Lévy measure density corresponding to inverse local time

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2012. 9.27
We are concerned with Lévy measure density corresponding to the inverse local time at the regular end point for harmonic transform of a one dimensional diffusion process. We show that the Lévy measure density is represented as a Laplace transform of the spectral measure corresponding to an original diffusion process, where the absorbing boundary condition is posed at the end point if it is regular.

\[ \mathbb{D}_{s,m,k} \leftrightarrow \mathbb{D}_{s,h,m_h,0} \leftrightarrow \mathbb{D}_{s,h,m_h,0} \]

absorbing \hspace{1cm} absorbing \hspace{1cm} reflecting

\[ n^*(\xi) \]
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One dimensional diffusion process

We set

$s : \text{continuous increasing fnc. on } I = (l_1, l_2), -\infty \leq l_1 < l_2 \leq \infty$

$m : \text{right continuous increasing fnc. on } I$

$k : \text{right continuous nondecreasing fnc. on } I$
One dimensional diffusion process

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  $k: \text{right continuous nondecreasing fnc. on } I$

- $G_{s,m,k}: 1\text{-dim diffusion operator with } s, m, \text{ and } k$

  \[ G_{s,m,k} u = \frac{dD_s u - u dk}{dm} \]
One dimensional diffusion process

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\( s : \) continuous increasing fnc. on \( I = (l_1, l_2), -\infty \leq l_1 < l_2 \leq \infty \)
\( m : \) right continuous increasing fnc. on \( I \)
\( k : \) right continuous nondecreasing fnc. on \( I \)

- \( \mathcal{G}_{s,m,k} : \) 1-dim diffusion operator with \( s, m, \) and \( k \)

\[
\mathcal{G}_{s,m,k} u = \frac{dD_s u - udk}{dm}
\]

- \( \mathcal{D}_{s,m,k} : \) 1-dim diffusion process with \( \mathcal{G}_{s,m,k} \)
  \( [l_1 \) is absorbing if \( l_1 \) is regular \]
One dimensional diffusion process

- \( p(t, x, y) \): transition probability w.r.t. \( dm \) for \( D_{s,m,k} \)

If \( l_1 \) is \((s, m, k)\)-regular,

\[
p(t, x, y) = \int_{[0, \infty)} e^{-\lambda t} \psi_0(x, \lambda) \psi_0(y, \lambda) \, d\sigma(\lambda), \quad t > 0, \ x, y \in I,
\]

(1)

where \( d\sigma(\lambda) \) is a Borel measure on \([0, \infty)\) satisfying

\[
\int_{[0, \infty)} e^{-\lambda t} \, d\sigma(\lambda) < \infty, \quad t > 0,
\]

(2)

and \( \psi_0(x, \lambda), \ x \in I, \ \lambda \geq 0 \), is the solution of the following integral equation

\[
\psi_0(x, \lambda) = s(x) - s(l_1) + \int_{(l_1, x]} \{s(x) - s(y)\} \psi_0(y, \lambda) \{-\lambda \, dm(y) + dk(y)\}
\]
One dimensional diffusion process

Proposition 2.1

Assume that $l_1$ is $(s, m, k)$-entrance and

$$
\int_{(l_1, c_0]} \{s(c_0) - s(x)\}^2 \, dm(x) < \infty. \quad (3)
$$

Then $p(t, x, y)$ is represented as (1) with $d\sigma(\lambda)$ satisfying (2) and $
\psi_o(x, \lambda)$ is the solution of the integral equation

$$
\psi_o(x, \lambda) = 1 + \int_{(l_1, x]} \{s(x) - s(y)\} \psi_o(y, \lambda) \{-\lambda \, dm(y) + dk(y)\}.
$$
Harmonic transform

- We set

\[ \mathcal{H}_{s,m,k,\beta} = \{ h > 0; \ G_{s,m,k} h = \beta h \}, \quad \text{for } \beta \geq 0 \]

For \( h \in \mathcal{H}_{s,m,k,\beta} \),

\[ ds_h(x) = h(x)^{-2} ds(x), \quad dm_h(x) = h(x)^2 dm(x) \]
Harmonic transform

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We obtain

$$\mathcal{G}_{s_h,m_h,0} : h \text{ transform of } \mathcal{G}_{s,m,k}$$

$$p_h(t,x,y) = e^{-\beta t} \frac{p(t,x,y)}{h(x) h(y)}$$
Harmonic transform

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- \( \mathbb{D}_{s_h,m_h,0} : 1\text{-dim diffusion process with } \mathcal{G}_{s_h,m_h,0} \)

\[ [l_1 \text{ is absorbing if } l_1 \text{ is regular }] \]
Harmonic transform

- $\mathcal{D}^*_{s_h,m,h,0}$: 1-dim diffusion process with $\mathcal{G}_{s_h,m,h,0}$
  
