d_μ. In this case, we use finite particle approximation of logarithmic derivative as we will show next section.

1.2 Finite particle approximation of logarithmic derivative

Let $\mu^N(N\in\mathbb{N})$ be a sequence of RPF such that μ^N has exactly N particles and $\mu^N\to\mu$ as N goes to infinity.

Decompose the logarithmic derivative d_{μ^N} as follows.

$$d_{\mu^N}(x,\mathsf{s})=u^N(x)+\mathsf{g}_r(x,\mathsf{s})+\mathsf{w}_r(x,\mathsf{s}),$$

where

$$u^N:\ \text{continuous},$$

$$\mathsf{g}_r(x,\mathsf{s})=\sum_{|x-s_i|<r} g(x,s_i),$$

$$\mathsf{w}_r(x,\mathsf{s})=\sum_{|x-s_i|\ge r} g(x,s_i),$$

for $g:S^2\to S$ continuous except diagonal set and $\mathsf{s}=\sum_i\delta_{s_i}$.

We assume two conditions.

(A1) $u^N\to u$ compact uniformly as $\mathrm{N}\to\infty$.

(A2) There exist $p>1$ and $w\in L^{\mathrm{p}}_{\mathrm{loc}}(\mathsf{S})$ such that

$$\lim_{r\to\infty}\limsup_{N\to\infty}\int_{S_R\times\mathsf{S}}|w_r(x,\mathsf{s})-\mathrm{w}(x)|^{\mathrm{p}}\,d\mu^{N,[1]}=0.$$

Lemma ('12 Osada)

Assume (A1) and (A2), then d_μ exists and given by

$$d_\mu=\mathrm{u}(\mathrm{x})+\lim_{r\to\infty}\mathsf{g}_r(x,\mathsf{s})+w(x).$$

1.3 Main Theorem

In previous section, we introduced finite approximation in logarithmic derivative sense. Our result is following. If finite approximation of logarithmic derivative is valid, then finite particle approximation of corresponding ISDE is also valid.

We introduce N-dim SDE corresponding to μ^N and ISDE corresponding to μ by following.

$$(1)\ \ dX_t^{\mathrm{N,i}}=\mathrm{dB}_t^{\mathrm{i}}+d_{\mu^N}\left(X_t^{N,i},X_t^{N,i\blacklozenge}\right)dt\quad(1\le i\le \mathrm{N}),$$

$$(2)\ \ dX_t^{\mathrm{i}}=\mathrm{dB}_t^{\mathrm{i}}+d_\mu\ (X_t^{\mathrm{i}},X_t^{\mathrm{i}\blacklozenge})dt\quad(i\in\mathbb{N}),$$

where $X_t^{N,i\blacklozenge}=\sum_{j\ne i}\delta_{X_t^{N,j}}$ and $X_t^{\mathrm{i}\blacklozenge}=\sum_{j\ne i}\delta_{X_t^j}$.

Define $l:\mathsf{S}\to S^\infty$ as labeled map and denote l^m as the first m particles of l.

Let $(X^{N,1},\dots,X^{N,N})$ be a solution of (1) whose initial distribution is $\mu^N\circ l^{-1}$.

Let $(X^1,X^2\dots)$ be a solution of (2) whose initial distribution is $\mu\circ l^{-1}$.

Thm 1 (K. -Osada)

Assume (A1) and (A2), (2) has the unique solution, and marginal assumptions.

Furthermore , we assume that

$$\mu^N\circ(l^m)^{-1}\to\mu\circ(l^m)^{-1}\text{ in distribution.}$$

Then for any $m\in\mathbb{N}$, we have

$$\lim_{N\to\infty}\left(X^{N,1},\dots,X^{N,m}\right)=(X^1,\dots,X^m)\ \text{ weakly in } C([0,\infty);S^m).$$

2. Examples of finite particle approximation

Example1 ISDE related to sine RPF

Consider Gaussian unitary ensemble (GUE), i.e. $N\times N$ Hermitian matrix whose entries are i.i.d Gaussian distribution.

Eigenvalues of GUE $\mathbf{x}_N=(x_1,\dots,x_N)$ have the following probability density.

$$m^N(d\mathbf{x}_N)=\frac{1}{Z}\prod_{i\ne j}|x_i-x_j|^2\exp\left(-\sum_{1\le i\le N}|x_i|^2\right)d\mathbf{x}_N.$$

Scaled empirical measure of GUE weakly converges to Wigner's semi-circle law.

$$\lim_{N\to\infty}\frac{1}{N}\sum_i\delta_{\frac{x_i}{\sqrt{N}}}=\frac{1}{\pi}\sqrt{2-x^2}dx.$$

Next we take bulk scaling of the semi-circle law. Fix θ satisfying $-\sqrt{2}<\theta<\sqrt{2}$. Put $x_i=\frac{s_i+N\theta}{\sqrt{N}}$ of m^N and denote μ_θ^N as the density with regard to $\mathbf{s}_N$.

$$\mu_\theta^N(d\mathbf{s}_N)=\frac{1}{Z}\prod_{i\ne j}|s_i-s_j|^2\exp\left(-\sum_{1\le i\le N}\frac{|s_i+N\theta|^2}{N}\right)d\mathbf{s}_N.$$

μ_θ^N converges to the sine RPF μ_θ as N goes to infinity.

