Super-Brownian motion in random environment and heat equation with noise.

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Introduction

In this talk, we focus on the following type SPDE (heat equation with noise):

$$u_t = \frac{1}{2} u_{xx} + a(u)\dot{W}(t, x),$$

where $a(\cdot)$ is a real valued continuous function and W is time space white noise.

Introduction

Super-Brownian motion $\{X_t(\cdot) : t \geq 0\}$ is a measure valued process which is characterized by several ways. (PDE, martingale problem...)
In this talk, we will characterize it as the unique solution of some martingale problem.

super-Brownian motion (SBM)

Super-Brownian motion $\{X_t(\cdot) : t \geq 0\}$ is the unique solution of the following martingale problem:

$$\begin{cases} \text{For all } \phi \in C_b^2\left(\mathbb{R}^d\right), \\ Z_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t \frac{1}{2} X_s\left(\Delta\phi\right) ds \\ \text{is an } \mathcal{F}_t^X\text{-martingale such that} \\ \langle Z(\phi)\rangle_t = \int_0^t X_s\left(\phi^2\right) ds. \end{cases}$$

Remark: $\{X_t(\cdot) : t \geq 0\} \in C([0,\infty), \mathcal{M}_F(\mathbb{R}^d))$.

We have some remarkable properties on SBM as follows.

Properties

1. ($d=1$, Konno-Shiga, Reimers) $X_t(\cdot)$ is absolutely continuous w.r.t. Lebesgue measure for all $t \in (0,\infty)$ almost surely and its density $u(t,x)$ (i.e. $X_t(dx) = u(t,x)dx$) satisfies the following SPDE:

$$u_t = \frac{1}{2}u_{xx} + \sqrt{u}\dot{W}(t,x), \ \lim_{t\to 0+} u(t,x)dx = X_0(dx),$$

where W is space-time white noise.

2. ($d \geq 2$, Perkins, Dawson-Perkins, et.al.) If $X_t(1) \neq 0$, then $X_t(\cdot)$ is singular w.r.t. Lebesgue measure. Also, the Hausdorff dimension of $\mathrm{supp}(X_t)$ is 2 a.s.

SPDE

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2. $a(u) = \sqrt{u - u^2} \Leftrightarrow$ the density of stepping stone model.

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Can we find any models associated to the sol. of SPDE for other $a(\cdot)$?

Suggestion by Mytnik

Mytnik constructed super-Brownian motion in random environment.

SBMRE(Mytnik)

For $d \geq 1$, we can construct SBMRE $\{X_t(\cdot) : t \geq 0\}$ as the limit of BBM in random environment which is the unique solution of the martingale problem:

$$\begin{cases} \text{For all } \phi \in C_b^2(\mathbb{R}^d), \\ Z_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s\left(\frac{1}{2}\Delta\phi\right) ds \\ \text{is an } \mathcal{F}_t^X\text{-martingale and} \\ \langle Z(\phi)\rangle_t = \int_0^t X_s\left(\phi^2\right) ds \\ \qquad\qquad + \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} g(x,y)\phi(x)\phi(y)X_s(dx)X_s(dy)ds, \end{cases}$$

where $g(x, y)$ is bounded symmetric continuous function.

Mytnik gave a remark in the paper that if g is replaced by δ_{x-y}, then a solution of the above martingale problem must have density a.s. and its density u is a solution of SPDE

$$u_t = \frac{1}{2}u_{xx} + \sqrt{u + u^2}\dot{W}(t, x). \qquad \text{(A)}$$

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Thus, we have a question: can we construct SBMRE which is a solution of SPDE (A)?

Branching Brownian motions

Also, SBM is obtained as the limit of the critical branching Brownian motions. Branching Brownian motions is defined as follows in this talk:

Branching Brownian motions (BBM)

1. There exist N particles at the origin at time 0.
2. Each particle at time $\frac{k}{N}$ independently performs Brownian motion up to time $t = \frac{k+1}{N}$ and it splits into two particles with probability $\frac{1}{2}$ or dies with probability $\frac{1}{2}$ independently.