  $[l_1$ is regular and reflecting boundary $]$
Harmonic transform

- $\mathbb{D}^*_{s_h,m_h,0}$: 1-dim diffusion process with $\mathcal{G}_{s_h,m_h,0}$
  
  $[l_1$ is regular and reflecting boundary $]$  

- $l^{(h*)}(t, \xi)$: local time for $\mathbb{D}^*_{s_h,m_h,0}$, that is,

\[
\int_0^t f(X(u)) \, du = \int_{l_1} l^{(h*)}(t, \xi) \, dm_h(\xi), \quad t > 0,
\]

for bounded continuous functions $f$ on $l_1$. 

Harmonic transform

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  [$l_1$ is regular and reflecting boundary]

- $l^{(h^*)}(t, \xi)$: local time for $\mathbb{D}^*_{s_h, m_h, 0}$, that is,
  
  $$\int_0^t f(X(u)) \, du = \int_{l_1} l^{(h^*)}(t, \xi) \, dm_h(\xi), \quad t > 0,$$

  for bounded continuous functions $f$ on $l$.

- $\tau^{(h^*)}(t)$: inverse local time $l^{(h^*)^{-1}}(t, l_1)$ at the end point $l_1$. 
Lévy measure density

**Proposition 2.2 (Itô and McKean)**

Assume the following conditions.

\[ l_1 \text{ is } (s, m, 0)-\text{regular and reflecting}, \ s(l_2) = \infty. \]

Then \([\tau^*(t), \ t \geq 0] \text{ is a Lévy process and there is a Lévy measure density } n^*(\xi) \text{ such that}\]

\[
E_{l_1}^* \left[ e^{-\lambda \tau^*(t)} \right] = \exp \left\{ -t \int_0^\infty (1 - e^{-\lambda \xi}) n^*(\xi) \, d\xi \right\}
\]

where \( E_{l_1}^* \) stands for the expectation with respect to \( P_{l_1}^* \),

\[
n^*(\xi) = \lim_{x, y \rightarrow l_1} D_s(x) D_s(y) p(\xi, x, y) = \int_{[0, \infty)} e^{-\lambda \xi} \, d\sigma(\lambda),
\]

where \( p(t, x, y) \) is the transition probability density for \( \mathbb{D}_{s, m, 0} \), and \( d\sigma(\lambda) \) is the Borel measure appeared in (1) satisfying (2).
Main theorem

Now we give a representation of $n^{(h^*)}(\xi)$ by means of items corresponding to the diffusion process $\mathcal{D}_{s,m,k}$. $l_1$ is $(s_h, m_h, 0)$-regular if and only if one of the following conditions is satisfied.

1. $l_1$ is $(s, m, k)$-regular and $h(l_1) \in (0, \infty)$.  \hspace{1cm} (4)
2. $l_1$ is $(s, m, k)$-entrance, $h(l_1) = \infty$, and $|m_h(l_1)| < \infty$.  \hspace{1cm} (5)
3. $l_1$ is $(s, m, k)$-natural, $h(l_1) = \infty$, and $|m_h(l_1)| < \infty$.  \hspace{1cm} (6)
Main theorem

Theorem 2.3
Let $h \in \mathcal{H}_{s,m,k,\beta}$. Assume one of (4), (5), and (6). Further assume that $l_1$ is reflecting and $s_h(l_2) = \infty$. Then there exists Lévy measure density $n^{(h^*)}(\xi)$. In particular, if (4) is satisfied, then

$$
n^{(h^*)}(\xi) = h(l_1)^2 e^{-\beta \xi} \int_{[0,\infty)} e^{-\xi \lambda} d\sigma(\lambda)
$$

$$
= h(l_1)^2 e^{-\beta \xi} \lim_{x,y \to l_1} D_s(x) D_s(y) p(\xi, x, y).
$$

If (5) is satisfied, then

$$
n^{(h^*)}(\xi) = D_s h(l_1)^2 e^{-\beta \xi} \int_{[0,\infty)} e^{-\xi \lambda} d\sigma(\lambda)
$$

$$
= D_s h(l_1)^2 e^{-\beta \xi} \lim_{x,y \to l_1} p(\xi, x, y).
$$
Example 2.4 (Bessel process)