Remark

In the limit $\mu_{\mathrm{sin},\theta}$, the particles density is depend on θ, but particles interact sine kernel which is independent of θ. In this sense, we say that bulk scaling has geometrical universality.

Next, consider dynamics corresponding to this geometrical universality. N-dim SDE (ISDE) corresponding to μ_θ^N (μ_θ) given by (3) ((4) respectively).

$$(3)\ dX_t^{\mathrm{N,i}}=\mathrm{dB}_t^{\mathrm{i}}+\sum_{1\le j\ne i\le N}\frac{1}{X_t^{N,i}-X_t^{N,j}}dt-\frac{X_t^{N,i}}{N}dt-\theta dt\quad(1\le i\le \mathrm{N}),$$

$$(4)\ dX_t^i=dB_t^i+\lim_{r\to\infty}\sum_{|X_t^i-X_t^j|<r}\frac{1}{X_t^i-X_t^j}dt\quad(i\in\mathbb{N}).$$

Thm (K. -Osada)

For these SDEs, Theorem 1 holds.

Remark

Roughly speaking, the limit formula of (3) is

$$(5)\ dX_t^{\mathrm{i}}=\mathrm{dB}_t^{\mathrm{i}}+\sum_{1\le j\ne i\le\infty}\frac{1}{X_t^i-X_t^j}dt-\theta dt\quad(i\in\mathbb{N}).$$

In other words, correct limit formula (4) and formal limit formula (5) are different. This phenomena is called SDE gap.

Example2 ISDE related to Airy RPF

Next, we take soft edge scaling in the semi-circle, i.e. take $x_i=\frac{1}{\sqrt{2}}(2\sqrt{N}+N^{-\frac{1}{6}}s_i)$ of m^N and denote μ_{Airy}^N as the density with regard to $\mathbf{s}_N$.

$$\mu_{\mathrm{Airy}}^N(d\mathbf{s}_N)=\frac{1}{Z}\prod_{i\ne j}|s_i-s_j|^2\exp\left(-\frac{1}{2}\sum_{1\le i\le N}\left(2\sqrt{N}+N^{-\frac{1}{6}}s_i\right)^2\right)d\mathbf{s}_N.$$

μ_{Airy}^N converges to Airy RPF μ_{Airy} as N goes to infinity. N-dim SDE and ISDE corresponding above RPFs are given by following respectively.

$$dX_t^{\mathrm{N,i}}=\mathrm{dB}_t^{\mathrm{i}}+\left(\sum_{1\le j\ne i\le N}\frac{1}{X_t^{N,i}-X_t^{N,j}}-N^{\frac{1}{3}}-\frac{X_t^{N,i}}{2N^{\frac{1}{3}}}\right)dt\quad(1\le i\le \mathrm{N}),$$

$$dX_t^{\mathrm{i}}=\mathrm{dB}_t^{\mathrm{i}}+\lim_{r\to\infty}\left(\sum_{|X_t^j|<r}\frac{1}{X_t^i-X_t^j}-\int_{|y|<r}\frac{1}{-y}\frac{\sqrt{-y}}{\pi}1_{[-\infty,0]}(y)dy\right)dt\quad(i\in\mathbb{N}).$$

Thm (K. -Osada)

For these SDEs, Theorem 1 holds.

Example3 Ruelle class potential

For $\beta>0$, let Ψ be interaction potentials as follows.

- Lennard Jones potential : $\Psi_{6-12}(\mathrm{x})=|\mathrm{x}|^{-12}-|\mathrm{x}|^{-6}$ for $d=3$.
- Riesz potential : $\Psi_a(x)=\left(\frac{\beta}{a}\right)|x|^{-a}$ for $d<a\in\mathbb{N}$.

For these potentials, corresponding N-dim SDE and ISDE are given by following.

$$dX_t^{\mathrm{N,i}}=\mathrm{dB}_t^{\mathrm{i}}-\frac{\beta}{2}\sum_{j\ne i}^{N}\nabla_x\Psi\left(X_t^{N,i}-X_t^{N,j}\right)dt\quad(1\le i\le \mathrm{N}),$$

$$dX_t^{\mathrm{i}}=\mathrm{dB}_t^{\mathrm{i}}-\frac{\beta}{2}\sum_{j\ne i}^{\infty}\nabla_x\Psi\left(X_t^{N,i}-X_t^{N,j}\right)dt\quad(i\in\mathbb{N}).$$

Thm (K. -Osada)

For these SDEs, Theorem 1 holds.