Lévy measure density corresponding to inverse local time

Tomoko Takemura and Matsuyo Tomisaki

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motivation

We are concerned with Lévy measure density corresponding to the inverse local time at the regular end point for harmonic transform of a one dimensional diffusion process. We show that the Lévy measure density is represented as a Laplace transform of the spectral measure corresponding to an original diffusion process, where the absorbing boundary condition is posed at the end point if it is regular.

	h transform		Itô and McKean	
$\mathbb{D}_{s,m,k}$	$\longleftrightarrow$	$\mathbb{D}_{s_h,m_h,0}$	$\longleftrightarrow$	$\mathbb{D}^*_{s_h,m_h,0}$
absorbing		absorbing		reflecting
				$n^*(\xi)$

Tabel contents

One dimensional diffusion process

- We set

$$s : \text{continuous increasing fnc. on } I = (l_1, l_2),\ -\infty \le l_1 < l_2 \le \infty$$
$$m : \text{right continuous increasing fnc. on } I$$
$$k : \text{right continuous nondecreasing fnc. on } I$$

One dimensional diffusion process

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- $\mathcal{G}_{s,m,k}$: 1-dim diffusion operator with s, m, and k

$$\mathcal{G}_{s,m,k} u = \frac{dD_s u - u dk}{dm}$$

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- $\mathbb{D}_{s,m,k}$: 1-dim diffusion process with $\mathcal{G}_{s,m,k}$
  [l_1 is absorbing if l_1 is regular]

One dimensional diffusion process

- $p(t, x, y)$: transition probability w.r.t. dm for $\mathbb{D}_{s,m,k}$

If l_1 is (s, m, k)-regular,

$$p(t, x, y) = \int_{[0,\infty)} e^{-\lambda t} \psi_o(x, \lambda) \psi_o(y, \lambda) \, d\sigma(\lambda), \qquad t > 0, \; x, y \in I, \tag{1}$$

where $d\sigma(\lambda)$ is a Borel measure on $[0, \infty)$ satisfying

$$\int_{[0,\infty)} e^{-\lambda t} \, d\sigma(\lambda) < \infty, \qquad t > 0, \tag{2}$$

and $\psi_o(x, \lambda)$, $x \in I$, $\lambda \geq 0$, is the solution of the following integral equation

$$\begin{aligned} \psi_o(x, \lambda) =& s(x) - s(l_1) \\ &+ \int_{(l_1, x]} \{s(x) - s(y)\} \psi_o(y, \lambda) \{-\lambda \, dm(y) + dk(y)\} \end{aligned}$$

One dimensional diffusion process

Proposition 2.1

Assume that l_1 is (s, m, k)-entrance and

$$\int_{(l_1, c_o]} \{s(c_o) - s(x)\}^2 \, dm(x) < \infty. \tag{3}$$

Then $p(t, x, y)$ is represented as (1) with $d\sigma(\lambda)$ satisfying (2) and $\psi_o(x, \lambda)$ is the solution of the integral equation

$$\psi_o(x, \lambda) = 1 + \int_{(l_1, x]} \{s(x) - s(y)\} \psi_o(y, \lambda) \{-\lambda \, dm(y) + dk(y)\}.$$

Harmonic transform

- We set

$$\mathcal{H}_{s,m,k,\beta} = \{h > 0;\ \mathcal{G}_{s,m,k} h = \beta h\}, \quad \text{for } \beta \geq 0$$

For $h \in \mathcal{H}_{s,m,k,\beta}$,

$$ds_h(x) = h(x)^{-2} ds(x), \quad dm_h(x) = h(x)^2 dm(x)$$

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- We obtain

$$\mathcal{G}_{s_h,m_h,0} : \text{h transform of } \mathcal{G}_{s,m,k} \quad \left[p_h(t,x,y) = e^{-\beta t} \frac{p(t,x,y)}{h(x)h(y)} \right]$$

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- $\mathbb{D}_{s_h,m_h,0}$: 1-dim diffusion process with $\mathcal{G}_{s_h,m_h,0}$
  [l_1 is absorbing if l_1 is regular]

Harmonic transform

- $\mathbb{D}^*_{s_h,m_h,0}$: 1-dim diffusion process with $\mathcal{G}_{s_h,m_h,0}$
  [l_1 is regular and reflecting boundary]

Harmonic transform

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  [l_1 is regular and reflecting boundary]
- $l^{(h*)}(t,\xi)$: local time for $\mathbb{D}^*_{s_h,m_h,0}$, that is,

$$\int_0^t f(X(u))\,du = \int_I l^{(h*)}(t,\xi)\,dm_h(\xi), \quad t > 0,$$

for bounded continuous functions f on I.

