Stochastic variational inequalities and applications to the total variation flow perturbed by linear multiplicative noise

Michael Röckner (University of Bielefeld)

joint work with

Viorel Barbu (Romanian Academy, Iasi)

Contents

1. Introduction and framework

Consider the nonlinear diffusion equation

$$
\begin{aligned}
dX(t) &= \mathrm{div}\Big[\mathrm{sign}\,(\nabla X(t))\Big]dt + X(t)dW(t) \text{ on } (0,T)\times\mathcal{O},\\
X &= 0 \text{ on } (0,T)\times\partial\mathcal{O}, \qquad\qquad \text{(SPDE)}\\
X(0) &= x \in L^2(\mathcal{O}),
\end{aligned}
$$

where $T > 0$ is arbitrary and $\mathcal{O} :=$ bounded, convex, open set in $\mathbb{R}^N$, $\partial\mathcal{O}$ smooth;

$$W(t,\xi) := \sum_{k=1}^{\infty} \mu_k e_k(\xi)\beta_k(t),\ (t,\xi) \in (0,\infty)\times\mathcal{O} \text{ with } \mu_k \in \mathbb{R},\ \beta_k,\ k \in \mathbb{N}, \text{ independent}$$

BM's on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and e_k, $k \in \mathbb{N}$, eigenbasis

of Dirichlet Laplacian Δ_D on $\mathcal{O}$. Furthermore, $\mathrm{sign}: \mathbb{R}^N \to 2^{\mathbb{R}^N}$ (multi-valued!)

$$\mathrm{sign}\, u := \begin{cases} \frac{u}{|u|}, & \text{if } u \in \mathbb{R}^N\setminus\{0\},\\ \{u \in \mathbb{R}^N : |u| \le 1\}, & \text{if } u = 0 \in \mathbb{R}^N. \end{cases}$$

Standing assumption:

(H1) $C_\infty^2 := \sum_{k=1}^{\infty} \mu_k^2 |e_k|_\infty^2 < \infty$ and $\sum_{k=1}^{\infty} \mu_k |\nabla e_k|_\infty < \infty$.

Set

$$\mu(\xi) := \sum_{k=1}^{\infty} \mu_k^2 e_k^2(\xi),$$

i.e. $\langle W(\cdot, \xi)\rangle_t = \mu(\xi) \cdot t,\ t \geq 0,\ \xi \in \mathcal{O}$.

Modelling:

(i) In nonlinear diffusion theory, (SPDE) is derived from the continuity equation perturbed by a Gaussian process proportional to the density $X(t)$ of the material, that is,

$$dX(t) = \operatorname{div} J(\nabla X(t))dt + X(t)dW(t),$$

where $J = \operatorname{sgn}$ is the flux of the diffusing material. (See [Y. Giga, R. Kobayashi 2003], [M.H. Giga, Y. Giga 2001], [Y. Giga, R.V. Kohn 2011].)

(ii) (SPDE) is also relevant as a mathematical model for faceted crystal growth under a stochastic perturbation as well as in material sciences (see [R. Kobayashi, Y. Giga 1999] for the deterministic model and complete references on the subject). As a matter of fact, these models are based on differential gradient systems corresponding to a convex and nondifferentiable potential (energy).

(iii) Other recent applications refer to the PDE approach to image recovery (see, e.g., [A. Chamballe, P.L. Lions 1997] and also [T. Barbu, V. Barbu, V. Biga, D. Coca 2009], [T. Chan, S. Esedogly, F. Park, A. Yip 2006]). In fact, if $x \in L^2(\mathcal{O})$ is the blurred image, one might find the restored image via the total variation flow $X = X(t)$ generated by the stochastic equation

$$\begin{aligned} &dX(t) = \operatorname{div}\left(\frac{\nabla X(t)}{|\nabla X(t)|}\right) dt + X(t)dW(t) \quad \text{in } (0,T)\times\mathcal{O}, \\ &X(0) = x \text{ in } \mathcal{O}. \end{aligned} \tag{SPDE'}$$

In its deterministic form, this is the so-called *total variation based image restoration model* and its stochastic version (SPDE') arises naturally in this context as a perturbation of the *total variation flow* by a Gaussian (Wiener) noise (which explains the title of the talk).

In [V. Barbu, G. Da Prato, M. R., SIAM 2009], a complete existence and uniqueness result was proved for variational solutions to (SPDE) in the case of additive noise, that is,

$$dX(t) - \mathrm{div}[\mathrm{sgn}(\nabla X(t))]dt = dW(t) \quad \text{in } (0,T)\times\mathcal{O},$$
$$X(0) = x \text{ in } \mathcal{O}, \qquad X(t) = 0 \text{ on } (0,T)\times\partial\mathcal{O},$$

if $1 \leq N \leq 2$. For the multiplicative noise $X(t)dW(t)$, only the existence of a variational solution was proved and uniqueness remained open. (See, however, the work [B. Gess, J.M. Tölle, arXiv 2011] for recent results on this line, if $x \in H_0^1(\mathcal{O})$.)

