Perturbation of Dirichlet forms and stability of fundamental solutions

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Introduction

Characterization of a symmetric Markov process $(\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in\mathbb{R}^d})$ on $\mathbb{R}^d$

- Self-adjoint operator (Non-positive generator) $\mathcal{L}$
- Strong continuous contraction semigroup $\{P_t\}_{t\geq 0}$
- Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\mathbb{R}^d)$

Relations among $\{X_t\}$, $\mathcal{L}$, $\{P_t\}_{t\geq 0}$ and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$

- $\mathcal{E}(u, v) = (-\mathcal{L}u, v)$
- $P_t = \exp(t\mathcal{L})$ (as an operator on $L^2(\mathbb{R}^d)$)
- $P_t f(x) = \mathbb{E}_x[f(X_t)]$

Beuling-Deny formula

$$\mathcal{E}(u,v) = \mathcal{E}^c(u,v) + \int_{\mathbb{R}^d\times\mathbb{R}^d} (u(y)-u(x))(v(y)-v(x))J(x,y)dxdy + \int_{\mathbb{R}^d} u(x)v(x)k(dx)$$

There are three parts in Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$

- $\mathcal{E}^c$: Strong local part. Corresponding to diffusion (continuous) part of X_t
- The second term: Jump part. Corresponding to discontinuous part of X_t
- The third term: Killing part. Corresponding to killing part of X_t

In the sequel, we consider the pure jump Markov process.

Setting of jump processes

Consider the jump Dirichlet form on $L^2(\mathbb{R}^d)$

$$\mathcal{E}(u,v) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(y)-u(x))(v(y)-v(x))J(x,y)dxdy$$

$$\frac{\kappa_1}{|x-y|^d \phi(|x-y|)} \leq J(x,y) \leq \frac{\kappa_2}{|x-y|^d \phi(|x-y|)},$$

where $J(x,y)$ is a symmetric function called jump intensity measure.
κ_1 and κ_2 are positive constants.
ϕ is a positive increasing function.

Let $\{X_t\}_{t\geq 0}$ be the corresponding Markov process.

(1) If $\phi(r) = r^\alpha$ for some $0 < \alpha < 2$, $\{X_t\}_{t\geq 0}$ is called α-stable-like.

(2) If $\phi(r) = r^\alpha(1 \vee \exp(n(r-1)))$ for some $0 < \alpha < 2$ and $n > 0$, $\{X_t\}_{t\geq 0}$ is called relativistic α-stable-like

In this talk, we consider these two cases.

Preceding results -Conservativeness-

Let ζ be the lifetime of Markov process $\{X_t\}_{t\geq 0}$.

Note that $J(x,y)$ is a symmetric function satisfying

$$\sup_{x\in\mathbb{R}^d}\int(1\wedge|x-y|^2)J(x,y)dy<\infty$$

$$\exp(-a|x|)\in L^1(\mathbb{R}^d)\quad(a>0)$$

It follows that $\{X_t\}$ is conservative by Masamune-Uemura 2011 and thus $\zeta=\infty$.

Preceding results -Heat kernel estimates-

Let $\{P_t\}$ be the semigroup associated with the Markov process $\{X_t\}$.

In both cases, Chen and Kumagai et al proved that the semigroup $\{P_t\}_{t>0}$ admits the jointly continuous transition density function $p(t,x,y)$ (or equivalently, the fundamental solution of $\frac{\partial u}{\partial t} = \mathcal{L}u$) defined on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$, and it has the two sided estimates.

Theorem (Chen and Kumagai 2003)

When $\{X_t\}_{t\geq 0}$ is α-stable-like process, $p(t,x,y)$ satisfies

$$C_1(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}) \leq p(t,x,y) \leq C_2(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}),$$

where C_1 and C_2 are positive constants depending on κ_1, κ_2, d and α.