Harmonic transform

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  for bounded continuous functions f on I.
- $\tau^{(h*)}(t)$: inverse local time ${l^{(h*)}}^{-1}(t,l_1)$ at the end point l_1

Lévy measure density

Proposition 2.2 (Itô and McKean)

Assume the following conditions.

$$l_1 \text{ is } (s, m, 0)\text{-regular and reflecting, } s(l_2) = \infty.$$

Then $[\tau^*(t),\ t \geq 0]$ *is a Lévy process and there is a Lévy measure density* $n^*(\xi)$ *such that*

$$E^*_{l_1}\left[e^{-\lambda\tau^*(t)}\right] = \exp\left\{-t\int_0^\infty (1 - e^{-\lambda\xi})n^*(\xi)\,d\,\xi\right\}$$

where $E^*_{l_1}$ *stands for the expectation with respect to* $P^*_{l_1}$*,*

$$n^*(\xi) = \lim_{x,y\to l_1} D_{s(x)}D_{s(y)}p(\xi, x, y) = \int_{[0,\infty)} e^{-\lambda\xi}d\sigma(\lambda),$$

where $p(t, x, y)$ *is the transition probability density for* $\mathbb{D}_{s,m,0}$*, and* $d\sigma(\lambda)$ *is the Borel measure appeared in (1) satisfying (2).*

Main theorem

Now we give a representation of $n^{(h*)}(\xi)$ by means of items corresponding to the diffusion process $\mathcal{D}_{s,m,k}$. l_1 is $(s_h, m_h, 0)$-regular if and only if one of the following conditions is satisfied.

$$l_1 \text{ is } (s, m, k)\text{-regular and } h(l_1) \in (0, \infty). \tag{4}$$

$$l_1 \text{ is } (s, m, k)\text{-entrance, } h(l_1) = \infty, \text{ and } |m_h(l_1)| < \infty. \tag{5}$$

$$l_1 \text{ is } (s, m, k)\text{-natural, } h(l_1) = \infty, \text{ and } |m_h(l_1)| < \infty. \tag{6}$$

Main theorem

Theorem 2.3

Let $h \in \mathcal{H}_{s,m,k,\beta}$. Assume one of (4), (5), and (6). Further assume that l_1 is reflecting and $s_h(l_2) = \infty$. Then there exists Lévy measure density $n^{(h*)}(\xi)$. In particular, if (4) is satisfied, then

$$\begin{aligned} n^{(h*)}(\xi) =& h(l_1)^2 e^{-\beta\xi} \int_{[0,\infty)} e^{-\xi\lambda}\, d\sigma(\lambda) \\ =& h(l_1)^2 e^{-\beta\xi} \lim_{x,y \to l_1} D_{s(x)} D_{s(y)} p(\xi, x, y). \end{aligned}$$

If (5) is satisfied, then

$$\begin{aligned} n^{(h*)}(\xi) =& D_s h(l_1)^2 e^{-\beta\xi} \int_{[0,\infty)} e^{-\xi\lambda}\, d\sigma(\lambda) \\ =& D_s h(l_1)^2 e^{-\beta\xi} \lim_{x,y \to l_1} p(\xi, x, y). \end{aligned}$$

Examples

Example 2.4 (Bessel process)

Let us consider the following diffusion operator $\mathcal{G}^{(\nu)}$ *on* $I = (0, \infty)$.

$$\mathcal{G}^{(\nu)} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu+1}{2x}\frac{d}{dx},$$

where $-\infty < \nu < \infty$.