1. Introduction and framework

In [V. Barbu, M. R., ArXiv 2012], we prove the existence and uniqueness of variational solutions to (SPDE) in all dimensions $N \geq 1$ and all initial conditions $x \in L^2(\mathcal{O})$. We would like to stress that one main difficulty is when $x \in L^2(\mathcal{O}) \setminus H_0^1(\mathcal{O})$, while the case $x \in H_0^1(\mathcal{O})$ is more standard. Furthermore, we prove the finite-time extinction of solutions with positive probability, if $N \leq 3$, generalizing corresponding results from [F. Andreu, V. Caselles, J. Díaz, J. Mazón, JFA 2002] and [F. Andreu-Vaillo, V. Caselles, J.M. Mazón, Birkhäuser 2004] obtained in the deterministic case.

Notation

$$L^p(\mathcal{O}) := \text{standard } L^p\text{-spaces with norm } |\cdot|_p,\ p \in [1,\infty]$$

$$W^{1,p}_{(0)}\mathcal{O} := \text{standard (Dirichlet) Sobolev spaces in } L^p(\mathcal{O}),\ p \in [1,\infty)$$

with norm

$$\|u\|_{1,p} := \left(\int_{\mathcal{O}} |\nabla u|^p d\xi\right)^{1/p} \quad (d\xi = \text{ Lebesgue measure on } \mathcal{O})$$

$$H^1_0(\mathcal{O}) := W^{1,2}_0(\mathcal{O}), \quad H^2(\mathcal{O}) := W^{2,2}(\mathcal{O}).$$

$$BV(\mathcal{O}) := \text{space of functions } u : \mathcal{O} \to \mathbb{R} \text{ with bounded variation}$$

$$\|Du\| := \sup\left\{\int_{\mathcal{O}} u \,\mathrm{div}\varphi \, d\xi : \varphi \in C_0^\infty(\mathcal{O};\mathbb{R}^N),\ |\varphi|_\infty \le 1\right\}$$

$$\left(= \int_{\mathcal{O}} |\nabla u| d\xi, \text{ if } u \in W^{1,1}(\mathcal{O})\right).$$

$$BV^0(\mathcal{O}) := \text{all } u \in BV(\mathcal{O}) \text{ vanishing on } \partial\mathcal{O}.$$

$-\mathbf{div}\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|}$ as subdifferential of $\|\mathbf{Du}\|$:
Consider $\phi_0 : L^1(\mathcal{O}) \to \overline{\mathbb{R}} = (-\infty, +\infty]$

$$\phi_0(u) := \begin{cases} \|Du\| & \text{if } \; u \in BV^0(\mathcal{O}), \\ +\infty & \text{otherwise,} \end{cases}$$

and let $\mathrm{cl}\,\phi_0$ denote the lower semicontinuous closure of ϕ_0 in $L^1(\mathcal{O})$, that is,

$$\mathrm{cl}\,\phi_0(u) := \inf\left\{\liminf \phi_0(u_n);\; u_n \to u \in L^1(\mathcal{O})\right\}.$$

Hence, by [H. Attouch, G. Buttazzo, M. Gerard 2006], for $u \in L^1(\mathcal{O})$,

$$\operatorname{cl}\phi_0(u) = \begin{cases} \|Du\| + \int_{\partial\mathcal{O}} |\gamma_0(u)| d\mathcal{H}^{N-1} & \text{if } u \in BV(\mathcal{O}), \\ +\infty & \text{otherwise}, \end{cases}$$

where $\gamma_0(u)$ is the trace of u on the boundary and $d\mathcal{H}^{N-1}$ is the Hausdorff measure. Let ϕ denote the restriction of $\operatorname{cl}\phi_0(u)$ to $L^2(\mathcal{O})$, i.e.,

$$\phi(u) := \begin{cases} \|Du\| + \int_{\partial\mathcal{O}} |\gamma_0(u)| d\mathcal{H}^{N-1}, & \text{if } u \in BV(\mathcal{O}) \cap L^2(\mathcal{O}), \\ +\infty, & \text{if } u \in L^2(\mathcal{O}) \setminus BV(\mathcal{O}). \end{cases}$$

Note that (as in the deterministic case) the initial Dirichlet boundary condition is lost during this procedure as a price for getting ϕ to be lower semicontinuous on $L^1(\mathcal{O})$. Hence in (SPDE) the boundary condition only holds in this generalized sense.

