Maximum and minimum of local times for two-dimensional random walk

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Aim and Setting

In this poster, I will describe the leading order of max and min of local times for the simple random walk (SRW) on 2D torus at time const $\times$ (cover time).

- 2D discrete torus: $\mathbb{Z}_N^2 := (\mathbb{Z}/N\mathbb{Z})^2$.
- $X = (X_t)_{t\geq 0}$: continuous-time SRW on $\mathbb{Z}_N^2$ (with exponential holding times of parameter 1).
- Local time: $L_t^N(x) := \int_0^t 1_{\{X_s = x\}} ds, \ \ t \geq 0$.
- Inverse local time: $\tau(t) := \inf\{s \geq 0 : L_s^N(o) > t\}, \ \ t \geq 0$ (o is the origin).
- Focus on time $\tau(t_\theta), \ \ t_\theta := \theta \cdot \frac{4}{\pi}(\log N)^2$.

Rem. Cover time estimate by Dembo-Peres-Rosen-Zeitouni(2004) implies $\tau(t_\theta) \approx \theta \cdot \inf\{t : \min_x L_t^N(x) > 0\}$.

Local time and GFF

I want to compare max of local times with that of 2D Gaussian free field (GFF).

$(h_x^N)_{x\in\mathbb{Z}_N^2}$: GFF on $\mathbb{Z}_N^2$.

i.e. centered Gaussian process with $\mathbb{E}(h_x^N h_y^N) = E_x[L_{H_o}^N(y)]$, where H_o is the hitting time of the origin.

Isomorphism theorem by Eisenbaum-Kaspi-Marcus-Rosen-Shi (2000)

For $\forall t \geq 0$,

$$\left\{L_{\tau(t)}^N(x) + \frac{1}{2}(h_x^N)^2 : x \in \mathbb{Z}_N^2\right\} \text{ under } P_o \times \mathbb{P} \overset{\text{law}}{=} \left\{\frac{1}{2}(h_x^N + \sqrt{2t})^2 : x \in \mathbb{Z}_N^2\right\}.$$

In particular, for fixed N, as $t \to \infty$,

$$\left(\frac{L_{\tau(t)}^N(x) - t}{\sqrt{2t}}\right)_{x\in\mathbb{Z}_N^2} \xrightarrow{\text{law}} (h_x^N)_{x\in\mathbb{Z}_N^2}. \quad (\star)$$

Max of 2D GFF

$V_N := [1,N]^2 \cap \mathbb{Z}^2$.

$\tilde{h}^N$: GFF on V_N with zero boundary conditions.

Leading order: Bolthausen-Deuschel-Giacomin (2001)

For $\forall\varepsilon > 0$,

$$\max_{x\in V_N} \tilde{h}_x^N = (1 \pm \varepsilon)2\sqrt{\frac{2}{\pi}}\log N, \text{ w.h.p.}$$

(i.e. the probability $\to 1$ as $N \to \infty$.)

Main result (A. 2015)

(i) For $\forall\varepsilon > 0$ and $\theta > 0$, w.h.p.,

$$\frac{\max_{x\in\mathbb{Z}_N^2} L_{\tau(t_\theta)}^N(x) - t_\theta}{\sqrt{2t_\theta}} = (1 + \frac{1}{2\sqrt{\theta}} \pm \varepsilon)2\sqrt{\frac{2}{\pi}}\log N.$$

(ii) For all $\varepsilon > 0$ and $\theta > 1$, w.h.p.,

$$\frac{\min_{x\in\mathbb{Z}_N^2} L_{\tau(t_\theta)}^N(x) - t_\theta}{\sqrt{2t_\theta}} = -(1 - \frac{1}{2\sqrt{\theta}} \pm \varepsilon)2\sqrt{\frac{2}{\pi}}\log N.$$

Rem. Recall $\tau(t_\theta) \approx \theta \cdot$ (cover time). Thus, for $\forall\theta \in (0,1)$, $\min_{x\in\mathbb{Z}_N^2} L_{\tau(t_\theta)}^N(x) = 0$.

