KPZ equation with fractional derivatives of white noise

Masato Hoshino

The University of Tokyo

1. Introduction

- We discuss the stochastic PDE
$$\partial_t h(t,x) = \partial_x^2 h(t,x) + (\partial_x h(t,x))^2 + \partial_x^\gamma \xi(t,x) \tag{1.1}$$
for $(t,x) \in [0,\infty) \times \mathbb{T}$ with $\gamma \geq 0$. ξ is a space-time white noise.
- $\partial_x^\gamma = -(-\partial_x^2)^{\frac{\gamma}{2}}$ is the fractional derivative.
- Since the solution of SPDE is very singular, we replace ξ by $\xi_\epsilon = \xi * \rho_\epsilon$. Here $\rho \in C_0^\infty(\mathbb{R}^2)$ and $\rho_\epsilon(t,x) = \epsilon^{-3}\rho(\epsilon^{-2}t, \epsilon^{-1}x)$.
- When $\gamma = 0$, this equation is called KPZ equation. M. Hairer showed that the renormalized equation
$$\partial_t h_\epsilon(t,x) = \partial_x^2 h_\epsilon(t,x) + (\partial_x h_\epsilon(t,x))^2 - C_\epsilon + \xi_\epsilon(t,x),$$
for some constant $C_\epsilon \sim \frac{1}{\epsilon}$, has a unique limit h. (2013)
- We can expect that the similar results hold if $\gamma < \frac{1}{2}$ by "local subcriticality".
- However, we can show the similar result only for $0 \leq \gamma < \frac{1}{4}$ by this theory.

Theorem 1.1. *Let $\rho = \rho(t,x)$ be a function on $\mathbb{R}^2$ such that smooth, compactly supported, symmetric in x, nonnegative, and $\int_{\mathbb{R}^2} \rho(t,x)dtdx = 1$. Let $0 \leq \gamma < \frac{1}{4}$ and $0 < \alpha < \frac{1}{2} - \gamma$. Then there exists a sequence of constants C_ϵ such that*
1. *For some constant C (depending on γ and ρ), $C_\epsilon \leq C\epsilon^{-1-2\gamma}$*
2. *For every initial condition $h_0 \in \mathcal{C}^\alpha(\mathbb{T})$, the sequence of solutions h_ϵ to the equation*
$$\partial_t h_\epsilon(t,x) = \partial_x^2 h_\epsilon(t,x) + (\partial_x h_\epsilon(t,x))^2 - C_\epsilon + \partial_x^\gamma \xi_\epsilon(t,x)$$
on $(t,x) \in [0,T) \times \mathbb{T}$ up to some random time T, converges to a unique stochastic process h, which is independent of the choice of ρ.

2. Regularity structures

- We write (1.1) in the mild form:
$$h_\epsilon = G * \{\mathbf{1}_{t>0}((\partial_x h_\epsilon)^2 + \partial_x^\gamma \xi_\epsilon)\} + Gh_0,$$
- G is the heat kernel on $[0,\infty) \times \mathbb{R}$, and Gh_0 is the smooth function solving the heat equation with the initial condition $h_0 \in \mathcal{C}^\eta(\mathbb{T})$.
- We reformulate (1.1) as the equation of a function H valued in the abstract liner space:
$$H = \mathcal{G}\mathbf{1}_{t>0}((\partial H)^2 + \Xi) + Gh_0. \tag{2.1}$$

2.1 Regularity structure for (1.1)

- We prepare dummy variables Ξ (noise term), $\mathbf{1}$ (constant), X_1 (time variable), X_2 (spatial variable), and operators $\mathcal{I}, \mathcal{I}'$ (convolution with G or $G' := \partial_x G$), ∂ (spatial derivative).
- We denote by $\mathcal{U}$ and $\mathcal{V}$ the minimal set of variables such that
$$\begin{aligned}&\Xi, \mathbf{1}, X_1, X_2 \in \mathcal{V},\\ &\mathcal{I}(X^k) = 0, \quad \partial\mathcal{I}\tau = \mathcal{I}'\tau, \quad \partial(X_0^{k_0}X_1^{k_1}) = k_1 X_0^{k_0}X_1^{k_1-1},\\ &\tau \in \mathcal{V} \Rightarrow \mathcal{I}\tau \in \mathcal{U}, \quad \tau_1, \tau_2 \in \mathcal{U} \Rightarrow \partial\tau_1\partial\tau_2 \in \mathcal{V}\end{aligned}$$
- We define the homogeneity of each variable by:
$$\begin{aligned}&|\Xi|_\mathfrak{s} = \alpha, \ |\mathbf{1}|_\mathfrak{s} = 0, \ |X_1|_\mathfrak{s} = 2, \ |X_2|_\mathfrak{s} = 1\\ &|\tau\bar{\tau}|_\mathfrak{s} = |\tau|_\mathfrak{s} + |\bar{\tau}|_\mathfrak{s}, \ |\mathcal{I}\tau|_\mathfrak{s} = |\tau|_\mathfrak{s} + 2, \ |\mathcal{I}'\tau|_\mathfrak{s} = |\tau|_\mathfrak{s} + 1.\end{aligned}$$
- α is a fixed number smaller than and sufficiently close to $-3/2 - \gamma$.
- We define a regularity structure (T,G) for (1.1) as the pair of a linear space T spanned by $\mathcal{U} \cup \mathcal{V}$ and a group G of linear operators on T.