Theorem (Chen, Kim and Kumagai 2011)
When $\{X_t\}_{t\geq 0}$ is relativistic α-stable-like process, $p(t,x,y)$ satisfies two sided estimates as follows;

(1) $0 < t \leq 1$ and $0 < |x-y| \leq 1$

$$c_1\left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \leq p(t,x,y) \leq c_2\left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}\right)$$

(2) $1 \vee |x-y| \leq t$

$$c_3 t^{-\frac{d}{2}} \exp\left(-\frac{c_4|x-y|^2}{t}\right) \leq p(t,x,y) \leq c_5 t^{-\frac{d}{2}} \exp\left(-\frac{c_6|x-y|^2}{t}\right)$$

(3) $1 \leq t \leq |x-y|$

$$c_7 t^{-\frac{d}{2}} \exp(-c_8|x-y|) \leq p(t,x,y) \leq c_9 t^{-\frac{d}{2}} \exp(-c_{10}|x-y|)$$

(4) $0 < t \leq 1$ and $1 \leq |x-y|$

$$c_{11}\, t \exp(-c_{12}|x-y|) \leq p(t,x,y) \leq c_{13}\, t \exp(-c_{14}|x-y|)$$

where c_i's are positive constants depending on $\kappa_1, \kappa_2, d, \alpha$ and n.

Problem

Summary of preceding results

Both upper and lower bounds of the heat kernel are the same function up to choice of positive constants.

Problem

Let μ be a positive Radon smooth measure on $\mathbb{R}^d$ satisfying Green tightness (abbreviation by $\mu \in \mathcal{K}_\infty$: The precise definition will be stated later.).
Consider the Schrödinger form (i.e. perturbation of Dirichlet form by μ)

$$\mathcal{E}^\mu(u,u) = \mathcal{E}(u,u) - \int_{\mathbb{R}^d} u^2 d\mu$$

Denote the corresponding generator by $\mathcal{L}^\mu$ and let $p^\mu(t,x,y)$ be the fundamental solution of equation

$$\frac{\partial u}{\partial t} = \mathcal{L}^\mu u$$

Problem

What conditions on μ are necessary and sufficient for $p^{\mu}(t,x,y)$ to have the same estimates as $p(t,x,y)$ does?

We call this phenomenon the stability of fundamental solution.

Remark

We consider the stability for global time.
If $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is associated with the standard Brownian motion, there is a preceding result as follows:

Theorem (Takeda 2007)

Assume that the Brownian motion is transient and μ belongs to the class $\mathcal{S}_{\infty}$(A precise definition is given later). Then the stability of fundamental solution holds if and only if μ satisfies

$$\inf\{\mathcal{E}(u,u) \mid u \in \mathcal{D}(\mathcal{E}),\ \int_{\mathbb{R}^d} u^2 d\mu = 1\} > 1$$

Main result

Assumption 1

The Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is transient and thus, we can define the Green kernel

$$G(x,y) := \int_0^\infty p(t,x,y)dt < \infty \quad (x \neq y)$$

Moreover, it follows that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} G(x,y)\mu(dx)\mu(dy) < \infty$$

Theorem (W. 2012)

Suppose $\mu \in \mathcal{K}_\infty$ and Assumption 1 holds. Then, $p^\mu(t,x,y)$ has the same two-sided estimates as $p(t,x,y)$ if and only if μ satisfies

$$\inf\{\mathcal{E}(u,u) \mid u \in \mathcal{D}(\mathcal{E}) \int_{\mathbb{R}^d} u^2 d\mu = 1\} > 1$$

Definition of the Kato class measure

Since we assume that $\{X_t\}$ is transient, we can define the Green kernel by

$$G(x,y) := \int_0^\infty p(t,x,y)dt < \infty.$$

Using $G(x,y)$, we define some classes of positive Radon smooth measure.

Definition

Let μ be a positive Radon smooth measure.
μ is in Kato class ($\mu \in \mathcal{K}$), if it holds that

$$\lim_{\beta\to\infty} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} G_\beta(x,y)\mu(dy) = 0,$$

where $G_\beta(x,y) = \int_0^\infty e^{-\beta t} p(t,x,y)dt$.

Definition of Green-tight measure

Definition

(1) μ belongs to $\mathcal{K}_\infty$ if $\mu \in \mathcal{K}$ and for arbitrary $\epsilon > 0$, there exist positive constant $\delta > 0$ and compact set K such that

$$\sup_{x \in \mathbb{R}^d} \int_{K^c \cup B} G(x,y)\mu(dy) < \epsilon$$

where B is an arbitrary set that satisfies $B \subset K$ and $\mu(B) < \delta$.