Comparison with 2D GFF

- $\dfrac{\max_{x\in\mathbb{Z}_N^2} L_{\tau(t_\theta)}^N(x) - t_\theta}{\sqrt{2t_\theta}} = (1 + \frac{1}{2\sqrt{\theta}} \pm \varepsilon)2\sqrt{\frac{2}{\pi}}\log N$, w.h.p.
- $\max_{x\in V_N} \tilde{h}_x^N = (1 \pm \varepsilon)2\sqrt{\dfrac{2}{\pi}}\log N$, w.h.p.

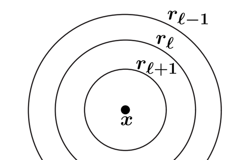
By $(\star)$, one may expect that the maximum of local times is close to that of 2D GFF, but there is a slight difference between the two maximum by the factor $\frac{1}{2\sqrt{\theta}}$.

Outline of proof

Only look at proof of maximum of local times.

Upper bound follows from the Bolthausen-Deuschel-Giacomin result and the isomorphism theorem.

Lower bound: Use method due to Dembo-Peres-Rosen-Zeitouni(2006): Reducing the task to estimates of # crossings of annuli as follows:



$r_\ell := e^{L-\ell}$, where $e^L \approx \frac{N}{\log N}$.

$\mathcal{N}_\ell^x :=$ # crossings from $\partial B(x, r_{\ell+1})$ to $\partial B(x, r_\ell)$ up to time $\tau(t_\theta)$.

Observation 1: By the law of large numbers,

$$L_{\tau(t_\theta)}^N(x) \approx \sum_{j=1}^{\mathcal{N}_{L-1}^x} L_x^{(j)} \approx \frac{2}{\pi}\mathcal{N}_{L-1}^x,$$

where $L_x^{(j)}$ is the local time at x of j th excursion from $\partial B(x, r_L)$ to $\partial B(x, r_{L-1})$.

Thus, $\frac{L_{\tau(t_\theta)}^N(x) - t_\theta}{\sqrt{2t_\theta}} \approx (1 + \frac{1}{2\sqrt{\theta}} - \varepsilon) \cdot 2\sqrt{\frac{2}{\pi}}\log N \iff \mathcal{N}_{L-1}^x \approx 2(\sqrt{\theta} + 1 - c\varepsilon)^2(\log N)^2 =: n_{L-1}$.

Observation 2: $\tau(t_\theta) \approx \frac{2}{\pi}N^2\mathcal{N}_0^x$ due to Dembo-Peres-Rosen-Zeitouni(2006).

Thus, $\mathcal{N}_0^x \approx 2\theta(\log N)^2 =: n_0$.

Observation 3: Conditioned on $\mathcal{N}_{L-1}^x \approx n_{L-1}$,

$(\sqrt{\mathcal{N}_\ell^x})_{0\leq\ell\leq L-1}$ behaves like a Brownian bridge from $\sqrt{n_0}$ to $\sqrt{n_{L-1}}$ by Belius-Kistler (2014+).

Thus, $(\sqrt{\mathcal{N}_\ell^x})_{0\leq\ell\leq L-1}$ will typically look like a linear function in ℓ.

By these observations, we reach the following definition of points with large value of local times:

Def. x is successful $\overset{\text{def}}{\iff}$ For $1 \leq \forall\ell \leq L-1$, $\sqrt{\mathcal{N}_\ell^x} \approx \sqrt{n_0}(1 - \frac{\ell}{L-1}) + \sqrt{n_{L-1}}\frac{\ell}{L-1}$.

(Recall: $n_0 = 2\theta(\log N)^2$, $n_{L-1} = 2(\sqrt{\theta} + 1 - c\varepsilon)^2(\log N)^2$.)