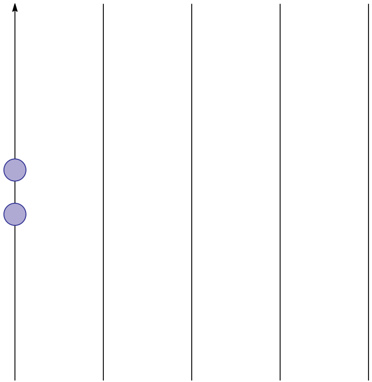
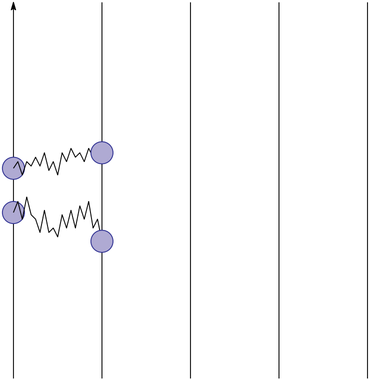


Figure : $N = 2$, $d = 1$



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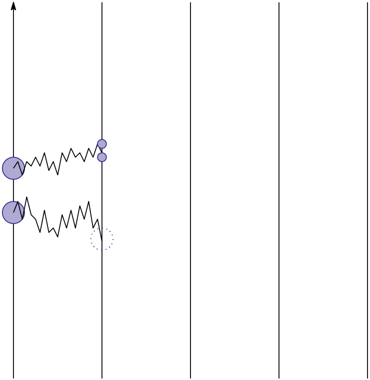

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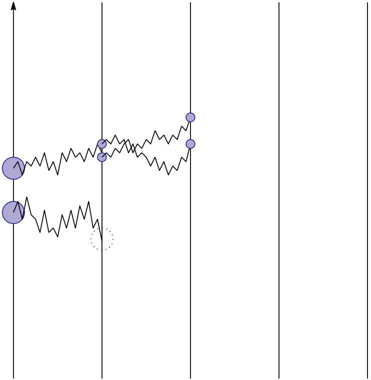

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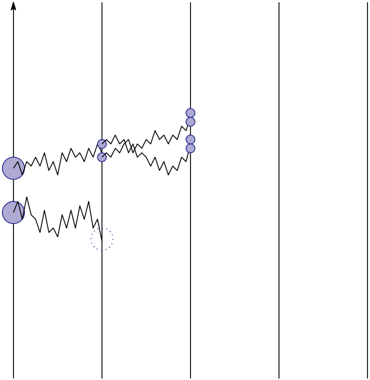

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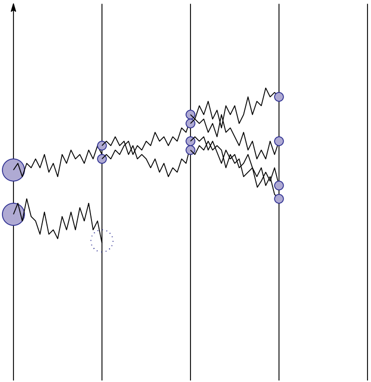

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Super-Brownian motion

We identify each particle as a Dirac mass, i.e. if a particle locates at site x, then we regard it as δ_x. We denote the positions of particles at time t by $\{x_t^1, \cdots, x_t^{B_t^{(N)}}\}$, where $B_t^{(N)}$ is the total number of particles at time t. Then we define the measure valued process $\{X_t^{(N)}(\cdot) : t \geq 0\}$ by

$$X_0^{(N)} = \delta_0,\ X_t^{(N)}(\cdot) = \frac{1}{N} \sum_{i=1}^{B_t^{(N)}} \delta_{x_t^i},$$

or for each $A \in \mathcal{B}(\mathbb{R}^d)$,

$$X_t^{(N)}(A) = \frac{\sharp\{\text{particles locates in } A\ \}}{N}.$$

Then, $\{X_t^{(N)}(\cdot) : t \geq 0\} \in D([0,\infty), \mathcal{M}_F(\mathbb{R}^d))$.