(2) μ belongs to $\mathcal{S}_\infty$ if $\mu \in \mathcal{K}$ and for arbitrary $\epsilon > 0$, there exist positive constant $\delta > 0$ and compact set K such that

$$\sup_{x,z \in \mathbb{R}^d} \int_{K^c \cup B} \frac{G(x,y)G(y,z)}{G(x,z)}\mu(dy) < \epsilon$$

where B is an arbitrary set that satisfies $B \subset K$ and $\mu(B) < \delta$.

equivalent definitions for $\mathcal{K}$

A_t^μ: Positive continuous additive functional in Revuz correspondence with μ.

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}_{g\cdot m}[\int_0^t f(X_s)dA_s^\mu] = \int_{\mathbb{R}^d} g(x)f(x)\mu(dx),$$

where m is the Lebesgue measure on $\mathbb{R}^d$, g is a γ-excessive function ($\gamma \geq 0$) and f is a bounded measurable function.

Proposition

The following assertions are equivalent each other.

$$(1)\mu \in \mathcal{K}$$

$$(2)\ \lim_{t\to 0}\sup_{x\in\mathbb{R}^d}\mathbb{E}_x[A_t^\mu] = 0$$

$$(3)\ \lim_{a\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|\leq a} G(x,y)\mu(dy) = 0$$

Outline of the proof

(1) $\Leftrightarrow$ (2): Note that

$$\mathbb{E}_x[A_t^\mu] = \int_0^t \int_{\mathbb{R}^d} p(s,x,y)\mu(dy)ds$$

and apply the argument of Kuwae and Takahashi 2006.

(2) $\Leftrightarrow$ (3): Based on the argument of Zhao 1991, we have only to prove

$$\alpha_0 := \sup_{t\geq 0} \inf_{r>0} \sup_{x\in\mathbb{R}^d} \mathbb{P}_x(\tau_{B(x,r)} > t) < 1$$

$$\beta_0 := \sup_{r>0} \inf_{t>0} \sup_{x\in\mathbb{R}^d} \mathbb{P}_x(\tau_{B(x,r)} < t) < 1$$

$$\lambda_0 := \sup_{u>0} \inf_{r>0} \sup_{x,y\in\mathbb{R}^d} \mathbb{P}_y(T_{B(x,r)} < \infty) < 1,$$

where

$$\tau_{B(x,r)} := \inf\{t > 0 \mid X_t \notin B(x,r)\}$$

$$T_{B(x,r)} := \inf\{t > 0 \mid X_t \in B(x,r)\}.$$

equivalent definitions for $\mathcal{K}_\infty$

Proposition

For $\mu \in \mathcal{K}$, the following assertions are equivalent each other.

(1) For arbitrary $\epsilon > 0$, there exist positive constant $\delta > 0$ and compact set K such that

$$\sup_{x \in \mathbb{R}^d} \int_{K^c \cup B} G(x,y)\mu(dy) < \epsilon$$

where B is an arbitrary set that satisfies $B \subset K$ and $\mu(B) < \delta$.

(2) For arbitrary $\epsilon > 0$, there exist a positive constant $\tilde{\delta} > 0$ and a set F of μ-finite measure such that

$$\sup_{x \in \mathbb{R}^d} \int_{F^c \cup B} G(x,y)\mu(dy) < \epsilon$$

(3) It holds that

$$\lim_{r \to \infty} \sup_{x \in \mathbb{R}^d} \int_{|y| \geq r} G(x,y)\mu(dy) = 0$$

The relation between $\mathcal{K}_\infty$ and $\mathcal{S}_\infty$

Proposition (W. 2012)
It holds that $\mathcal{K}_\infty = \mathcal{S}_\infty$.

Outline of the proof
In general, it holds that $\mathcal{S}_\infty \subset \mathcal{K}_\infty$ by Chen and Song 2002.
Thus we have only to prove $\mathcal{K}_\infty \subset \mathcal{S}_\infty$.
By the argument of $3G$-theorem, it is sufficient to prove that

$$\frac{G(x,y)G(y,z)}{G(x,z)} \leq C_0(G(x,y)+G(y,z))$$

for some positive constant C_0.