Let us consider the following diffusion operator \( G(\nu) \) on \( I = (0, \infty) \).

\[
G(\nu) = \frac{1}{2} \frac{d^2}{dx^2} + \frac{2\nu + 1}{2x} \frac{d}{dx},
\]

where \(-\infty < \nu < \infty\).

\[
ds^{(\nu)}(x) = x^{-2\nu-1} \, dx, \quad dm^{(\nu)}(x) = 2x^{2\nu+1} \, dx.
\]

The killing measure is null. The state of the end point 0 depends on \( \nu \), that is,

- it is \((s^{(\nu)}, m^{(\nu)}, 0)\)-entrance if \( \nu \geq 0 \),
- it is \((s^{(\nu)}, m^{(\nu)}, 0)\)-regular if \(-1 < \nu < 0\),
- it is \((s^{(\nu)}, m^{(\nu)}, 0)\)-exit if \( \nu \leq -1 \).
Examples

Further

\[ \int_0^1 \left\{ s^{(\nu)}(1) - s^{(\nu)}(x) \right\}^2 \, dm^{(\nu)}(x) < \infty \iff |\nu| < 1. \]

The end point \( \infty \) is \((s^{(\nu)}, m^{(\nu)}, 0)\)-natural for all \( \nu \), and in particular,

\[ s^{(\nu)}(\infty) = \infty \iff \nu \leq 0. \]

Let

\[ \mathbb{D}^{(\nu)} : \text{the diffusion process on } I \text{ with } G^{(\nu)} \]

\(( 0 \text{ being absorbing if } -1 < \nu < 0)\)

\[ p^{(\nu)}(t, x, y) : \text{the transition probability density w.r.t. } dm^{(\nu)}. \]
Examples

(1) $-1 < \nu < 0 \ [0 : (s^{(\nu)}, m^{(\nu)}, 0)$-regular ]

$D^{(\nu,*)}$ : the diffusion process on $I$ with $G^{(\nu)}$

( 0 being reflecting)

$n^{(\nu,*)}$ : the Lévy measure density corresponding to the inverse local time at 0 for $D^{(\nu,*)}$

Since $s^{(\nu)}(\infty) = \infty$,

$$n^{(\nu,*)}(\xi) = \lim_{x,y \to 0} D_s^{(\nu)}(x) D_s^{(\nu)}(y) p^{(\nu)}(\xi, x, y)$$

$$= \int_0^{\infty} e^{-\xi \lambda} \sigma^{(\nu)}(\lambda) d\lambda = 2^{-|\nu|+1} \frac{|\nu|}{\Gamma(|\nu|)} \xi^{-(|\nu|+1)}.$$
Examples

(2) \(-1 < \nu < 1.\)
[ 0 : \((s^{(\nu)}, m^{(\nu)}, 0)\)-regular or -entrance, and (3) is satisfied ]

For \(\beta > 0\), we put

\[
h(x) = \left( \frac{\beta}{2} \right)^{1/2} x^{-\nu} K_{|\nu|}(\sqrt{2\beta}x) \]

Then \(h(x) \in \mathcal{H}_{s^{(\nu)}, m^{(\nu)}, 0, \beta}\) and

\[
G^{(\nu)}_h = \frac{1}{2} \frac{d^2}{dx^2} + \left\{ \frac{1}{2x} + \sqrt{2\beta} \frac{K'_{\nu}(\sqrt{2\beta}x)}{K_{\nu}(\sqrt{2\beta}x)} \right\} \frac{d}{dx},
\]

\[
ds^{(\nu, \beta)}(x) = h(x)^{-2} ds^{(\nu)}(x), \quad dm^{(\nu, \beta)}(x) = h(x)^2 dm^{(\nu)}(x).
\]
Examples

The end point 0 is \((s^{(ν,β)}, m^{(ν,β)}, 0)\)-regular. We consider the diffusion process \(\mathbb{D}_{h}^{(ν,*)}\) with \(G_{h}^{(ν)}\) as the generator and with the end point 0 being reflecting. Let \(n_{h}^{(ν,*)}\) be the Lévy measure density corresponding to the inverse local time at 0 for \(\mathbb{D}_{h}^{(ν,*)}\).