2.2 Solution map

- We say the pair of
$$\Pi : \mathbb{R}^2 \times T \ni (z,\tau) \to \Pi_z\tau \in \mathcal{D}'(\mathbb{R}^2), \quad \Gamma : \mathbb{R}^2 \times \mathbb{R}^2 \times T \ni (z,\bar{z},\tau) \to \Gamma_{z,\bar{z}}\tau \in G$$
(with some properties) a model (Π,Γ) for the regularity structure (T,G).
- We say that a model (Π,Γ) is "admissible", if and only if $(\Pi_z X^k)(\bar{z}) = 0$ and
$$\begin{aligned}(\Pi_z\mathcal{I}\tau)(\bar{z}) &= G * (\Pi_z\tau)(\bar{z}) - \sum_{|l|_\mathfrak{s} < |\tau|_\mathfrak{s}+2} \frac{(\bar{z}-z)^l}{l!}\partial^l G * (\Pi_z\tau)(\bar{z}),\\ (\Pi_z\mathcal{I}'\tau)(\bar{z}) &= G' * (\Pi_z\tau)(\bar{z}) - \sum_{|l|_\mathfrak{s} < |\tau|_\mathfrak{s}+1} \frac{(\bar{z}-z)^l}{l!}\partial^l G' * (\Pi_z\tau)(\bar{z}).\end{aligned}$$
- We denote by $\mathcal{D}_P^{\theta,\eta}(\mathcal{F})$ ($\mathcal{F} = \mathcal{U}, \mathcal{V}$) the space of all functions $f : \mathbb{R}^2 \to T$ of the form
$$f(t,x) = \bigoplus_{\tau \in \mathcal{F}} f_\tau(t,x)\tau \quad (t \neq 0),$$
where each f_τ satisfies "$\theta - |\tau|_\mathfrak{s}$-Hölder" property with the coefficient proportionally to $|t|^{\frac{\eta-\theta}{2}}$.

Theorem 2.1. *If $\theta > 0$, $\eta \leq \theta$, and $\alpha \wedge \eta > -2$, then there exists a unique continuous linear map $\mathcal{R} : \mathcal{D}_P^{\theta,\eta}(\mathcal{V}) \to \mathcal{C}_\mathfrak{s}^{\alpha\wedge\eta}$ such that $\mathcal{R}f$ is "near" to $\Pi_z f(z)$ in $\mathcal{C}_\mathfrak{s}^{\alpha\wedge\eta}$, at each $z \in \mathbb{R}^2$. Furthermore, the map $\mathcal{M} \ltimes \mathcal{D}_P^{\theta,\eta}(\mathcal{V}) \ni (Z,f) \to \mathcal{R}^Z f \in \mathcal{C}_\mathfrak{s}^{\alpha\wedge\eta}$ is locally uniformly continuous.*