Lemma

(1) Let $\{X_t\}_{t\geq 0}$ be α-stable-like process. Then there exists a positive constants C_1, C_2 such that

$$\frac{C_1}{|x-y|^{d-\alpha}} \leq G(x,y) \leq \frac{C_2}{|x-y|^{d-\alpha}}$$

(2) Let $\{X_t\}_{t\geq 0}$ be relativistic α-stable-like process. Then there exists positive constants C_1, C_2 such that

$$C_1\Big(\frac{1}{|x-y|^{d-\alpha}} \vee \frac{1}{|x-y|^{d-2}}\Big) \leq G(x,y) \leq C_2\Big(\frac{1}{|x-y|^{d-\alpha}} \vee \frac{1}{|x-y|^{d-2}}\Big)$$

In both cases, there exists a positive decreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$C_3 \leq \frac{g(r)}{g(2r)} \leq C_4$$

for some positive constants C_3, C_4 and

$$C_1 g(|x-y|) \leq G(x,y) \leq C_2 g(|x-y|)$$

Noting that at least either $|x-y| \geq |x-z|/2$ or $|y-z| \geq |x-z|/2$ holds, the above formula is valid.

Perturbation theory

In the sequel we consider the Schrödinger form for $\mu \in \mathcal{K}_\infty$:

$$\mathcal{E}^\mu(u,u) = \mathcal{E}(u,u) - \int_{\mathbb{R}^d} u^2 d\mu$$

Denote the corresponding semigroup by P_t^μ. Then it follows that

$$P_t^\mu f(x) = \mathbb{E}_x[\exp(A_t^\mu) f(X_t)]$$

Note that P_t^μ is represented by Feynman-Kac formula with exponential growth.

Proposition(Albeverio Blanchard Ma 1991)
P_t^μ admits jointly continuous integral kernel $p^\mu(t,x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

Outline of the proof for main theorem -only if part-

Theorem (W. 2012)

Suppose $\mu \in \mathcal{K}_\infty$ and Assumption 1 holds. Then, $p^\mu(t,x,y)$ has the same two-sided estimates as $p(t,x,y)$ if and only if μ satisfies

$$\inf\{\mathcal{E}(u,u) \mid u \in \mathcal{D}(\mathcal{E}) \text{and} \int u^2 d\mu = 1\} > 1$$

(Outline of the proof)

Since the stability of fundamental solution holds, it follows that

$$G^\mu(x,y) := \int_0^\infty p^\mu(t,x,y)dt < \infty$$

from the Green kernel estimates.

Proposition (Takeda 2002)
For $\mu \in \mathcal{S}_\infty$, the following assertions are equivalent.

$$(1) G^\mu(x,y) < \infty \textit{ for } x, y \in \mathbb{R}^d \textit{ with } x \neq y$$

$$(2) \inf\{\mathcal{E}(u,u) \mid u \in \mathcal{D}(\mathcal{E}) \text{and} \int u^2 d\mu = 1\} > 1$$

$$(3) \sup_{x \in \mathbb{R}^d} \mathbb{E}_x[\exp(A^\mu_\infty)] < \infty$$

Noting that $\mathcal{K}_\infty = \mathcal{S}_\infty$, we see that the only if part is valid.

Remark
μ is said to satisfy the gaugeability if the third formula holds.

Outline of the proof for main theorem -if part-

Outline of the proof
Following the arguments of Takeda 2006.

Let $h(x) = \mathbb{E}_x[\exp(A^{\mu}_{\infty})]$.
Then the gaugeability implies that $1 \leq h(x) \leq C_0$ for some positive constant.

Proposition (Chen-Zhang 2002)
If $h(x) = \exp(u(x))$ for some $u \in \mathcal{D}_e(\mathcal{E})$, there exists an appropriate multiplicative functional L_t such that

$$Q_t f(x) := \mathbb{E}_x[L_t f(X_t)]$$

is strong continuous semigroup on $L^2(h^2 m)$.
Moreover, the associated Dirichlet form $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ has a representation

$$\tilde{\mathcal{E}}(v, v) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(y) - v(x))^2 J(x, y) h(x) h(y) dxdy$$
$$\mathcal{D}(\tilde{\mathcal{E}}) = \mathcal{D}(\mathcal{E})$$

The existence of $u \in \mathcal{D}_e(\mathcal{E})$

Define

$$G\mu(x) = \int_{\mathbb{R}^d} G(x,y)\mu(dy).$$

By Stollmann-Voigt 1996, it is known that

$$\int_{\mathbb{R}^d} u^2 d\mu \le \|G\mu\|_\infty \mathcal{E}(u,u) \quad (u \in \mathcal{D}_e(\mathcal{E})).$$