\[
n_{h}^{(ν,*)} = 2^{-|ν|} Γ(|ν| + 1) ξ^{-(|ν|+1)} e^{-βξ}.
\]

(3) \(0 < ν < 1\)

We put

\[
h^{(0)}(x) = \{s^{(ν)}(∞) − s^{(ν)}(x)\}/\{s^{(ν)}(∞) − s^{(ν)}(1)\} = x^{-2ν}.
\]

Denote by \(G_{h}^{(ν,0)}\) the harmonic transform of \(G^{(ν)}\) based on \(h^{(0)} \in \mathcal{H}_{s^{(ν)}, m^{(ν)}, 0,0}\), that is,

\[
G_{h}^{(ν,0)} = \frac{1}{2} \frac{d^2}{dx^2} + \frac{-2ν + 1}{2x} \frac{d}{dx}.
\]
Examples

\[ ds^{(\nu,0)}(x) = h^{(0)}(x)^{-2} \quad ds^{(\nu)}(x) = x^{2\nu-1} \, dx, \]
\[ dm^{(\nu,0)}(x) = h^{(0)}(x)^2 \quad dm^{(\nu)}(x) = 2x^{-2\nu+1} \, dx. \]

The end point 0 is \((s^{(\nu,0)}, m^{(\nu,0)}, 0)\)-regular. We consider the diffusion process \(\mathbb{D}^{(\nu,0,*)}_h\) with \(G^{(\nu,0)}_h\) as the generator and with the end point 0 being reflecting. Let \(n^{(\nu,0,*)}_h\) be the Lévy measure density corresponding to the inverse local time at 0 for \(\mathbb{D}^{(\nu,0,*)}_h\).

\[ n^{(\nu,0,*)}_h = 2^{-\nu+1} \frac{\nu}{\Gamma(\nu)} \xi^{-\nu-1}. \]
Example 2.5 (Radial Ornstein-Uhlenbeck process)

Let us consider the following diffusion operator \( G^{(\nu,\kappa)} \) on \( I = (0, \infty) \).

\[
G^{(\nu,\kappa)} = \frac{1}{2} \frac{d^2}{dx^2} + \left( \frac{2\nu + 1}{2x} - \kappa x \right) \frac{d}{dx},
\]

where \(-\infty < \nu < \infty\) and \(\kappa > 0\).

\[
ds^{(\nu,\kappa)}(x) = x^{-2\nu-1} e^{\kappa x^2} dx, \quad dm^{(\nu,\kappa)}(x) = 2x^{2\nu+1} e^{-\kappa x^2} dx.
\]

The killing measure is null. The state of the end point 0 depends on \(\nu\), that is,

- it is \((s^{(\nu,\kappa)}, m^{(\nu,\kappa)}, 0)\)-entrance if \(\nu \geq 0\),
- it is \((s^{(\nu,\kappa)}, m^{(\nu,\kappa)}, 0)\)-regular if \(-1 < \nu < 0\),
- it is \((s^{(\nu,\kappa)}, m^{(\nu,\kappa)}, 0)\)-exit if \(\nu \leq -1\).
Examples

Further

\[ \int_0^1 \left\{ s^{(\nu,\kappa)}(1) - s^{(\nu,\kappa)}(x) \right\}^2 dm^{(\nu,\kappa)}(x) < \infty \iff |\nu| < 1. \]

The end point \( \infty \) is always \((s^{(\nu,\kappa)}, m^{(\nu,\kappa)}, 0)\)-natural for all \( \nu \), and

\[ s^{(\nu,\kappa)}(\infty) = \infty. \]

Let

\[ \mathbb{D}^{(\nu,\kappa)} : \text{the diffusion process on } I \text{ with } \mathcal{G}^{(\nu,\kappa)} \]

(0 being absorbing if \(-1 < \nu < 0\))

\[ p^{(\nu,\kappa)}(t, x, y) : \text{the transition probability density w.r.t. } dm^{(\nu,\kappa)}. \]
Examples

(1) $-1 < \nu < 0$ \[ 0 : (s^{(\nu,\kappa)}, m^{(\nu,\kappa)}, 0)\text{-regular} \]

$\mathbb{D}^{(\nu,\kappa,\ast)}$ : the diffusion process on $I$ with $G^{(\nu,\kappa)}$

