Infinite particle systems of long range jumps with long range interactions

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Construction of unlabeled infinite particle systems with interaction

- Brownian motion : Osada(1996), Osada(2013), ...
- Jump type : Kondratiev-Lytvynov-Röckner(2007), E.Lytvynov and N.Ohlerich(2008), ...

Our problem

Construct unlabeled infinite particle systems of Lévy processes with a "long range" interaction.

Examples of a "long range" interaction

- Dyson interaction
- Ginibre interaction
- Airy interaction

↑ For each operator K, $\mathbf{Spec}(K)$ contains 1.

Notation

- S : the state space (e.g. $\mathbb{R}^d$)
- $\mathfrak{M} = \{\xi;\ \xi$ is a non negative integer valued Radon measure. $\}$: configuration space
- $\mathfrak{D}_\circ = \{f : \mathfrak{M} \to \mathbb{R};\ f$ is local and smooth$\}$
- $U_r = \{x \in S; |x| \le r\}$, $\mathfrak{M}_r^i = \{\xi \in \mathfrak{M}; \xi(U_r) = i\}$,
- $\pi_r(\xi) = \xi(\cdot \cap U_r)$, $\pi_r^c(\xi) = \xi(\cdot \cap U_r^c)$.
- μ : a prob. meas. on $\mathfrak{M}$

Our bilinear form $(\mathfrak{E}, \mathfrak{D}_\infty)$

For $f, g \in \mathfrak{D}_\circ$ we set $\mathbb{D}[f,g] : \mathfrak{M} \to \mathbb{R}$ by the following.

$$\mathbb{D}[f,g](\xi) = \frac{1}{2}\sum_i \int_S (f(\xi^{s_i,y_i}) - f(\xi))(g(\xi^{s_i,y_i}) - g(\xi))\nu(\xi, s_i; y_i)dy_i,$$

where $s_i \in S$, $\xi = \sum_i \delta_{s_i}$, $\xi^{s_i,y_i} = \xi + \delta_{y_i} - \delta_{s_i}$, $\nu(\xi, s_j; y)$ is a density of a (finite or infinite) measure s.t.

$$\int_S (1 \wedge |y - s_j|^2)\, \nu(\xi, s_j; y)dy < \infty, \quad \text{for all } \xi,\ j.$$

$$\mathfrak{E}(f,g) = \int_{\mathfrak{M}} \mathbb{D}[f,g](\xi)d\mu, \quad f, g \in \mathfrak{D}_\infty,$$

$$\mathfrak{D}_\infty = \{f \in \mathfrak{D}_\circ \cap L^2(\mathfrak{M}, \mu); \mathfrak{E}(f,f) < \infty\}.$$

assumptions $\mathbf{A}$

(A.0) There exist a k-density function σ_r^k of μ on U_r and a k-correlation function ρ^k for all $k \in \mathbb{N}$.

(A.1) $(\mathfrak{E}, \mathfrak{D}_\infty)$ is closable on $L^2(\mathfrak{M}, \mu)$.

(A.2) $\sigma_r^k \in L^p(U_r^k, dx)$ for all $k, r \in \mathbb{N}$ with some $1 < p \le \infty$.

(A.3) $\sum_{i=1}^\infty i\mu(\mathfrak{M}_r^i) < \infty$ for all $r \in \mathbb{N}$.

assumptions $\mathbf{B}$

(B.0) There exists a function $p(r)$ on $(0, \infty)$ such that $\nu(\xi, x; y) \le C_1 p(|x - y|)$ for μ-a.s. $\xi \in \mathfrak{M}$ and dx-a.e. $x, y \in S$.

(B.1) $\rho^1(x) = O(|x|^\kappa)$ as $|x| \to \infty$ for some $\kappa \ge 0$.

(B.2) $p(r) = O(r^{-(d+\alpha)})$ as $r \to \infty$ for some $\alpha > \kappa$.

(B.3) $p(r) = O(r^{-(d+\gamma)})$ as $r \to +0$ for some $0 < \gamma < 2$.

(B.4) $\frac{\mathrm{Var}[\xi(U_r)]}{(\mathbb{E}[\xi(U_r)])^2} = O(r^{-\delta})$ as $r \to \infty$ for some $\delta > 0$.

Remark 1

(i) The LHS of (B.4) is equal to

$$\frac{\int_{U_r} \rho^1(x)dx - \int_{U_r^2} (\rho^1(x_1)\rho^1(x_2) - \rho^2(x_1, x_2))\, dx_1 dx_2}{\left(\int_{U_r} \rho^1(x)dx\right)^2}.$$

→ (B.4) holds if μ is the Poisson random point field with respect to Lebesgue measure or μ is a determinantal point field.