1. Introduction and framework

By $\partial\phi : D(\partial\phi) \subset L^2(\mathcal{O}) \to L^2(\mathcal{O})$, we denote the subdifferential of ϕ, that is,

$$\partial\phi(u) := \{\eta \in L^2(\mathcal{O});\ \phi(u) - \phi(v) \le \langle \eta, u - v\rangle,\ \forall v \in D(\phi)\},$$

where

$$D(\phi) := \{u \in L^2(\mathcal{O});\ \phi(u) < \infty\} = BV(\mathcal{O}) \cap L^2(\mathcal{O}).$$

Then it turns out that

$$\partial\phi(u) := \{-\mathrm{div}\, z | z \in L^\infty(\mathcal{O};\ \mathbb{R}^N),\ |z|_\infty \le 1,\ \langle z,\ \underbrace{\nabla u}_{\text{measure!}}\ \rangle = \phi(u)\}$$

(where div and pairing $\langle\ ,\ \rangle$ in sense of Schwartz distributions).
Heuristically,

$$\text{“}\int_{\mathcal{O}} |\nabla u| d\xi\text{”} \overset{\text{int. by parts}}{=} \text{“}\int_{\mathcal{O}} \left(-\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right) u\, d\xi\text{”}.$$

Rigorously, for $-\mathrm{div}\, z \in \partial\phi(u)$

$$\phi(u) = \langle z,\ \underbrace{\nabla u}_{\text{measure!}}\ \rangle \overset{\text{Def. of div}}{=} \int_{\mathcal{O}} (-\mathrm{div}\ \underbrace{z}\ ) u\, d\xi.$$

$\xi \mapsto z(\xi)$ is section of

$\xi \mapsto \dfrac{\nabla u}{|\nabla u|} = \mathrm{sign}\,(\nabla u)$ (multi-valued!)

Hence (again) we can rewrite (SPDE) as

$$\begin{aligned} & dX(t) + \partial\phi(X(t))dt \ni X(t)dW(t), \quad t \in [0, T], \\ & X(0) = x \in L^2(\mathcal{O}). \end{aligned} \tag{SPDE''}$$

However, since the multi-valued mapping $\partial\phi : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ is highly singular, at present for arbitrary initial conditions $x \in L^2(\mathcal{O})$ no general existence result for stochastic infinite dimensional equations of subgradient type is applicable to the present situation. Our approach is to rewrite (SPDE") (hence (SPDE), (SPDE')) as a stochastic variational inequality (SVI).

2. Definition of (solutions to) SVI and the main existence and uniqueness result

Definition 1

Let $0 < T < \infty$ and let $x \in L^2(\mathcal{O})$. A stochastic process $X : [0, T] \times \Omega \to L^2(\mathcal{O})$ is said to be a **variational solution** to (SPDE) if the following conditions hold.

(i) X is $(\mathcal{F}_t)$-adapted, has $\mathbb{P}$-a.s. continuous sample paths in $L^2(\mathcal{O})$ and $X(0) = x$.

(ii) $X \in L^2([0, T] \times \Omega; L^2(\mathcal{O}))$, $\phi(X) \in L^1([0, T] \times \Omega)$.

(iii) For each $(\mathcal{F}_t)$- progressively measurable process $G \in L^2([0, T] \times \Omega; L^2(\mathcal{O}))$ and each $(\mathcal{F}_t)$-adapted $L^2(\mathcal{O})$-valued process Z with $\mathbb{P}$-a.s. continuous sample paths such that $Z \in L^2([0, T] \times \Omega; H_0^1(\mathcal{O}))$ and, solving the equation

$$Z(t) - Z(0) + \int_0^t G(s)ds = \int_0^t Z(s)dW(s), \quad t \in [0, T],$$

we have

$$\frac{1}{2}\,\mathbb{E}|X(t)-Z(t)|_2^2+\mathbb{E}\int_0^t\phi(X(\tau))d\tau\leq\frac{1}{2}\,\mathbb{E}|x-Z(0)|_2^2$$
$$+\mathbb{E}\int_0^t\phi(Z(\tau))d\tau+\frac{1}{2}\,\mathbb{E}\int_0^t\int_{\mathcal{O}}\mu\,(X(\tau)-Z(\tau))^2d\xi\,d\tau \tag{SVI}$$
$$+\mathbb{E}\int_0^t\langle X(\tau)-Z(\tau),G(\tau)\rangle d\tau,\ \ t\in[0,T].$$

Remark. The relationship between (SPDE) and (SVI) becomes more transparent if we recall that (SPDE) can be rewritten as (SPDE") and so we have

$$d(X - Z) + (\partial\phi(X) - G)dt \ni (X - Z)dW.$$

If we (heuristically) apply the Itô formula to $\frac{1}{2}\,|X - Z|_2^2$ and take into account the definition of $\partial\phi$, we obtain just (SVI) after taking expectation. It should be emphasized, however, that X arising in Definition 1 is not a strong solution to (SPDE) (or (SPDE")) in the standard sense, that is,

$$X(t) - x \in -\int_0^t \partial\phi(X(s))ds + \int_0^t X(s)dW(s),\ \forall t \in (0, T).$$

2. Definition of (solutions to) SVI and the main existence and uniqueness result

Theorem I

Let $\mathcal{O}$ be a bounded and convex open subset of $\mathbb{R}^N$ with smooth boundary and $T > 0$. For each $x \in L^2(\mathcal{O})$ there is a variational solution X to (SPDE), and X is the unique solution in the class of all solutions X such that, for some $\delta > 0$,

$$X \in L^{2+\delta}(\Omega; L^2([0,T]; L^2(\mathcal{O}))).$$

Furthermore, X has the following properties:

(i) $X \in L^2(\Omega; C([0,T]; L^2(\mathcal{O})))$.