Note that by the above observations, w.h.p.,

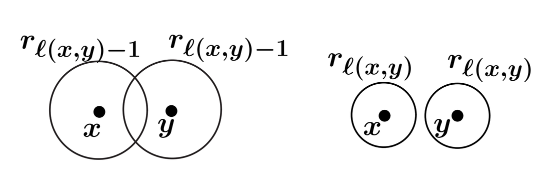
$$\{x : x \text{ is successful}\} \subset \left\{x : \frac{L_{\tau(t_\theta)}^N(x) - t_\theta}{\sqrt{2t_\theta}} \geq (1 + \frac{1}{2\sqrt{\theta}} - \varepsilon) \cdot 2\sqrt{\frac{2}{\pi}}\log N\right\}. \quad (*)$$

Set $Z_N :=$ #successful points. By $(*)$, we have

$$P\left(\frac{\max_{x\in\mathbb{Z}_N^2} L_{\tau(t_\theta)}^N(x) - t_\theta}{\sqrt{2t_\theta}} \geq (1 + \frac{1}{2\sqrt{\theta}} - \varepsilon) \cdot 2\sqrt{\frac{2}{\pi}}\log N\right) \geq P(Z_N \geq 1) \geq \frac{\{E(Z_N)\}^2}{E(Z_N^2)}.$$

Want: $\liminf_{N\to\infty} \frac{\{E(Z_N)\}^2}{E(Z_N^2)} = 1. \quad (\#)$

Proof of $(\#)$.



Fix $x, y \in \mathbb{Z}_N^2$. Let $\ell(x,y)$ be the minimum of ℓ such that $B(x, r_\ell)$ and $B(y, r_\ell)$ are disjoint. By the overlap structure of balls, we can regard that $\mathcal{N}_i^x \approx \mathcal{N}_i^y$ for $1 \leq \forall i \leq \ell(x,y) - 1$, and that $\{\mathcal{N}_i^x : \ell(x,y) \leq i \leq L-1\}$ and $\{\mathcal{N}_i^y : \ell(x,y) \leq i \leq L-1\}$ are almost independent. Thus,

$$\begin{aligned}
&P(x \text{ and } y \text{ are successful}) \\
&\leq P(x \text{ is successful}) \cdot P\left(\sqrt{\mathcal{N}_i^y} \approx \sqrt{n_i} \ \ \forall i > \ell(x,y) \middle| \sqrt{\mathcal{N}_{\ell(x,y)}^y} \approx \sqrt{n_{\ell(x,y)}}\right) \\
&\leq P(x \text{ is successful})^2 \cdot e^{2(1-c\varepsilon)\ell(x,y)},
\end{aligned}$$

where $\sqrt{n_i} := \sqrt{n_0}(1 - \frac{i}{L-1}) + \sqrt{n_{L-1}}\frac{i}{L-1}$.

Recall: $r_\ell = e^{L-\ell}$, $e^L \approx \frac{N}{\log N}$.

Note that for $x, y \in \mathbb{Z}_N^2$ with $|x - y| \geq 2r_0$, we can regard that $\{x \text{ is successful}\}$ and $\{y \text{ is successful}\}$ are almost independent because $B(x, r_0)$ and $B(y, r_0)$ are disjoint. Thus,

$$\begin{aligned}
E[Z_N^2] &\leq \sum_{|x-y|\geq 2r_0} P(x \text{ and } y \text{ are successful}) + \sum_{\ell=1}^{L}\sum_{\ell(x,y)=\ell} P(x \text{ and } y \text{ are successful}) \\
&\leq (1 + o(1))N^4 P(x_0 \text{ is successful})^2 + \sum_{\ell=1}^{L} N^2 e^{2L-2\ell} P(x_0 \text{ is successful})^2 \cdot e^{2(1-c\varepsilon)\ell} \\
&\leq (1 + o(1))\{E(Z_N)\}^2 + \frac{1}{(\log N)^2}\{E(Z_N)\}^2, \text{ where } x_0 \text{ is a fixed point.}
\end{aligned}$$

Thus, $\liminf_{N\to\infty} \frac{\{E(Z_N)\}^2}{E(Z_N^2)} = 1$, and hence

$$\lim_{N\to\infty} P\left(\frac{\max_{x\in\mathbb{Z}_N^2} L_{\tau(t_\theta)}^N(x) - t_\theta}{\sqrt{2t_\theta}} \geq (1 + \frac{1}{2\sqrt{\theta}} - \varepsilon) \cdot 2\sqrt{\frac{2}{\pi}}\log N\right) = 1.$$