(ii) Condition (B.1) and (B.2) imply that

$$\int_S \rho^1(x)p(x, A)dx < \infty, \tag{1}$$

for all compact subset A. The property (1) is necessary to construct the infinite particle systems of independent jump type processes. Hence Condition (B.1) and (B.2) are reasonable.

Theorem 1 (E.)

Assume (A.0)–(A.3) and (B.0)–(B.4). Let $(\mathfrak{E}, \mathfrak{D})$ be the closure of $((\mathfrak{E}, \mathfrak{D}_\infty), L^2(\mathfrak{M}, \mu))$. Then $(\mathfrak{E}, \mathfrak{D})$ is a quasi-regular Dirichlet form on $L^2(\mathfrak{M}, \mu)$. Hence there exists a special standard process $\{\mathbb{P}_\xi\}_{\xi \in \mathfrak{M}}$ generated by $((\mathfrak{E}, \mathfrak{D}), L^2(\mathfrak{M}, \mu))$.

For a random point field μ we set

$$\mu_{r,\xi}^m(\cdot) = \mu(\pi_r(\eta) \in \cdot | \eta(U_r) = m, \pi_r^c(\eta) = \pi_r^c(\xi))$$

Let $\Psi : S \to \mathbb{R} \cup \{\infty\}$ (interaction)

$$\mathcal{H}_r = \sum_{s_i, s_j \in S_r, i<j} \Psi(s_i - s_j)$$

Definition 1

μ is a Ψ-quasi-Gibbs measure if there exists $c_{r,\xi}^m$ s.t.

$${c_{r,\xi}^m}^{-1} e^{-\mathcal{H}_r} d\Lambda_r^m \le \mu_{r,\xi}^m \le c_{r,\xi}^m e^{-\mathcal{H}_r} d\Lambda_r^m.$$

Here $\Lambda_r^m = \Lambda_r(\cdot | \xi(U_r) = m)$ and Λ_r is the Poisson random point field with $1_{U_r} dx$.

Remark 2

The above definition is a simplified version.

Proposition 3

μ : quasi-Gibbs meas. + additional assumption for $\Psi \Rightarrow$ (A.1)

Example of our results

μ : Sine random point field, Ginibre random point field
→ quasi-Gibbs measures($\Rightarrow$ (A.0)–(A.3) holds) and $\rho^1 \equiv \mathbf{const.}$
Then the assumption (B.1) is satisfied for $\kappa = 0$. Hence we can take $0 < \alpha, \gamma < 2$. Therefore we can construct interacting symmetric α-stable processes for any $0 < \alpha < 2$.

μ : Airy random point field
→ quasi-Gibbs measures($\Rightarrow$ (A.0)–(A.3) holds) and
$\rho^1(x) = O(|x|^{1/2})$ as $x \to -\infty$.
Then the assumption (B.1) is satisfied for $\kappa = \frac{1}{2}$. Hence we can take $\frac{1}{2} < \alpha < 2$, $0 < \gamma < 2$. Therefore we can construct interacting symmetric α-stable processes for any $\frac{1}{2} < \alpha < 2$.

Future problems

- labeled dynamics and SDE representations
- scaling limits
- some relations to Determinantal processes

The L^2-generators Ω

Let

$$c(\xi, x; y) = \nu(\xi, x; y) + \nu(\xi^{xy}, y; x)\frac{d\mu^y}{d\mu^x}(\xi \setminus x)\frac{\rho^1(y)}{\rho^1(x)}, \quad \text{if } \xi(\{x\}) \ge 1,$$

where μ_x is the reduced Palm measure. Then

$$\Omega f(\xi) = \frac{1}{2}\sum_{i=1}^\infty \int_S (f(\xi^{s_i,y_i}) - f(\xi))c(\xi, s_i, y_i)dy_i,$$

for $\xi = \sum_{j=1}^\infty \delta_{s_j}$.

The SDE representations

$\{X_t^i\}_{i\in\mathbb{N}}$: the labeled dynamics. Then

$$X_j(t) = X_j(0) + \int_0^t \int_S \int_0^{\frac{1}{2}c(\Xi(s-), X_j(s-), X_j(s-)+u)} uN(dsdudr),$$

for all $j \in \mathbb{N}$, where $\Xi(t) = \sum_i \delta_{X_i(t)}$, $N(dsdudr)$ is the Poisson point process on $[0, \infty) \times \mathbb{R}^d \times \mathbb{R}$ with intensity $dsdudr$.

Ginibre interaction

$$c(\xi, x; y) = 1 + \lim_{r\to\infty} \prod_{|s_i|<r} \frac{|y - s_i|^2}{|x - s_i|^2}.$$