(ii) $\sup_{t\in[0,T]} \mathbb{E}[|X(t)|_2^p] \leq \exp\left[C_\infty^2 \, \frac{p}{2}\,(p-1)\right] \|x\|_2^p,$ *for all* $p \in [2,\infty)$.

(iii) *Let $x, y \in L^2(\mathcal{O})$ and X^x, X^y be the corresponding variational solutions with initial conditions x, y, respectively, then, for some positive constant $C = C(N, C_\infty^2)$,*

$$\mathbb{E}\left[\sup_{\tau\in[0,T]} |X^x(\tau) - X^y(\tau)|_2^2\right] \leq 2|x-y|_2^2 e^{CT}.$$

(iv) *If $x \geq 0$, then $X(t) \geq 0 \;\forall\, t \in [0,T]$.*

(v) If $N \leq 3$, then

$$\mathbb{E}\left[\sup_{\tau\in[0,T]} |X^x(\tau) - X^y(\tau)|_N^N\right] \leq 2|x-y|_N^N e^{CT} \text{ for all } x, y \in L^N(\mathcal{O}).$$

(vi) If $x \in H_0^1(\mathcal{O})$, then for some $C > 0$ (independent of x)

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X(t)\|_{1,2}^2\right] \leq C\|x\|_{1,2}^2,$$

hence $X \in L^2(\Omega; L^\infty([0,T]; H_0^1(\mathcal{O})))$.

Remark. From (iv) one can deduce that, if the initial condition x is in $H_0^1(\mathcal{O})$, then the corresponding solution X in Theorem I is, in fact, an ordinary variational solution of the (multivalued) (SPDE) (not just in the sense of SVI as in Definition 1). Our main point is, however, here to have existence and uniqueness for all starting points $x \in L^2(\mathcal{O})$. Therefore, we skip the details on the simpler and more standard case of special initial conditions in $H_0^1(\mathcal{O})$.

3. The equivalent random PDE

Substituting $Y := e^{-W}X$ in (SPDE) and heuristically applying Itô's product rule we find that Y satisfies the following deterministic PDE with random coefficients:

$$\begin{aligned}\frac{dY}{dt} =& e^{-W(t)}\mathrm{div}\left(\mathrm{sign}(\nabla(e^{W(t)}Y(t)))\right) - \frac{1}{2}\mu Y(t) \text{ on } (0,T)\times\mathcal{O}, \\ Y =& 0 \text{ on } (0,T)\times\partial\mathcal{O} \\ Y(0) =& x \in L^2(\mathcal{O})\end{aligned} \tag{PDE}$$

Our next aim is to (rigorously!) prove that

$$\text{(SPDE)} \Leftrightarrow \text{(PDE)}$$

Definition 2

Let $0 < T < \infty$ and let $x \in L^2(\mathcal{O})$. A stochastic process $Y : [0, T] \times \Omega \to L^2(\mathcal{O})$ is said to be a *variational solution* to (PDE) if the following conditions hold:

(i) Y is $(\mathcal{F}_t)$-adapted, has $\mathbb{P}$-a.s. continuous sample paths, and $Y(0) = x$.

(ii) $e^W Y \in L^2([0, T] \times \Omega; L^2(\mathcal{O}))$, $\phi(e^W Y) \in L^1([0, T] \times \Omega)$.

(iii) For each $(\mathcal{F}_t)$-progressively measurable process $G \in L^2([0, T] \times \Omega; L^2(\mathcal{O}))$ and each $(\mathcal{F}_t)$-adapted, $L^2(\mathcal{O})$-valued process $\widetilde{Z}$ with $\mathbb{P}$-a.s. continuous sample paths such that $e^W \widetilde{Z} \in L^2([0, T] \times \Omega; H_0^1(\mathcal{O}))$ and solving the equation

$$\widetilde{Z}(t) - \widetilde{Z}(0) + \int_0^t e^{-W(s)} G(s) ds + \frac{1}{2} \int_0^t \mu \widetilde{Z}(s) ds = 0, \quad t \in [0, T], \ \mathbb{P}\text{-a.s.},$$

we have

$$\begin{aligned}
&\frac{1}{2}\,\mathbb{E}|e^{W(t)}(Y(t)-\widetilde{Z}(t))|_2^2+\mathbb{E}\int_0^t \phi(e^{W(\tau)}Y(\tau))d\tau \\
&\quad\le \frac{1}{2}\,\mathbb{E}|x-\widetilde{Z}(0)|_2^2+\mathbb{E}\int_0^t \phi(e^{W(\tau)}\widetilde{Z}(\tau))d\tau \\
&\quad+\frac{1}{2}\,\mathbb{E}\int_0^t\int_{\mathcal{O}} \mu e^{2W(\tau)}(Y(\tau)-\widetilde{Z}(\tau))^2 d\xi\, d\tau \\
&\quad+\mathbb{E}\int_0^t \langle e^{W(\tau)}(Y(\tau)-\widetilde{Z}(\tau)),G(\tau)\rangle d\tau,\ \ t\in[0,T].
\end{aligned} \tag{VI}$$

Proposition 1 $X:[0,T]\times\Omega\to L^2(\mathcal{O})$ is a variational solution to (SPDE) if and only if $Y:=e^{-W}X$ is a variational solution to (PDE).