Applying this formula, we obtain

$$\int_{\mathbb{R}^d} \psi d\mu_F \le (\mu(F))^{1/2}(\int_{\mathbb{R}^d} \psi^2 d\mu_F)^{1/2} \le (\mu(F))\|G\mu_F\|_\infty^{1/2}\mathcal{E}(\psi,\psi)^{1/2}$$

where $\mu_F(A) = \mu(F \cap A)$ and F is of finite μ-measure.
μ_F is of finite energy integral.

$$\int_{\mathbb{R}^d} \psi d\mu_F \leq \mathcal{E}(G\mu_F, G\mu_F)^{1/2}\mathcal{E}(\psi,\psi)^{1/2}$$
$$\leq \Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} G(x,y)d\mu_F(x)d\mu_F(y)\Big)^{1/2}\mathcal{E}(\psi,\psi)^{1/2}.$$

Since we assume $\iint_{\mathbb{R}^d\times\mathbb{R}^d} G(x,y)\mu(dx)\mu(dy) < \infty$
μ is also of finite energy integral and $G\mu \in \mathcal{D}_e(\mathcal{E})$

Let $K_t = \mathbb{E}_x[\exp(A^{\mu}_{\infty})|\mathcal{M}_t]$ $(\{\mathcal{M}_t\}$: filtration). Noting that

$$h(X_t) = \exp(-A^{\mu}_t)K_t \qquad \mathbb{E}_x[\int_0^t h(X_s)dA^{\mu}_s] = h(x) - \mathbb{E}_x[h(X_t)]$$

$$\lim_{t\to\infty} h(X_t) = 1$$

It follows that

$$h(x) = 1 + G(h\mu)(x)$$

Noting that $G(h\mu) \in \mathcal{D}_e(\mathcal{E})$, we can define $u := \log(1 + G(h\mu)) \in \mathcal{D}_e(\mathcal{E})$

Calculation of L_t based on Chen-Zhang 2002

Since $G(h\mu) \in \mathcal{D}_e(\mathcal{E})$, we can consider the Fukushima's decomposition as follows:

$$G(h\mu)(X_t) - G(h\mu)(X_0) = M_t^h + N_t^h$$

where M_t^h is a martingale additive functional and N_t^h is a continuous additive functional of zero energy. Note that

$$h(X_t) - h(X_0) = M_t^h + N_t^h$$

We define the martingale

$$M_t = \int_0^t \frac{1}{h(X_{s-})} dM_s^h$$

and consider the Doléans-Dade formula

$$Z_s = 1 + \int_0^s Z_{t-} dM_t$$

L_t : Unique solution for the previous formula.

$$L_t = \exp(M_t - \frac{1}{2}\langle M^c\rangle_t) \prod_{0<s\leq t} (1 + \Delta M_s) \exp(-\Delta M_s))$$
$$= \exp(M_t - \frac{1}{2}\langle M^c\rangle_t) \prod_{0<s\leq t} \frac{h(X_s)}{h(X_{s-})} \exp\Big(1 - \frac{h(X_s)}{h(X_{s-})}\Big)$$

where $\Delta M_s = M_s - M_{s-}$.
$\langle M^c\rangle$ is the sharp bracket for continuous part of M_t.

Applying the Ito formula to the semimartingale $h(X_t)$ and $\log x$, we have

$$L_t = \frac{h(X_t)}{h(X_0)} \exp(A_t^\mu)$$

Now we see that L_t defines the Girsanov transformation in the sense of Chen-Zhang 2002 and

$$Q_t f(x) = \mathbb{E}_x[L_t f(X_t)] = \mathbb{E}_x[\frac{h(X_t)}{h(X_0)} \exp(A_t^\mu) f(X_t)]$$

$$\tilde{\mathcal{E}}(v,v) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v(y) - v(x))^2 J(x,y) h(x) h(y) dxdy$$

$$\mathcal{D}(\tilde{\mathcal{E}}) = \mathcal{D}(\mathcal{E})$$

For appropriate κ_1' and κ_2',

$$\frac{\kappa_1'}{|x-y|^d \phi(|x-y|)} \leq J(x,y)h(x)h(y) \leq \frac{\kappa_2'}{|x-y|^d \phi(|x-y|)}$$

Integral kernel for $\{Q_t\}$: $h(x)p^\mu(t,x,y)h(y)$ with respect to Lebesgue measure
By Chen Kumagai 2003 and Chen, Kim and Kumagai 2011, $p^\mu(t,x,y)$ has the same two-sided estimates as $p(t,x,y)$.

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