(0 being reflecting)

$n^{(\nu,\kappa,\ast)}$ : the Lévy measure density corresponding to

the inverse local time at 0 for $\mathbb{D}^{(\nu,\kappa,\ast)}$

Since $s^{(\nu,\kappa)}(\infty) = \infty$

$$n^{(\nu,\kappa,\ast)}(\xi) = \lim_{x,y \rightarrow 0} D_s^{(\nu,\kappa)}(x) D_s^{(\nu,\kappa)}(y) p^{(\nu,\kappa)}(\xi, x, y)$$

$$= 2^{-|\nu|+1} \frac{|\nu|}{\Gamma(|\nu|)} \left( \frac{\kappa}{\sinh(\kappa \xi)} \right)^{|\nu|+1} e^{\kappa(\nu+1)\xi}. $$
Examples

(2) \(-1 < \nu < 1\) 
\[ 0 : (s^{(\nu,\kappa)}, m^{(\nu,\kappa)}, 0)\text{-regular or -entrance, and (3) is satisfied} \]

For \(\beta > 0\), we put

\[ h(x) = \kappa^{\frac{|\nu|}{2}} \frac{1}{2} x^{-\nu-1} e^{\frac{\kappa x^2}{2}} W^{\frac{\beta}{2\kappa} + \frac{\nu + 1}{2}, \frac{|\nu|}{2}}(\kappa x^2). \]

Then \(h(x) \in \mathcal{H}_{s^{(\nu,\kappa)}, m^{(\nu,\kappa)}, 0, \beta}\) and

\[ G^{(\nu,\kappa)}_h = -\frac{1}{2} \frac{d^2}{dx^2} + \left\{ -\frac{1}{2} + 2\kappa x \frac{W^{\frac{\beta}{2\kappa} + \frac{\nu + 1}{2}, \frac{|\nu|}{2}}(\kappa x^2)}{W^{\frac{\beta}{2\kappa} + \frac{\nu + 1}{2}, \frac{|\nu|}{2}}(\kappa x^2)} \right\} \frac{d}{dx}, \]

\[ ds^{(\nu,\kappa)}_h(x) = h(x)^{-2} ds^{(\nu,\kappa)}(x), \quad dm^{(\nu,\kappa)}_h(x) = h(x)^2 dm^{(\nu,\kappa)}(x). \]
Examples

The end point 0 is \((s_h^{(\nu,\kappa), m_h^{(\nu,\kappa)}, 0})\)-regular and \(s_h^{(\nu,\kappa)}(\infty) = \infty\).

We consider the diffusion process \(D_h^{(\nu,\kappa,*)}\) with \(G_h^{(\nu,\kappa)}\) as the generator and with the end point 0 being reflecting. Let \(n_h^{(\nu,\kappa,*)}\) be the Lévy measure density corresponding to the inverse local time at 0 for \(D_h^{(\nu,\kappa,*)}\).

\[
n_h^{(\nu,\kappa,*)}(\xi) = 2^{-|\nu|-1}\Gamma(|\nu| + 1) \left( \frac{\kappa}{\sinh(\kappa \xi)} \right)^{|\nu|+1} e^{\{\kappa(\nu+1)-\beta\} \xi}.
\]
Examples

We finally consider the special case $\beta = \kappa(\nu + 1) > 0$. Then $G_h^{(\nu,\kappa)}$ is reduced to

$$G_h^{(\nu,\kappa)} = \frac{1}{2} \frac{d^2}{dx^2} + \left\{ -\frac{1}{2x} + 2\kappa x \frac{W'_{0,|\nu|/2}(\kappa x^2)}{W_{0,|\nu|/2}(\kappa x^2)} \right\} \frac{d}{dx}$$

$$= \frac{1}{2} \frac{d^2}{dx^2} + \left\{ \frac{1}{2x} + \kappa x \frac{K'_{|\nu|/2}(\kappa x^2/2)}{K_{|\nu|/2}(\kappa x^2/2)} \right\} \frac{d}{dx},$$

Lévy measure density corresponding to the inverse local time at 0 for $D_h^{(\nu,\kappa,*)}$ is given by

$$n_h^{(\nu,\kappa,*)}(\xi) = 2^{-|\nu|-1} \Gamma(|\nu| + 1) \left( \frac{\kappa}{\sinh(\kappa \xi)} \right)^{|\nu|+1}.$$