4. Method of proof

We approximate (SPDE) by

$$\begin{aligned} &dX_\lambda = \mathrm{div}\ \widetilde{\psi}_\lambda(\nabla X_\lambda)dt + X_\lambda dW \ \text{ in } \ (0,T)\times\mathcal{O}, \\ &X_\lambda = 0 \ \text{ on } \ (0,T)\times\partial\mathcal{O}, \\ &X_\lambda(0) = x \in L^2(\mathcal{O}), \end{aligned} \qquad (\mathrm{SPDE}_\lambda)$$

and the corresponding rescaled (PDE) by

$$\begin{aligned} &\frac{dY_\lambda}{dt} = e^{-W}\mathrm{div}(\widetilde{\psi}_\lambda(\nabla(e^W Y_\lambda))) - \frac{1}{2}\mu Y_\lambda \\ &\qquad\qquad \text{in } (0,T)\times\mathcal{O}, \\ &Y_\lambda = 0 \text{ on } (0,T)\times\partial\mathcal{O}, \\ &Y_\lambda(0) = x \in L^2(\mathcal{O}), \end{aligned} \qquad (\mathrm{PDE}_\lambda)$$

where $\lambda \in (0,1]$, $\widetilde{\psi}_\lambda(u) = \psi_\lambda(u) + \lambda u,\ \forall u \in \mathbb{R}^N$. Here, ψ_λ is the Yosida approximation of the function $\psi(u) = \mathrm{sgn}\ u$, that is,

$$\psi_\lambda(u) = \begin{cases} \frac{1}{\lambda}u & \text{if } |u| \le \lambda, \\ \frac{u}{|u|} & \text{if } |u| > \lambda. \end{cases}$$

Proposition 2

(i) For each $\lambda \in (0,1]$ and each $x \in L^2(\mathcal{O})$, there is a unique strong solution X_λ to (SPDE$_\lambda$) which satisfies $X_\lambda(0) = x$, that is, X_λ is $\mathbb{P}$-a.s. continuous in $L^2(\mathcal{O})$ and $\{\mathcal{F}_t\}$-adapted such that

$$X_\lambda \in L^2([0,T] \times \Omega; H_0^1(\mathcal{O})),$$

$$X_\lambda(t) = x + \int_0^t \operatorname{div} \widetilde{\psi}_\lambda (\nabla X_\lambda(s))\, ds + \int_0^t X_\lambda(s) dW(s),$$

$$t \in [0,T], \ \mathbb{P}\text{-a.s.}$$

(ii) $Y_\lambda := e^{-W} X_\lambda$ is an $(\mathcal{F}_t)$-adapted process $Y_\lambda : [0,T] \times \Omega \to L^2(\mathcal{O})$ with $\mathbb{P}$-a.s. continuous paths which is the **unique** solution of (PDE$_\lambda$), i.e., it satisfies $\mathbb{P}$-a.s. (PDE$_\lambda$) with $Y_\lambda(0) = x$ and

$$Y_\lambda \in L^2([0,T]; H_0^1(\mathcal{O})) \cap C([0,T]; L^2(\mathcal{O})) \cap W^{1,2}([0,T]; H^{-1}(\mathcal{O})),$$

a.e. $t \in [0,T]$. Furthermore, if $x \in H_0^1(\mathcal{O})$, then

$$Y_\lambda \in C([0,T];\ H_0^1(\mathcal{O}))\ \mathbb{P}\text{-a.s.}$$

(iii) (**Crucial!**) If $x \in H_0^1(\mathcal{O})$, then $\mathbb{P}$-a.s.

$$X_\lambda \in C([0,T]; H_0^1(\mathcal{O})).$$

Furthermore, for some $C > 0$

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|X_\lambda(t)\|_{1,2}^2\Big] + \lambda \mathbb{E}\int_0^T |\Delta X_\lambda(t)|_2^2\, dt \le C\|x\|_{1,2}^2 \quad \forall x \in H_0^1(0),\ \lambda \in (0,1],$$

Proof of Proposition 2

(i) standard

(ii) Last part: deterministic maximal regularity!