Super-Brownian motion

Theorem A (Watanabe '68)

$\left\{ X_t^{(N)}(\cdot) \right\} \Rightarrow \{X_t(\cdot) : t \geq 0\}$, where X is the unique solution of the martingale problem:

$$\begin{cases} \text{For all } \phi \in C_b^2\left(\mathbb{R}^d\right), \\ Z_t(\phi) = X_t(\phi) - \phi(0) - \int_0^t \frac{1}{2} X_s\left(\Delta\phi\right) ds \\ \text{is an } \mathcal{F}_t^X\text{-martingale such that} \\ \langle Z(\phi)\rangle_t = \int_0^t X_s\left(\phi^2\right) ds. \end{cases}$$

Construction by Mytnik

Let $\xi = \{\xi(x) : x \in \mathbb{R}^d\}$ be a random field such that

1. $P(\xi(x) > z) = P(\xi(x) < -z)$ for all $x \in \mathbb{R}^d$ and $z \in \mathbb{R}$.
2. $g(x, y) = E[\xi(x)\xi(y)]$.

Let $\{\xi_k : k \in \mathbb{N}\}$ be independent copies of ξ. Then, the limit of the following BBM in random environment is SBMRE(Mytnik).

BBMRE

1. There exist N particles at time 0.
2. Particles independently perform Brownian motion in $t \in \left[\frac{k}{N}, \frac{k+1}{N}\right)$. Then, at time $t = \frac{k+1}{N}$, a particle independently splits into two particles with probability $\frac{1}{2} + \frac{\xi_{k+1}(x)}{2N^{1/2}}$ or dies out with probability $\frac{1}{2} - \frac{\xi_{k+1}(x)}{2N^{1/2}}$, where x is the site it reached at time $t = \frac{k+1}{N}$.

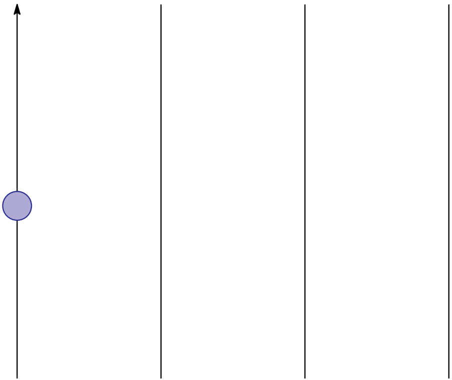
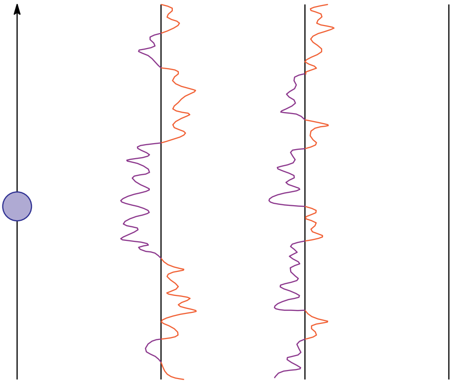


Figure : $N = 1$, $d = 1$



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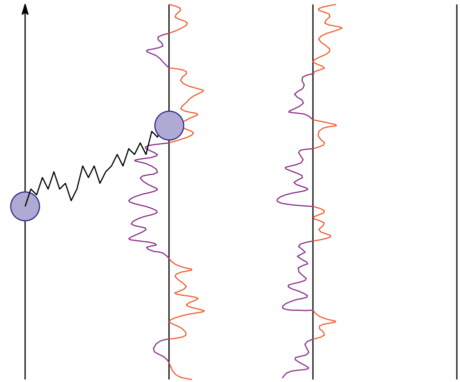

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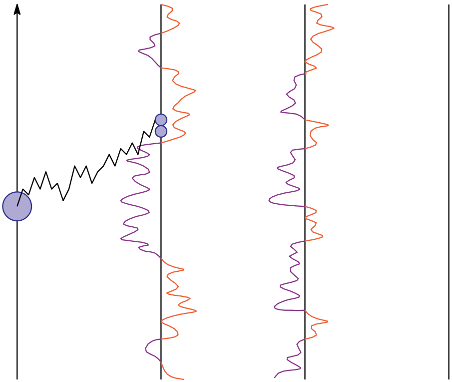

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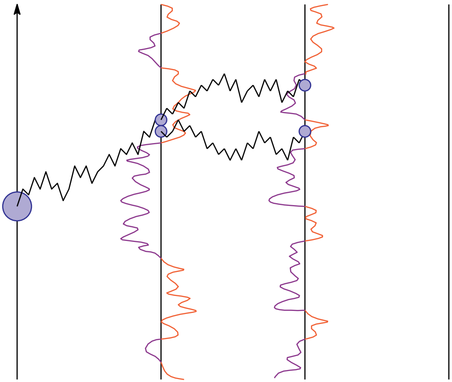