$$ds^{(\nu)}(x) = x^{-2\nu-1}\,dx, \quad dm^{(\nu)}(x) = 2x^{2\nu+1}\,dx.$$

The killing measure is null. The state of the end point 0 depends on ν*, that is,*

$$\begin{aligned} &\text{it is } (s^{(\nu)}, m^{(\nu)}, 0)\text{-entrance if } \nu \geq 0, \\ &\text{it is } (s^{(\nu)}, m^{(\nu)}, 0)\text{-regular if } -1 < \nu < 0, \\ &\text{it is } (s^{(\nu)}, m^{(\nu)}, 0)\text{-exit if } \nu \leq -1. \end{aligned}$$

Examples

Further

$$\int_0^1 \{s^{(\nu)}(1) - s^{(\nu)}(x)\}^2 \, dm^{(\nu)}(x) < \infty \iff |\nu| < 1.$$

The end point ∞ is $(s^{(\nu)}, m^{(\nu)}, 0)$-natural for all ν, and in particular,

$$s^{(\nu)}(\infty) = \infty \iff \nu \leq 0.$$

Let

$$\mathbb{D}^{(\nu)} : \text{the diffusion process on } I \text{ with } \mathcal{G}^{(\nu)}$$

$$(\text{ 0 being absorbing if } -1 < \nu < 0)$$

$$p^{(\nu)}(t, x, y) : \text{the transition probability density w.r.t. } dm^{(\nu)}.$$

Examples

(1) $-1 < \nu < 0$ [0 : $(s^{(\nu)}, m^{(\nu)}, 0)$-regular]

$\mathbb{D}^{(\nu,*)}$: the diffusion process on I with $\mathcal{G}^{(\nu)}$
(0 being reflecting)

$n^{(\nu,*)}$:the Lévy measure density corresponding to
the inverse local time at 0 for $\mathbb{D}^{(\nu,*)}$

Since $s^{(\nu)}(\infty) = \infty$,

$$n^{(\nu,*)}(\xi) = \lim_{x,y\to 0} D_{s^{(\nu)}(x)} D_{s^{(\nu)}(y)} p^{(\nu)}(\xi, x, y)$$

$$= \int_0^\infty e^{-\xi\lambda}\sigma^{(\nu)}(\lambda)\, d\lambda = 2^{-|\nu|+1}\frac{|\nu|}{\Gamma(|\nu|)}\xi^{-(|\nu|+1)}.$$

Examples

(2) $-1 < \nu < 1$.
[0 : $(s^{(\nu)}, m^{(\nu)}, 0)$-regular or -entrance, and (3) is satisfied]
For $\beta > 0$, we put

$$h(x) = \left(\frac{\beta}{2}\right)^{\frac{|\nu|}{2}} x^{-\nu} K_{|\nu|}(\sqrt{2\beta}x)$$

Then $h(x) \in \mathcal{H}_{s^{(\nu)}, m^{(\nu)}, 0, \beta}$ and

$$\mathcal{G}_h^{(\nu)} = \frac{1}{2}\frac{d^2}{dx^2} + \left\{ \frac{1}{2x} + \sqrt{2\beta}\,\frac{K_\nu'\left(\sqrt{2\beta}\,x\right)}{K_\nu\left(\sqrt{2\beta}\,x\right)} \right\} \frac{d}{dx},$$

$$ds^{(\nu,\beta)}(x) = h(x)^{-2}\, ds^{(\nu)}(x), \quad dm^{(\nu,\beta)}(x) = h(x)^2\, dm^{(\nu)}(x).$$

Examples

The end point 0 is $(s^{(\nu,\beta)}, m^{(\nu,\beta)}, 0)$-regular. We consider the diffusion process $\mathbb{D}_h^{(\nu,*)}$ with $\mathcal{G}_h^{(\nu)}$ as the generator and with the end point 0 being reflecting. Let $n_h^{(\nu,*)}$ be the Lévy measure density corresponding to the inverse local time at 0 for $\mathbb{D}_h^{(\nu,*)}$.