First part: Below we use $\langle \cdot, \cdot \rangle_2$ to denote the inner product in $L^2(\mathcal{O})$, in order to avoid confusion with the quadratic variation process.
Let $\varphi \in H_0^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$. Then, for every $t \in [0, T]$,

$$\langle \varphi, e^{-W(t)} X_\lambda(t) \rangle_2 = \sum_{j=1}^{\infty} \langle e_j, e^{-W(t)} \varphi \rangle_2 \, \langle e_j, X_\lambda(t) \rangle_2 .$$

Furthermore, by Itô's formula, we have for all $\xi \in \mathcal{O}, t \in [0, T]$,

$$e^{-W(t,\xi)} = 1 - \int_0^t e^{-W(s,\xi)} dW(s,\xi) + \frac{1}{2} \mu(\xi) \int_0^t e^{-W(s,\xi)} ds.$$

Now, fix $j \in \mathbb{N}$. Then we have $\mathbb{P}$-a.e. that, for all $t \in [0, T]$,

$$\begin{aligned}\langle e_j, e^{-W(t)}\varphi\rangle_2 &= \langle e_j, \varphi\rangle_2 - \sum_{k=1}^{\infty} \mu_k \int_{\mathcal{O}} e_j(\xi)\varphi(\xi)e_k(\xi) \int_0^t e^{-W(s,\xi)} d\beta_k(s) d\xi \\ &\quad + \frac{1}{2}\int_0^t \langle e_j, \mu\, e^{-W(s)}\varphi\rangle_2 ds \\ &= \langle e_j, \varphi\rangle - \sum_{k=1}^{\infty} \mu_k \int_0^t \langle e_j, e_k e^{-W(s)}\varphi\rangle_2 d\beta_k(s) \\ &\quad + \frac{1}{2}\int_0^t \langle e_j, \mu\, e^{-W(s)}\varphi\rangle_2 ds,\end{aligned}$$

where we used the stochastic Fubini Theorem in the second equality and the sums converge in $L^2(\Omega; C([0, T]; \mathbb{R}))$.

By Itô's product rule we hence obtain $\mathbb{P}$-a.s. that, for all $t \in [0, T]$,

$$\begin{aligned}
\langle e_j, e^{-W(t)}\varphi\rangle_2 \, \langle e_j, X_\lambda(t)\rangle_2 &= \langle e_j, \varphi\rangle_2 \, \langle e_j, x\rangle_2 \\
&+ \int_0^t \langle e_j, e^{-W(s)}\varphi\rangle_2 \, \langle e_j, \operatorname{div} \widetilde{\psi}_\lambda(\nabla X_\lambda(s))\rangle ds \\
&+ \sum_{k=1}^\infty \mu_k \int_0^t \langle e_j, e^{-W(s)}\varphi\rangle_2 \, \langle e_j, X_\lambda(s) e_k\rangle_2 d\beta_k(s) \\
&- \sum_{k=1}^\infty \mu_k \int_0^t \langle e_j, e_k e^{-W(s)}\varphi\rangle_2 \, \langle e_j, X_\lambda(s)\rangle_2 d\beta_k(s) \\
&+ \frac{1}{2} \int_0^t \langle e_j, \mu \; e^{-W(s)}\varphi\rangle_2 \, \langle e_j, X_\lambda(s)\rangle_2 \; ds \\
&- \sum_{k=1}^\infty \mu_k^2 \int_0^t \langle e_j, X_\lambda(s) e_k\rangle_2 \, \langle e_j, e_k e^{-W(s)}\varphi\rangle_2 \; ds,
\end{aligned}$$

where all the sums converge in $L^2(\Omega; C([0,T];\mathbb{R}))$ and interchanging the infinite sums with stochastic differentials is justified.
Now, we sum the above equation from $j = 1$ to $j = \infty$ and interchange this summation both with the sum over k and with the deterministic and stochastic integrals (which is justified). Then, because the two terms involving the stochastic integrals cancel, we obtain

$$\langle \varphi, e^{-W(t)} X_\lambda(t)\rangle_2 = \langle \varphi, x\rangle_2 + \int_0^t \langle \varphi, e^{-W(s)} \mathrm{div}\, \widetilde{\psi}_\lambda(\nabla X_\lambda(s))\rangle ds$$
$$+\frac{1}{2}\int_0^t \langle \varphi, \mu\, e^{-W(s)} X_\lambda(s)\rangle_2 ds - \sum_{k=1}^{\infty} \mu_k^2 \int_0^t \langle \varphi, e_k^2 e^{-W(s)} X_\lambda(s)\rangle_2 ds,$$

which immediately implies that $Y_\lambda = e^{-W} X_\lambda$ solves (PDE$_\lambda$).