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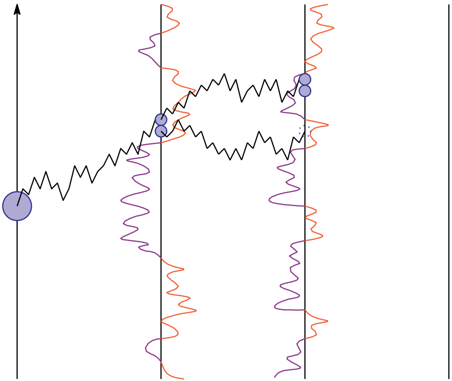

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Superprocesses in random environment (by Mytnik)

When we define the measure valued processes $\{X_t^{(N)}(\cdot) : t \geq 0\}$ in the same way, it weakly converges to the SBMRE given as above.

Question (again)

Can we construct SBMRE which is a solution of SPDE

$$u_t = \frac{1}{2}u_{xx} + \sqrt{u+u^2}\dot{W}(t,x)?$$

Idea

Idea 1

Replace $g(x, y)$ by δ_{x-y}.
$\Rightarrow$ No. (Branchings have no interaction since particles cannot reach the same site at each branching time a.s.)

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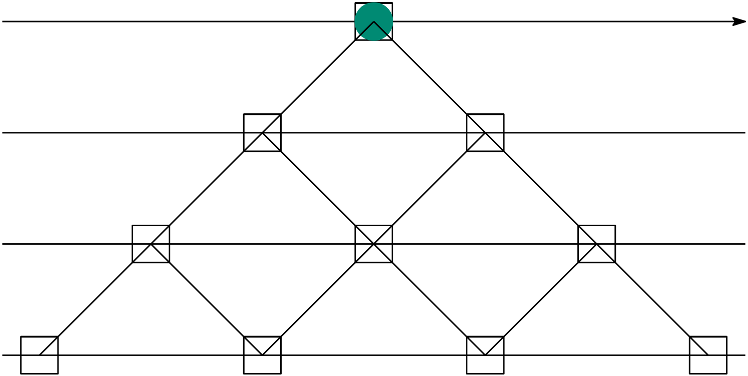
Idea 2

Construct SBMRE as a limit of some branching processes in which particles can reach the same site with positive probability.
$\Rightarrow$ Branching random walks in random environment.

Let $\{\xi(n,x) : (n,x) \in \mathbb{N} \times \mathbb{Z}\}$ be $\{-1,1\}$-valued i.i.d. random variables with $P(\xi(n,x) = 1) = \frac{1}{2}$. BRWRE is defined by the following way.

BRWRE

1. There are N particles at the origin at time 0.
2. Each particle at site x at time n moves to an independently and uniformly chosen nearest neighbor site and then it splits into two particles with probability $\frac{1}{2} + \frac{\beta\xi(n,x)}{2N^{1/4}}$ or dies out with probability $\frac{1}{2} - \frac{\beta\xi(n,x)}{2N^{1/4}}$, where $\beta \in \mathbb{R}$ is a constant.



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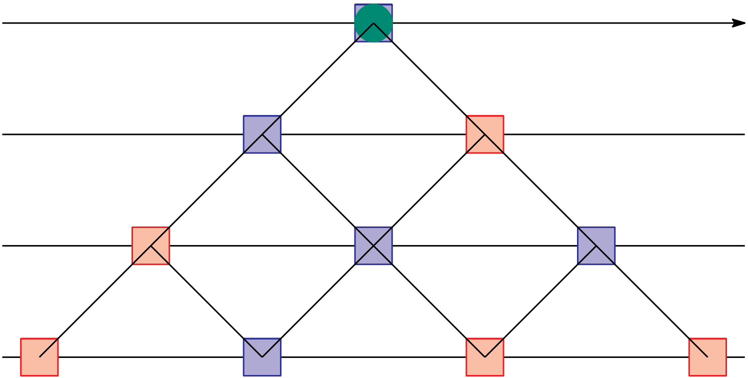