$$n_h^{(\nu,*)} = 2^{-|\nu|-1}\Gamma(|\nu|+1)\xi^{-(|\nu|+1)}e^{-\beta\xi}.$$

(3) $0 < \nu < 1$
We put
$h^{(0)}(x) = \{s^{(\nu)}(\infty) - s^{(\nu)}(x)\}/\{s^{(\nu)}(\infty) - s^{(\nu)}(1)\} = x^{-2\nu}$.
Denote by $\mathcal{G}_h^{(\nu,0)}$ the harmonic transform of $\mathcal{G}^{(\nu)}$ based on $h^{(0)} \in \mathcal{H}_{s^{(\nu)},m^{(\nu)},0,0}$, that is,

$$\mathcal{G}_h^{(\nu,0)} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{-2\nu+1}{2x}\frac{d}{dx}.$$

Examples

$$ds^{(\nu,0)}(x) = h^{(0)}(x)^{-2}\, ds^{(\nu)}(x) = x^{2\nu-1}\, dx,$$
$$dm^{(\nu,0)}(x) = h^{(0)}(x)^{2}\, dm^{(\nu)}(x) = 2x^{-2\nu+1}\, dx.$$

The end point 0 is $(s^{(\nu,0)}, m^{(\nu,0)}, 0)$-regular. We consider the diffusion process $\mathbb{D}_h^{(\nu,0,*)}$ with $\mathcal{G}_h^{(\nu,0)}$ as the generator and with the end point 0 being reflecting. Let $n_h^{(\nu,0,*)}$ be the Lévy measure density corresponding to the inverse local time at 0 for $\mathbb{D}_h^{(\nu,0,*)}$.

$$n_h^{(\nu,0,*)} = 2^{-\nu+1}\frac{\nu}{\Gamma(\nu)}\xi^{-\nu-1}.$$

Examples

Example 2.5 (Radial Ornstein-Uhlenbeck process)

Let us consider the following diffusion operator $\mathcal{G}^{(\nu,\kappa)}$ *on* $I = (0, \infty)$.

$$\mathcal{G}^{(\nu,\kappa)} = \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{2\nu+1}{2x} - \kappa x\right)\frac{d}{dx},$$

where $-\infty < \nu < \infty$ *and* $\kappa > 0$.

$$ds^{(\nu,\kappa)}(x) = x^{-2\nu-1}e^{\kappa x^2}\,dx, \quad dm^{(\nu,\kappa)}(x) = 2x^{2\nu+1}e^{-\kappa x^2}\,dx.$$

The killing measure is null. The state of the end point 0 depends on ν, *that is,*

$$\begin{aligned}
&\text{it is } (s^{(\nu,\kappa)}, m^{(\nu,\kappa)}, 0)\text{-entrance if } \nu \geq 0,\\
&\text{it is } (s^{(\nu,\kappa)}, m^{(\nu,\kappa)}, 0)\text{-regular if } -1 < \nu < 0,\\
&\text{it is } (s^{(\nu,\kappa)}, m^{(\nu,\kappa)}, 0)\text{-exit if } \nu \leq -1.
\end{aligned}$$

Examples

Further

$$\int_0^1 \{s^{(\nu,\kappa)}(1) - s^{(\nu,\kappa)}(x)\}^2 \, dm^{(\nu,\kappa)}(x) < \infty \iff |\nu| < 1.$$

The end point ∞ is always $(s^{(\nu,\kappa)}, m^{(\nu,\kappa)}, 0)$-natural for all ν, and

$$s^{(\nu,\kappa)}(\infty) = \infty.$$

Let

$$\mathbb{D}^{(\nu,\kappa)} \text{ : the diffusion process on } I \text{ with } \mathcal{G}^{(\nu,\kappa)}$$

$$(\text{ 0 being absorbing if } -1 < \nu < 0)$$

$$p^{(\nu,\kappa)}(t,x,y) \text{ :the transition probability density w.r.t. } dm^{(\nu,\kappa)}.$$

Examples

(1) $-1 < \nu < 0$ [0 : $(s^{(\nu,\kappa)}, m^{(\nu,\kappa)}, 0)$-regular]

$\mathbb{D}^{(\nu,\kappa,*)}$: the diffusion process on I with $\mathcal{G}^{(\nu,\kappa)}$
(0 being reflecting)