(iii): Yoshida approximation for Δ_D, via its resolvent

$$J_\varepsilon := (\mathrm{I}d - \varepsilon\, \Delta_D)^{-1},\ \varepsilon > 0,$$

and the following crucial result due to H. Brezis (private communication):

Proposition 3

(i) $$\int_{\mathcal{O}} |\nabla J_\varepsilon(u)| d\xi \leq \int_{\mathcal{O}} |\nabla u| d\xi \quad \forall\, u \in W_0^{1,1}(\mathcal{O}),\ \varepsilon > 0.$$

Hence:

(ii) $$\phi(J_\varepsilon(u)) \leq \phi(u) \quad \forall\, u \in BV(\mathcal{O}),\ \varepsilon > 0;$$

and:

(iii) For all $g : [0, \infty) \to [0, \infty)$ continuous, convex with $g(0) = 0$, of quadratic growth,

$$\int_{\mathcal{O}} g\left(|\nabla J_\varepsilon(u)|\right) d\xi \leq \int_{\mathcal{O}} g\left(|\nabla u|\right) d\xi \quad \forall\, u \in H_0^1(\mathcal{O}),\ \varepsilon > 0.$$

Remark. As (iii) one proves for $p \in [1, \infty)$

$$\int_{\mathcal{O}} |\nabla J_\varepsilon(u)|^p d\xi \leq \int_{\mathcal{O}} |\nabla u|^p d\xi \quad \forall\, u \in W_0^{1,p}(\mathcal{O}),\ \varepsilon > 0.$$

($p = \infty$ is also true and was proved already in [H. Brezis, G. Stampacchia 1968].)

Proof of Theorem I.
Existence: Prove that X_λ, $\lambda \in (0,1]$, is Cauchy in $L^2\left(\Omega;\ C([0,T];\ L^2(\mathcal{O}))\right)$
Uniqueness: Use (SVI) with

$$\tilde{Z} = J_\varepsilon(Y_\lambda)$$

where $J_\varepsilon := (\mathrm{I}d - \varepsilon\,\Delta_D)^{-1}$, and let first $\varepsilon \to 0$ and then $\lambda \to 0$
Technically **very** hard! □

Theorem II

Let $2 \leq N \leq 3$. *Let* X *be as in Theorem* I, *with initial condition* $x \in L^N(\mathcal{O})$, *and let* $\tau := \inf\{t \geq 0; |X(t)|_N = 0\}$. *Then, we have*

$$\mathbb{P}[\tau \leq t] \geq 1 - \rho^{-1} \left(\int_0^t e^{-C^* s} ds \right)^{-1} |x|_N, \ \forall t \geq 0.$$

Here $\rho := \inf\{|y|_{W_0^{1,1}(\mathcal{O})} / |y|_{\frac{N}{N-1}}; y \in W_0^{1,1}(\mathcal{O})\}$ *and* $C^* := \frac{C_\infty^2}{2}(N-1)$. *In particular, if* $|x|_N < \rho / C^*$, *then* $\mathbb{P}[\tau < \infty] > 0$.

Remark. The case $N = 1$ is similar, but one proves extinction in $L^2(\mathcal{O})$-norm rather than $L^1(\mathcal{O})$-norm (see [V. Barbu, G. Da Prato, M. R. 2011]).

5. Extinction in finite time

We fix $\lambda \in (0,1]$ and start with the following lemma, which is one of the main ingredients of the proof.

Lemma

Let $x \in H_0^1(\mathcal{O})$. Then:

(i) $e^{-NC^*t}|X_\lambda(t)|_N^N$, $t \geq 0$, *is an $\{\mathcal{F}_t\}$-supermartingale, and hence so is* $e^{-C^*t}|X_\lambda(t)|_N$, $t \geq 0$.

(ii) *We have $\mathbb{P}$-a.s.*

$$|X_\lambda(t)|_N^N + N\rho \int_s^t |X_\lambda(r)|_N^{N-1} dr$$

$$\leq |X_\lambda(s)|_N^N + NC^* \int_s^t |X_\lambda(r)|_N^N dr + N(N-1)\lambda \int_s^t |X_\lambda(r)|_{N-2}^{N-2} dr$$

$$+N \int_s^t \langle |X_\lambda(r)|^{N-2} X_\lambda(r), X_\lambda(r) dW(r) \rangle, \ \forall s,t \in [0,T], \ s \leq t.$$

Proof of the Lemma. By the Itô-formula for L^p-norms in [N.V. Krylov 2010] and stopping, interpolation etc. we get

$$|X_\lambda(t)|_N^N + N(N-1)\int_s^t \int_{\mathcal{O}} |X_\lambda(r)|^{N-2} \nabla X_\lambda(r) \cdot \widetilde{\psi}_\lambda(\nabla X_\lambda(r)) d\xi\, dr$$

$$= |X_\lambda(s)|_N^N + \frac{1}{2} N(N-1) \int_s^t \int_{\mathcal{O}} \mu\, |X_\lambda(r)|^N d\xi\, dr \qquad (L^p\text{-Itô})$$

$$+ N\int_s^t \langle |X_\lambda(r)|^{N-2} X_\lambda(r), X_\lambda(r) dW(r)\rangle,\ \forall s,t \in [0,T],\ s \le t.$$

Since $\widetilde{\psi}_\lambda(u) \cdot u \geq 0 \quad \forall\, u \in \mathbb{R}^N$ (L^p-Itô) implies that $\mathbb{P}$-a.s. $\forall\, s \leq t$

$$e^{-NC^* t}|X_\lambda(t)|_N^N \leq e^{-NC^* s}|X_\lambda(s)|_N^N + \int_s^t e^{-NC^* r}\langle |X_\lambda(r)|^{N-2} X_\lambda(r),\, X_\lambda(r)\, dW(r)\rangle,$$

which in turn implies (i).