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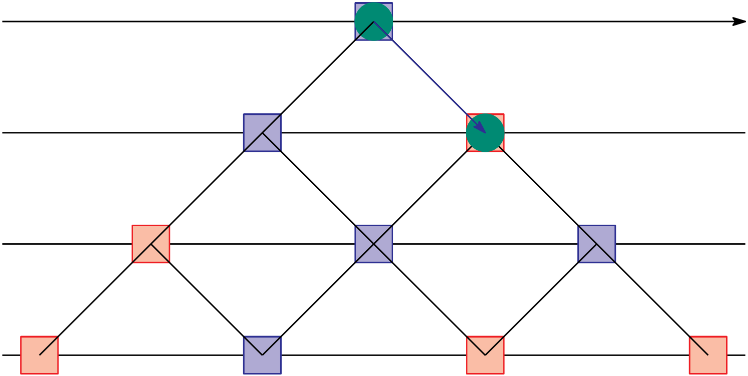

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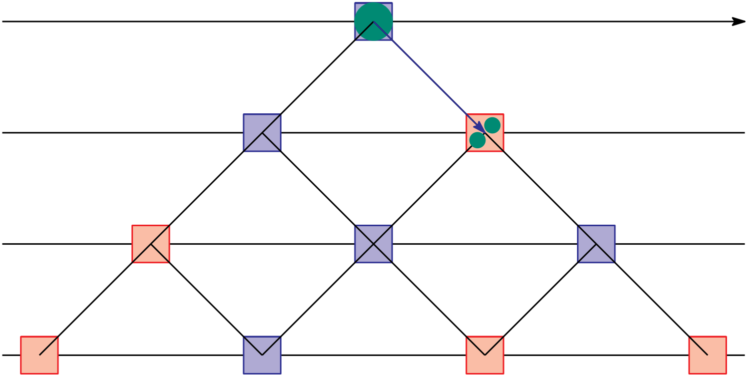

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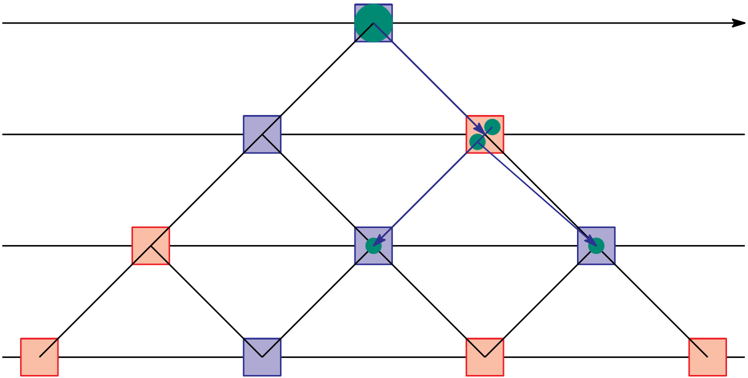

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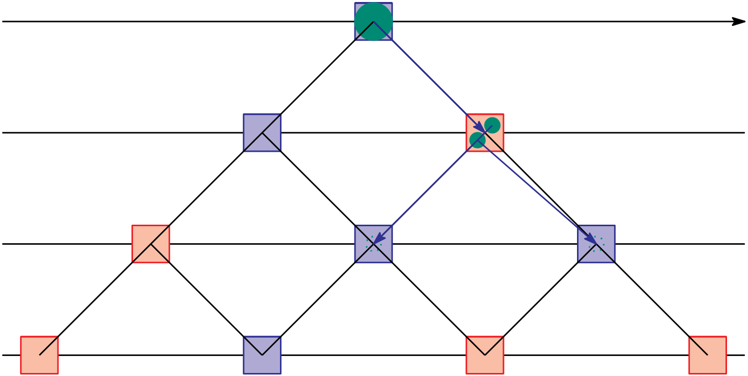

Figure : $N = 1$

We define BRWRE as measure valued processes like BBM. For $A \in \mathcal{B}(\mathbb{R})$

$$X_t^{(N)}(A) = \sum_{x \in \sqrt{N}A} \frac{B^{(N)}_{\lfloor Nt \rfloor, x}}{N} = \frac{\sharp \left\{ \text{ particles locates in } \sqrt{N}A \text{ at time } \lfloor Nt \rfloor \right\}}{N},$$

where

$$B^{(N)}_{n,x} = \sharp \{\text{particles at site } x \text{ at time } n.\}$$

Main result

Theorem [N '12]

