TECHNISCHE UNIVERSITÄT DRESDEN

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Non-Dominated Nonlocal Equations

Comparison Principle for Viscosity Solutions

Motivation

Nonlinear equations with a **non-dominated nonlocal part** (in the sense that the resulting operators do not satisfy dominated convergence) play an important role in the recent studies of **processes with jumps under uncertainty** as in [1]. We generalize the existing theory from [2] in order to obtain a comparison principle for such equations (and hence uniqueness of the boundary value problem) with (possibly) unbounded solutions.

Notion & Assumptions

Definition (Viscosity Solutions). An upper (lower) semicontinuous function $u \in \mathrm{SC}_p(\mathbb{R}^d)$ is a *viscosity subsolution* (*viscosity supersolution*) in Ω of the nonlocal equation

$$F(x, u(x), Du(x), D^2u(x), u(\cdot)) = 0 \tag{1}$$

if for every $\phi \in C_p^2(\mathbb{R}^d)$ such that $u-\phi$ has a global maximum (global minimum) in $x \in \Omega$, the inequality

$$F(x, u(x), D\phi(x), D^2\phi(x), \phi(\cdot)) \leq 0 \;(\geq 0)$$

holds. A *viscosity solution* of (1) is both a viscosity subsolution and a viscosity supersolution.

Remark (Assumptions on F).

- **Consistency**
  $F^\kappa(x,r,q,X,\phi,\phi) = F(x,r,q,X,\phi)$ for $\phi \in C_p^2(\Omega)$
- **Maximum Principle**
  $F^\kappa(x,r,q,X,u,\phi) \geq F^\kappa(x,r,q,Y,v,\psi)$
  for $X \leq Y$ and $x \in \arg\max(u-v) \cap \arg\max(\phi-\psi)$
- **Translation Invariance**
  $F^\kappa(x,r,q,X,u+c_1,\phi+c_2) = F^\kappa(x,r,q,u,\phi)$ for $c_i \in \mathbb{R}$
- **Continuity**
  $F^\kappa(x_n,r_n,q_n,X_n,u_n,\phi_n) \longrightarrow F^\kappa(x,r,q,X,u,\phi)$
  for $(x_n,r_n,q_n,X_n) \to (x,r,q,X)$, $u_n \to u$ locally uniform with $u \in C_p(\mathbb{R}^d)$ and $\sup_{n\in\mathbb{N}} \|u_n\|_{C_p} < \infty$, as well as $D^k\phi_n \to D^k\phi$ locally uniform for $k \leq 2$
- **Monotonicity**
  $F^\kappa(x,r,q,X,u,\phi) \leq F^\kappa(x,s,q,X,u,\phi)$ for $r \leq s$

Results

Theorem (Domination Principle). Suppose that for $1 \leq i \leq k$ $u_i \in \mathrm{USC}_p([0,T]\times\mathbb{R}^d)$ are viscosity solutions in $(0,T)\times\mathbb{R}^d$ of

$$\partial_t u_i + G_i(t,x,u_i,Du_i,D^2u_i,u_i(t,\cdot)) \leq 0$$

that satisfy the polynomial growth conditions

$$\max_i |u_i(t,x)| \leq C(1+|x|^p)$$
$$\max_i |u_i(0,x) - u_i(0,y)| \leq C(1+|x|^{p-1}+|y|^{p-1})|x-y|$$

for all $t \in [0,T]$, $x,y \in \mathbb{R}^d$ and some $C \geq 0$. If the operators $(G_i)_i$ satisfy an additional regularity condition for $\beta_1,\dots,\beta_k > 0$ and $\sum_{i=1}^k \beta_i u_i(0,x) \leq 0$ for all $x \in \mathbb{R}^d$, then $\sum_{i=1}^k \beta_i u_i(t,x) \leq 0$ for all $(t,x) \in (0,T)\times\mathbb{R}^d$.