Since $\widetilde{\psi}_\lambda(u) \cdot u \geq |u| - \lambda$, we have

$$\begin{aligned}
(N-1)|X_\lambda|^{N-2}\nabla X_\lambda(r) \cdot \widetilde{\psi}_\lambda(\nabla X_\lambda) \\
&\geq (N-1)|X_\lambda|^{N-2}(|\nabla X_\lambda| - \lambda) \\
&= |\nabla(|X_\lambda|^{N-1})| - (N-1)\lambda|X_\lambda|^{N-2}.
\end{aligned}$$

Hence, the second term on the left hand side of (L^p-Itô) is bigger than

$$N\rho\int_s^t \int_{\mathcal{O}} |X_\lambda(r)|_N^{N-1} dr - N(N-1)\lambda \int_s^t |X_\lambda(r)|_{N-2}^{N-2} dr,$$

where we used Sobolev's embedding theorem in $W_0^{1,1}(\mathcal{O})$, i.e.,

$$\rho|y|_{\frac{N}{N-1}} \leq \|y\|_{1,1}, \ \forall y \in W_0^{1,1}(\mathcal{O}),$$

in the last step. Plugging this into (L^p-Itô) implies (ii). □

Proof of Theorem II (Sketch):
By approximation we may assume that $x \in H_0^1(\mathcal{O})$. Let X_λ^x be the solution to (SPDE$_\lambda$) with initial condition x. Applying Itô's formula to (L^p-Itô) and the function $\varphi_\varepsilon(r) = (r+\varepsilon)^{\frac{1}{N}}$, $\varepsilon \in (0,1)$, and proceeding as in the proof of the previous lemma, we obtain $\mathbb{P}$-a.s.

$$\begin{aligned}
&\varphi_\varepsilon(|X_\lambda^x(t)|_N^N) + \rho \int_0^t |X_\lambda^x(r)|_N^{N-1}(|X_\lambda^x(r)|_N^N+\varepsilon)^{-\frac{N-1}{N}}\,dr \\
&\quad \le \varphi_\varepsilon(|x|_N^N) + C^* \int_0^t |X_\lambda^x(r)|_N dr \\
&\quad +\lambda(N-1)\int_0^t |X_\lambda^x(r)|_{N-2}^{N-2}(|X_\lambda^x(r)|_N^N+\varepsilon)^{-\frac{N-1}{N}}\,dr \\
&\quad +\int_0^t \langle X_\lambda^x(r)|X_\lambda^x(r)|^{N-2}(|X_\lambda^x(r)|_N^N+\varepsilon)^{-\frac{N-1}{N}}, X_\lambda^x(r)dW(r)\rangle, \\
&\qquad\qquad t \ge 0.
\end{aligned} \tag{$*$}$$

Since $x \in H_0^1(\mathcal{O})$, by Proposition 2(iii) and interpolation we have, for $N = 3$ and some $C > 0$,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} |X_\lambda^x(t) - X^x(t)|_N^2\right] \\
&\quad \leq C\left(\mathbb{E}\left[\sup_{t\in[0,T]} |X_\lambda^x(t) - X^x(t)|_2^2\right]\right)^{\frac{1}{2}} \|x\|_{1,2} \ \forall \lambda \in (0,1] \\
&\qquad \to 0, \ \text{as } \lambda \to 0,
\end{aligned}
$$

where X^x is the solution to (SPDE) with initial condition x.

Hence taking expectation in $(*)$ by Fatou's Lemma we may let $\lambda \to 0$ in $(*)$, and subsequently let $\varepsilon \to 0$ to arrive at

$$e^{-C^* t}\mathbb{E}|X^x(t)|_N + \rho \int_0^t e^{-C^* \theta}\mathbb{P}[|X^x(\theta)|_N > 0]\, d\theta \le |x|_N, \ \forall t > 0. \qquad (**)$$

But, the process $t \to e^{-C^* t}|X^x(t)|_N$ is an L^1-limit of supermartingales, hence itself a supermartingale. Hence

$$|X^x(t)|_N = 0 \text{ for } t \geq \tau = \inf\{t \geq 0 : |X^x(t)|_N = 0\},$$

and thus $\mathbb{P}[|X^x(\theta)|_N > 0] = \mathbb{P}[\tau > \theta]$. By $(**)$, this yields

$$\mathbb{P}[\tau > t] \leq \left(\rho \int_0^t e^{-C^* \theta} d\theta\right)^{-1} |x|_N,$$

as claimed. □

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