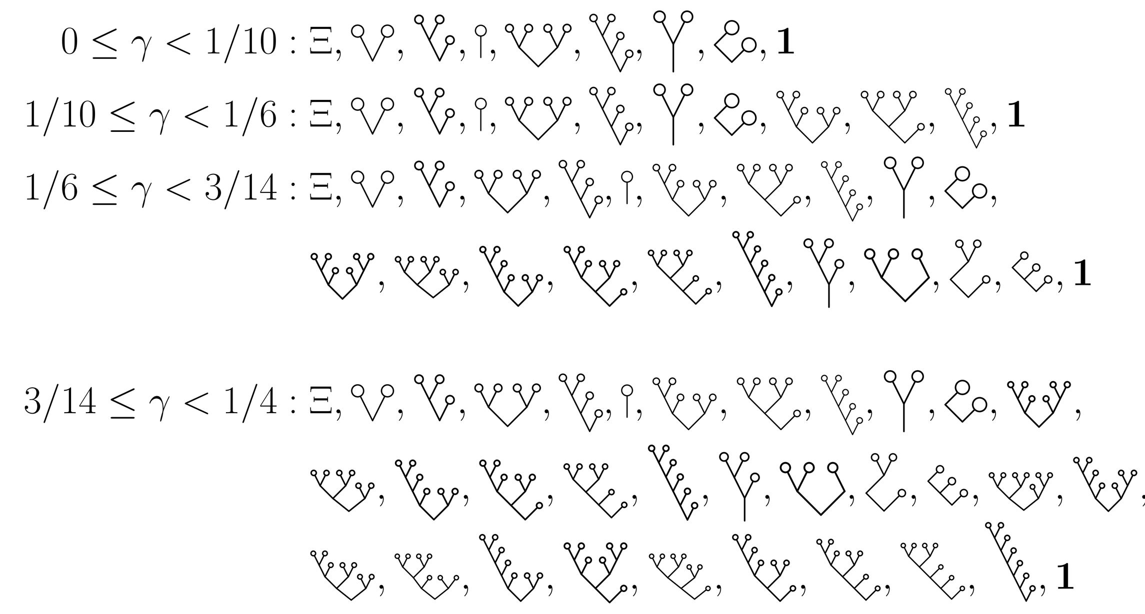
Theorem 2.2. *Let $0 \leq \gamma < \frac{1}{2}$, and choose α smaller than but sufficiently close to $-3/2 - \gamma$. Let $\theta > -\alpha$, and $0 < \eta < \alpha + 2$. Then, for every periodic initial condition $h_0 \in \mathcal{C}^\eta(\mathbb{R})$, every admissible model Z, there exists a time $T = T(h_0, Z) > 0$ and a unique solution $H = \mathcal{S}(h_0, Z) \in \mathcal{D}_P^{\theta,\eta}(\mathcal{U})$ to (2.1) on $[0,T]$. Moreover, the solution map $(h_0, Z) \mapsto H$ is jointly uniformly continuous in a neighborhood around (h_0, Z).*

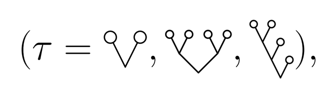
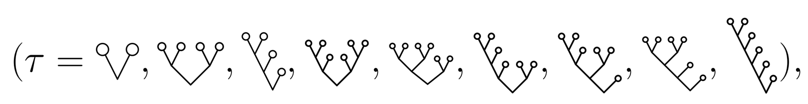
3. Renormalization

- We define the natural model $Z^{(\epsilon)} = (\Pi^{(\epsilon)}, \Gamma^{(\epsilon)})$ by postulating admissibility and $\Pi_z^{(\epsilon)}\Xi(\bar{z}) = \partial_x^\gamma \xi(\bar{z})$.

3.1 Renormalization map

- We use the shorthand notations to write the elements of $\mathcal{F} = \mathcal{U} \cup \mathcal{V}$.
- Each circle represents Ξ.
- For any trees τ, we draw $\mathcal{I}'(\tau)$ by adding a downward straight line starting at the root of τ.
- For any trees τ and $\bar{\tau}$, we draw $\tau\bar{\tau}$ by jointing the trees at their roots.
- In the following, the elements of negative homogeneities play important roles.



- For any given constants $\{C_\tau^{(\epsilon)}\}$, we define the renormalized model $\widehat{Z}^{(\epsilon)} = (\widehat{\Pi}^{(\epsilon)}, \widehat{\Gamma}^{(\epsilon)})$ by
$$\widehat{\Pi}_z^{(\epsilon)}\tau = \Pi_z^{(\epsilon)}\tau - C_\tau^{(\epsilon)} \quad (\tau = \text{see below}), \qquad \widehat{\Pi}_z^{(\epsilon)}\tau = \Pi_z^{(\epsilon)}\tau \quad \text{(otherwise)}$$

when $0 \leq \gamma < 1/6$, or by
$$\widehat{\Pi}_z^{(\epsilon)}\tau = \Pi_z^{(\epsilon)}\tau - C_\tau^{(\epsilon)} \quad (\tau = \text{see below}),\qquad \widehat{\Pi}_z^{(\epsilon)}\tau = \Pi_z^{(\epsilon)}\tau \quad \text{(otherwise)}$$

when $1/6 \leq \gamma < 1/4$.

Theorem 3.1. *Let $\mathcal{S} : (h_0, Z) \mapsto H$ be a solution map defined in Theorem 2.2. Then $h_\epsilon^M = \mathcal{R}\mathcal{S}(h_0, Z^{(\epsilon),M})$ solves the equation*
$$\partial_t h_\epsilon^M = \partial_x^2 h_\epsilon^M + (\partial_x h_\epsilon^M)^2 - C^{(\epsilon)} + \partial_x^\gamma \xi_\epsilon.$$
Here, $C^{(\epsilon)}$ is a constant depending on $\{C_\tau^{(\epsilon)}\}$.