Corollary (Comparison Principle). Suppose that functions $u^\varphi \in C_p([0,T]\times\mathbb{R}^d)$ are viscosity solutions in $(0,T)\times\mathbb{R}^d$ of

$$\partial_t u^\varphi + G(t,x,u^\varphi,Du^\varphi,D^2u^\varphi,u^\varphi(t,\cdot)) = 0$$

with $u^\varphi(0,\cdot) = \varphi \in C_p(\mathbb{R}^d)$ such that

$$|\varphi(x)-\varphi(y)| \leq C(1+|x|^{p-1}+|y|^{p-1})|x-y|$$

for all $x,y \in \mathbb{R}^d$ and some $C \geq 0$. If the operator G satisfies an additional regularity condition and $\varphi \leq \psi$, then

$$u^\varphi(t,x) \leq u^\psi(t,x)$$

for all $(t,x) \in (0,T)\times\mathbb{R}^d$. Moreover, if $G(t,x,\cdot)$ is superadditive and concave, then $\varphi \mapsto u^\varphi$ is subadditive and convex.

Examples

Remark (Hamilton–Jacobi–Bellman Equation). A Hamilton–Jacobi–Bellman equation is a (nonlinear) parabolic nonlocal equation $\partial_t u + G(t,x,u,Du,D^2u,u(t,\cdot)) = 0$ with

$$G(t,x,r,p,X,u,\phi) = \inf_{\alpha\in\mathcal{A}} G_\alpha(t,x,r,p,X,u,\phi)$$

for a family of linear nonlocal operators of the form

$$\begin{aligned} G_\alpha(r,p,X,\phi) &= f_\alpha - \mathcal{L}_\alpha(r,p,X) - \mathcal{I}_\alpha(\phi)\\ \mathcal{L}_\alpha(r,p,X) &= c_\alpha r + b_\alpha^T p + \operatorname{tr}\left(\sigma_\alpha\sigma_\alpha^T X\right)\\ \mathcal{I}_\alpha(\phi)(t,x) &= \int \big[\, \phi(t,x+j_\alpha(z)) - \phi(t,x)\\ &\qquad - D\phi(t,x)j_\alpha(z)\mathbb{1}_{|z|\leq 1} \,\big]\, m_\alpha(dz), \end{aligned}$$

where the coefficients satisfy the following assumptions:

- **Boundedness**
  $\sup_\alpha(|f_\alpha(t,x)| + |c_\alpha(t,x)| + |b_\alpha(t,x)| + |\sigma_\alpha(t,x)|) < \infty$
  $\sup_\alpha \int [\,|z|^2\mathbb{1}_{|z|\leq 1} + (1+|z|^p)\,\mathbb{1}_{|z|>1}\,]\, m_\alpha(dz) < \infty$
- **Tightness**
  $\lim_{\kappa\to 0}\sup_\alpha \int_{|z|\leq\kappa} |z|^2\, m_\alpha(dz) = 0$
  $\lim_{R\to\infty}\sup_\alpha \int_{R<|z|} [\,1+|z|^p\,]\, m_\alpha(dz) = 0$
- **Continuity**
  $\sup_\alpha |j_\alpha(t,x,z) - j_\alpha(s,y,z)| \leq (\omega(|t-s|) + C|x-y|)\,|z|$
  $\sup_\alpha |\phi_\alpha(t,x) - \phi_\alpha(s,y)| \leq \omega(|t-s|) + C|x-y|$
  for $\phi \in \{b,\sigma,f,c\}$, some $C \geq 0$ and $\omega(0+) = \omega(0) = 0$
- **Growth Condition**
  $|j_\alpha(t,x,z)| \leq C|z|$
- **Monotonicity**
  $c_\alpha(t,x) \leq 0$

References

[1] A. Neufeld and M. Nutz, *Nonlinear Lévy Processes and their Characteristics*, Forthcoming in 'Transactions of the American Mathematical Society'

[2] E. Jakobsen and K. Karlsen, *A 'maximum principle for semicontinuous functions' applicable to integro-partial differential equations*, NoDEA, 13(2), 137-165, 2006

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