$\left(X_t^{(N)}(\cdot) : t \geq 0\right)$ is tight and its limit point $(X_t(\cdot) : t \geq 0)$ is a solution of the following martingale problem:

$$\begin{cases} \text{For all } \phi \in C_b^2(\mathbb{R}), \\ X_0(\phi) = \phi(0), \\ Z_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s\left(\frac{1}{2}\Delta\phi\right) ds \\ \text{is an } \mathcal{F}_t^X\text{-martingale and} \\ \langle Z(\phi) \rangle = \int_0^t X_s(\phi^2) ds \\ \qquad\qquad + \frac{\beta^2}{2} \int_0^t \int_{\mathbb{R}\times\mathbb{R}} \delta_{x-y}\phi(x)\phi(y) X_s(dx) X_s(dy) ds. \end{cases}$$

Conclusion

Thus, we can construct a SBMRE which is a solution of SPDE:

$$u_t = \frac{1}{2}u_{xx} + \sqrt{u + \frac{\beta^2}{2}u^2}\dot{W}(t,x), \ \lim_{t \to 0+} u(t,x)dx = \delta_0.$$

Moreover, we can construct SBMRE by the same way which is a solution of SPDE:

$$u_t = \frac{1}{2}u_{xx} + \sqrt{\gamma u + \beta^2 u^2}\dot{W}(t,x), \ \lim_{t \to 0+} u(t,x)dx = m(dx)$$

for $\gamma > 0, \ \beta \in \mathbb{R}, \ m \in \mathcal{M}_F(\mathbb{R})$.

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Uniqueness?

Sketch of proof

For $\phi \in C_b^2(\mathbb{R})$, the martingale part of $X^N_{\frac{k}{N}}(\phi)$ is divided into three parts:

$$M^{(b,N)}_{\cdot}(\phi) : \text{branching term}$$
$$M^{(r,N)}_{\cdot}(\phi) : \text{random walk term}$$
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Then, $\left\langle M^{(b,N)}(\phi) \right\rangle_{\cdot} \Rightarrow \int_0^{\cdot} X_t(\phi)dt$ as $N \to \infty$ and $\left\langle M^{(r,N)}(\phi) \right\rangle_{\cdot} \Rightarrow 0.$

Sketch of proof

The quadratic variation of the last one is written as

$$\left\langle M^{(e,N)}(\phi) \right\rangle_{n/N} = \frac{\beta^2}{N^2} \sum_{k=0}^{n-1} \sum_{x \in \mathbb{Z}} \phi\left(\frac{x}{N^{1/2}}\right)^2 \frac{\left(B_{k,x}^{(N)}\right)^2}{N^{1/2}}.$$

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We approximate $B_{k,x}^{(N)}$ as "density". We set

$$u^{(N)}(t,z) = \frac{B_{k,x}^{(N)}}{2N^{1/2}}\mathbf{1}\left\{\left[\frac{k}{N},\frac{k+1}{N}\right)\times\left[\frac{x-1}{N^{1/2}},\frac{x+1}{N^{1/2}}\right)\ni(t,z)\right\}.$$

Remark: $\displaystyle\int_{z\in[\frac{x-1}{N^{1/2}},\frac{x+1}{N^{1/2}})} u^{(N)}(t,z)dz = \frac{1}{N}B_{k,x}^{(N)}$ for $t\in[\frac{k}{N},\frac{k+1}{N})$.

Sketch of proof

Then, we have that

$$\left\langle M^{(e,N)}(\phi)\right\rangle_{n/N} \sim \frac{\beta^2}{2}\int_0^{n/N}\int_{\mathbb{R}}\phi(z)^2\left(u^{(N)}(t,z)\right)^2 dzdt$$

If $u^{(N)}(\cdot,\cdot) \Rightarrow u(\cdot,\cdot)$, then this term converges to $\frac{\beta^2}{2}\int_0^{\cdot}\int_{\mathbb{R}}\phi(x)^2 u(t,x)^2 dxdt$.

Thank you for your attention!