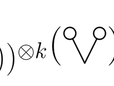
Theorem 3.2. *Let $\{C_\tau^{(\epsilon)}\}$ be sequences of constants defined later. Then, there exists an admissible random model $\widehat{Z}$ independent of the choice of ρ, such that $\widehat{Z}^{(\epsilon)}$ converge to $\widehat{Z}$ in probability, as $\epsilon \to 0$.*

4. Proof of Theorem 3.2

- We denote by $I_k : (L^2(\mathbb{R} \times \mathbb{T}))^{\otimes k} \to L^2(\Omega, \mathbb{P})$ the multiple Wiener-Itô integral.
- For any τ, we have the expansion
$$(\Pi_0^{(\epsilon)}\tau)(z) = \sum_k I_k\Big(\sum_l m_{k,l}\mathcal{W}_{k,l}^{(\epsilon)}(\tau)(z)\Big).$$
for some integers $m_{k,l}$ and kernels $\mathcal{W}_{k,l}^{(\epsilon)}(\tau)(z)$.
- We set the constant
$$C_\tau^{(\epsilon)} := \sum_{\mathcal{W}_{l,i}^{(\epsilon,0)}(\tau)(z) = \text{const}} \mathcal{W}_{l,i}^{(\epsilon,0)}(\tau)(z).$$
- The limit model $\Pi^{(\epsilon)}$ has the form
$$(\widehat{\Pi}_0\tau)(z) = \sum_k I_k\Big(\sum_l m_{k,l}\widehat{\mathcal{W}}_{k,l}(\tau)(z)\Big).$$

Theorem 3.2 is a consequence of the following theorem.

Theorem 4.1. *Let $0 \leq \gamma < 1/4$. Then there exist some $\kappa, \theta > 0$ such that*
$$\mathbb{E}|(\widehat{\Pi}_0\tau)(\phi_0^\lambda)| \lesssim \lambda^{2|\tau|_\mathfrak{s}+\kappa}, \quad \mathbb{E}|(\widehat{\Pi}_0\tau - \widehat{\Pi}_0^{(\epsilon)}\tau)(\phi_0^\lambda)| \lesssim \epsilon^{2\theta}\lambda^{2|\tau|_\mathfrak{s}+\kappa} \tag{4.1}$$
for any τ with negative homogeneities, and uniformly over $0 < \lambda$, $\phi : ||\phi||_{C^2} \leq 1$. Here, $\phi_0^\lambda(t,x) = \lambda^{-3}\phi(\lambda^{-2}t, \lambda^{-1}x)$.

proof. For any τ, we have the bounds
$$\begin{aligned}&\mathbb{E}|(\widehat{\Pi}_0\tau)(\phi_0^\lambda)|^2 \lesssim \sum_{k,l}\iint \phi_0^\lambda(z)\phi_0^\lambda(\bar{z})(\widehat{\mathcal{W}}_{k,l}(\tau)(z), \widehat{\mathcal{W}}_{k,l}(\tau)(\bar{z}))_{(L^2(\mathbb{R}\times\mathbb{T}))^{\otimes k}}dzd\bar{z},\\ &\mathbb{E}|(\widehat{\Pi}_0\tau - \widehat{\Pi}_0^{(\epsilon)}\tau)(\phi_0^\lambda)|^2\\ &\lesssim \sum_{k,l}\iint \phi_0^\lambda(z)\phi_0^\lambda(\bar{z})(\delta\widehat{\mathcal{W}}_{k,l}^{(\epsilon)}(\tau)(z), \delta\widehat{\mathcal{W}}_{k,l}^{(\epsilon)}(\tau)(\bar{z}))_{(L^2(\mathbb{R}\times\mathbb{T}))^{\otimes k}}dzd\bar{z}\end{aligned}$$
Here, $\delta\widehat{\mathcal{W}}_{l,i}^{(\epsilon,k)}(\tau) = \widehat{\mathcal{W}}_{l,i}^{(k)}(\tau) - \widehat{\mathcal{W}}_{l,i}^{(\epsilon,k)}(\tau)$. We have only to bound the inner products of kernels in the right hand side. In particular, we have
$$|(\widehat{\mathcal{W}}_2(\tau)(z), \widehat{\mathcal{W}}_2(\tau)(\bar{z}))_{(L^2(\mathbb{R}\times\mathbb{T}))^{\otimes k}}(\tau_0;z;\bar{z})| \lesssim ||z-\bar{z}||_\mathfrak{s}^{-2-4\gamma}.$$

uniformly over $z, \bar{z}$. So γ must satisfy $-2-4\gamma > -3 \Leftrightarrow \gamma < 1/4$. □