Convergence Implications via Dual Flow Method

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1 Main Results.

1.1 Stochastic Flow on $[0,+\infty)$ with an Absorbing Barrier.

Let σ, $b : [0,+\infty) \to \mathbb{R}$ be Borel-measurable functions such that

(i) $\sigma(x) > 0$ for $x \in (0,+\infty)$, (ii) $\sigma|_{(0,+\infty)}$, $b|_{(0,+\infty)} \in C^2(0,+\infty)$ and σ', b' are bounded on $[1,+\infty)$,

(iii) It holds that $\displaystyle\int_{0+}^1 \frac{1}{\sigma(x)^2}\exp\Big\{-\int_x^1 \frac{2b(y)}{\sigma(y)^2}\mathrm{d}y\Big\}\mathrm{d}x + \int_{0+}^1 \exp\Big\{\int_x^1 \frac{2b(y)}{\sigma(y)^2}\mathrm{d}y\Big\}\mathrm{d}x < +\infty$,

(iv) The condition (1) still holds if replacing b by $\widehat{b}$, where $\widehat{b}(x) := \sigma(x)\sigma'(x) - b(x)$.

For each $s \in \mathbb{R}$, we consider the following stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = \sigma(X_t)\mathrm{d}w^{(s)}(t) + b(X_t)\mathrm{d}t, & t \geq s, \\ X_s = x \in (0,+\infty), \end{cases} \tag{1.1}$$

where $w = (w(t))_{t\in\mathbb{R}}$ is a one-dimensional Wiener process and $w^{(s)}(t) := w(t) - w(s)$.

Let $x \in (0,+\infty)$ and $s \in \mathbb{R}$. Under the condition (i) and (ii), the stochastic differential equation (1.1) with the driving process $w^{(s)}$ admits a unique strong solution $X = (X_t)_{t\geq s}$ with $X_s = x$, as long as X moves in $(0,+\infty)$. Then by the condition (iii), the boundary point 0 is a regular boundary for the corresponding generator $L = (\sigma^2/2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b\frac{\mathrm{d}}{\mathrm{d}x}$, so that there is a finite random time $\tau^{s,x} > s$ such that $\lim_{t\to\tau^{s,x}} X_t = 0$ a.s. Now we define

$$X_{s,t}(x) := \begin{cases} X_t & \text{if } t \in [s,\tau^{s,x}), \\ 0 & \text{if } t \geq \tau^{s,x}. \end{cases}$$

Then we further set $X_{s,t}(0) := X_{s,t}(0+)$.

So we obtain a stochastic flow $\{X_{s,t}\}_{s\leq t}$ which we call the **(σ,b,w)-stochastic flow with the absorbing barrier at zero.** Then the stochastic flow $\{X^*_{s,t}\}_{s\leq t}$ defined by $X^*_{s,t} := X^{-1}_{-t,-s}$ (the right-continuous inverse) is called the **dual stochastic flow** and it is known that each one-point motion of $\{X^*_{s,t}\}_{s\leq t}$ solves a corresponding stochastic differential equation of Skorokhod-type (Akahori-Watanabe '02).

1.2 Euler-Maruyama Approximation.

Fix $T > 0$, and let $t_k := kT/n$ for $k \in \mathbb{Z}$. We write $\Delta t_k := t_k - t_{k-1}$ and $\Delta w_k := w(t_k) - w(t_{k-1})$. For integers $k \leq l$, we define $X_{k,l}(x)$ for $k \leq l$ and $x \in (0,+\infty)$, by $X_{k,k}(x) := x$ and for $k < l$,

$$X_{k,l}(x) := 1_{\{X_{k,l-1}(x)>0\}} \times \max\big\{0, X_{k,l-1}(x) + \sigma(X_{k,l-1}(x))\Delta w_l + b(X_{k,l-1}(x))\Delta t_l\big\}.$$

For $x = 0$, we define $X_{k,l}(0) := X_{k,l}(0+)$.

To assure that each $X_{k,l}$ values in $\mathcal{T}$, we need the following property: for each $k \leq l$,

$$X_{k,l} : [0,+\infty) \to [0,+\infty) \textit{ is nondecreasing a.s.}. \tag{1.2}$$

Definition 1.2.1. We call $\{X_{k,l}\}_{k\leq l}$ the **Euler-Maruyama approximation** of the (σ,b,w)-stochastic flow with the absorbing barrier at 0 if the condition (1.2) holds.

Let $\{X_{s,t}\}_{s\leq t}$ be the (σ,b,w)-stochastic flow with the absorbing barrier at 0 and let $\{X^n_{k,l}\}_{k\leq l}$ be the Euler-Maruyama approximation of $\{X_{s,t}\}_{s\leq t}$ with time-step $h = T/n$.

1.3 Convergence Implications for Errors.

Theorem 1.3.1. *Assume the conditions (i)–(iv) in the subsection 1.1. (1) Let $f : [0,+\infty) \to \mathbb{R}$ be a differentiable function with $f(0) = 0$ and with compact support. For $x > 0$, we have*

$$\mathbf{E}[f(X^*_{0,T}(x))] - \mathbf{E}[f(X^{n*}_{0,n}(x))] = \int_0^{+\infty} f'(y)\{\mathbf{P}(X^n_{0,n}(y) > x) - \mathbf{P}(X_{0,T}(y) > x)\}\mathrm{d}y.$$

(2) If σ and b are smooth, then $X^*_{0,T}$ is absolutely continuous and we have for each $K > 0$,

$$\mathbf{E}\Big[\sup_{0\leq x\leq K} |X^*_{0,T}(x) - X^{n*}_{0,n}(x)|\Big] \leq \mathbf{E}\Big[\Big\{1 + \sup_{0\leq x\leq K} |(X^*_{-T,0})'(x)|\Big\} \sup_{0\leq x\leq m} |X_{0,T}(x) - X^n_{0,n}(x)|\Big],$$

where $m := \min\{X^*_{-T,0}(K), X^{n*}_{-n,0}(K)\}$.

(3) If further σ is a constant and b has the bounded derivative, then for each $K > 0$, we have

$$\mathbf{E}\Big[\sup_{0\leq x\leq K} |X^*_{0,T}(x) - X^{n*}_{0,n}(x)|\Big] \leq \mathrm{const.}\mathbf{E}\Big[\sup_{0\leq x\leq m} |X_{0,T}(x) - X^n_{0,n}(x)|\Big].$$

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