$n^{(\nu,\kappa,*)}$:the Lévy measure density corresponding to the inverse local time at 0 for $\mathbb{D}^{(\nu,\kappa,*)}$

Since $s^{(\nu,\kappa)}(\infty) = \infty$,

$$n^{(\nu,\kappa,*)}(\xi) = \lim_{x,y\to 0} D_{s^{(\nu,\kappa)}(x)} D_{s^{(\nu,\kappa)}(y)} p^{(\nu,\kappa)}(\xi, x, y)$$

$$= 2^{-|\nu|+1} \frac{|\nu|}{\Gamma(|\nu|)} \left(\frac{\kappa}{\sinh(\kappa\xi)} \right)^{|\nu|+1} e^{\kappa(\nu+1)\xi}.$$

Examples

(2) $-1 < \nu < 1$
[0 : $(s^{(\nu,\kappa)}, m^{(\nu,\kappa)}, 0)$-regular or -entrance, and (3) is satisfied]
For $\beta > 0$, we put

$$h(x) = \frac{\kappa^{\frac{|\nu|}{2}-\frac{1}{2}}}{2}\Gamma\left(\frac{|\nu|}{2} - \frac{\nu}{2} + \frac{\beta}{2\kappa}\right) x^{-\nu-1} e^{\frac{\kappa x^2}{2}} W_{-\frac{\beta}{2\kappa}+\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x^2).$$

Then $h(x) \in \mathcal{H}_{s^{(\nu,\kappa)},m^{(\nu,\kappa)},0,\beta}$ and

$$\mathcal{G}_h^{(\nu,\kappa)} = \frac{1}{2}\frac{d^2}{dx^2} + \left\{-\frac{1}{2x} + 2\kappa x \frac{W'_{-\frac{\beta}{2\kappa}+\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x^2)}{W_{-\frac{\beta}{2\kappa}+\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x^2)}\right\}\frac{d}{dx},$$

$$ds_h^{(\nu,\kappa,)}(x) = h(x)^{-2}\, ds^{(\nu,\kappa)}(x), \quad dm_h^{(\nu,\kappa,)}(x) = h(x)^2\, dm^{(\nu,\kappa)}(x).$$

Examples

The end point 0 is $(s_h^{(\nu,\kappa)}, m_h^{(\nu,\kappa)}, 0)$-regular and $s_h^{(\nu,\kappa)}(\infty) = \infty$. We consider the diffusion process $\mathbb{D}_h^{(\nu,\kappa,*)}$ with $\mathcal{G}_h^{(\nu,\kappa)}$ as the generator and with the end point 0 being reflecting. Let $n_h^{(\nu,\kappa,*)}$ be the Lévy measure density corresponding to the inverse local time at 0 for $\mathbb{D}_h^{(\nu,\kappa,*)}$.

$$n_h^{(\nu,\kappa,*)}(\xi) = 2^{-|\nu|-1}\Gamma(|\nu|+1)\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1} e^{\{\kappa(\nu+1)-\beta\}\xi}.$$

Examples

We finally consider the special case $\beta = \kappa(\nu+1) > 0$. Then $\mathcal{G}_h^{(\nu,\kappa)}$ is reduced to

$$\begin{aligned}\mathcal{G}_h^{(\nu,\kappa)} =& \frac{1}{2}\frac{d^2}{dx^2} + \left\{ -\frac{1}{2x} + 2\kappa x \frac{W'_{0,\frac{|\nu|}{2}}(\kappa x^2)}{W_{0,\frac{|\nu|}{2}}(\kappa x^2)} \right\} \frac{d}{dx} \\ =& \frac{1}{2}\frac{d^2}{dx^2} + \left\{ \frac{1}{2x} + \kappa x \frac{K'_{|\nu|/2}(\kappa x^2/2)}{K_{|\nu|/2}(\kappa x^2/2)} \right\} \frac{d}{dx},\end{aligned}$$

Lévy measure density corresponding to the inverse local time at 0 for $\mathbb{D}_h^{(\nu,\kappa,*)}$ is given by

$$n_h^{(\nu,\kappa,*)}(\xi) = 2^{-|\nu|-1}\Gamma(|\nu|+1)\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1}.$$