

**On spectral bounds for symmetric  
Markov chains with coarse Ricci  
curvatures**

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# 1 Aim

Under the coarse Ricci curvature lower bound,

- (1) Upper estimate of (non-linear) spectral radius
- (2) Lower estimate of (non-linear) spectral gap
- (3) Strong  $L^p$ -Liouville property for  $P$ -harmonic maps

## 2 **Plan of talk**

- (1) Wasserstein distance (Historical Remark)
- (2) Coarse Ricci curvature
- (3) CAT(0)-space, 2-uniformly convex space
- (4) Main Theorems

### 3 Wasserstein space

Def 3.1 (Wasserstein distance)

$(E, d)$ : Polish space,  $p \in [1, \infty[$ .

$$\mathcal{P}^p(E) := \{\mu \in \mathcal{P}(E) \mid \int_E d^p(\cdot, \exists/\forall x_0) d\mu < \infty\},$$

For  $\mu, \nu \in \mathcal{P}^p(E)$ ,

$$d_{W_p}(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left( \int_{E \times E} d^p(x, y) \pi(dx dy) \right)^{1/p}$$

:  $p$ -Wasserstein distance.

## Rem 3.1

(1)  $d_{W_1}$  is nothing but the Kantorovich-Rubinstein distance.  $d_{W_p}$  was (re)discovered by various authors independently:

**Gini** ('14):  $d_{W_1}$  on discrete prob. on  $\mathbb{R}$ .

**Kantorovich** ('42):  $d_{W_1}$  on prob. on cpt sp

**Salvemini** ('43): For discrete  $\mu, \nu \in \mathcal{P}(E)$ ,

**Dall'Aglio** ('56): For general  $\mu, \nu \in \mathcal{P}^p(E)$ ,

$$d_{W_p}(\mu, \nu)^p = \int_0^1 |F_\mu^{-1}(t) - F_\nu^{-1}(t)|^p dt.$$

**Fréchet** ('57): metric properties of  $d_{W_p}$ .

**Kantorovich–Rubinshtein** ('58):

$$d_{W_1}(\mu, \nu) = \sup_{f:1\text{-Lip}} \left( \int_E f d\mu - \int_E f d\nu \right)$$

**Vasershtein** ('69):

$$d_{W_1}(\mu, \nu) := \inf_{X \sim \mu, Y \sim \nu} \mathbf{E}[d(X, Y)]$$

**Dobrushin** ('70) named '**Vasershtein distance**'

**Mallows** ('72):  $d_{W_2}$  in statistical context

**Tanaka** ('73):  $d_{W_2}$ , Boltzmann equation

**Bickel–Freedman** ('80):  $d_{W_2}$  was named as

Mallows metric

(2) In English literatures, the German spelling 'Wasserstein'<sup>1</sup> is used (attributed to the name 'Vasershtein' being of Germanic origin).

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<sup>1</sup>Vaserstein himself uses the terminology 'Wasserstein distance' in <http://www.math.psu.edu/vstein/>

## 4 Coarse Ricci curvature

$(E, d)$ : Polish space,  $\mathcal{E} = \mathcal{B}(E)$ : Borel field.  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ .

$X = (\Omega, X_n, \mathcal{F}_n, \mathcal{F}_\infty, P_x)_{x \in E}$ :

conservative Markov chain on  $(E, \mathcal{E})$ .

$\Omega := E^{\mathbb{N}_0}$ : set of all  $E$ -valued sequences

$\omega = \{\omega(n)\}_{n \in \mathbb{N}_0}$ .  $X_n(\omega) := \omega(n)$ ,  $n \in \mathbb{N}_0$ .



$P(x, dy) := P_x(X_1 \in dy), x \in E :$

*transition kernel* of  $X$ :

$P(x, dy)$  satisfies the following:

(P1) For each  $x \in E$ ,  $P(x, \cdot) \in \mathcal{P}(E)$ .

(P2) For each  $A \in \mathcal{E}$ ,  $P(\cdot, A) \in \mathcal{E}$ .

Further we impose the following:

(P3) For each  $x \in E$ ,  $P(x, \cdot) \in \mathcal{P}^1(E)$ .

We set  $P_x(A) := P(x, A)$ ,  $A \in \mathcal{E}$  and  $Pf(x) := \int_E f(y)P_x(dy) = \mathbf{E}_x[f(X_1)]$ . For the given Markov chain  $X$  as above and a fixed  $n \in \mathbb{N}$ , a Markov chain  $X^n = (\Omega, X_k^n, \mathcal{F}_k^n, \mathcal{F}_\infty^n, P_x^n)_{x \in E}$  with state space  $(E, d)$  defined by the transition kernel  $P^n(x, dy)$  is called an *n-step Markov chain*.

Def 4.1 (Ollivier (2009))

The *coarse Ricci curvature*  $\kappa(x, y)$  along  $(xy)$  for  $x \neq y$  is defined by

$$\kappa(x, y) := 1 - \frac{d_{W_1}(P_x, P_y)}{d(x, y)} (\leq 1)$$

and  $\kappa := \inf \{ \kappa(x, y) \mid (x, y) \in E^2 \setminus \mathbf{diag} \}$  is said to be the *lower bound of the coarse Ricci curvature*.  $\kappa \in [-\infty, 1]$ .

The *n*-step coarse Ricci curvature  $\kappa_n(x, y)$  of  $X$  along  $(xy)$  is defined to be

$$\kappa_n(x, y) := 1 - \frac{d_{W_1}(P_x^n, P_y^n)}{d(x, y)}$$

and  $\kappa_n := \inf\{\kappa_n(x, y) \mid (x, y) \in E^2 \setminus \mathbf{diag}\}$

is its lower bound.  $\kappa_n(x, y)$  is nothing

but the coarse Ricci curvature for  $X^n$  and

$\kappa_1(x, y) = \kappa(x, y)$  for  $(x, y) \in E^2 \setminus \mathbf{diag}$ .

Note that  $\kappa_n \geq 1 - (1 - \kappa)^n$  holds.

Recent works on coarse Ricci curvature:

**Lin-Yau** (2010): locally finite graphs

$$\kappa(x, y) \geq -2 \left( 1 - \frac{1}{d_x} - \frac{1}{d_y} \right)$$

**Lin-Lu-Yau** (2011): New def for  $\kappa(x, y)$ .

**Jost-Liu** (2011): locally finite graphs

$$\kappa(x, y) \geq -2 \left( 1 - \frac{1}{d_x} - \frac{1}{d_y} \right)_+$$

**Bauer-Jost-Liu** (2011): graphs with loops

$$1 - (1 - \kappa_n)^{\frac{1}{n}} \leq \lambda_1 \leq \cdots \leq \lambda_{N-1} \leq 1 + (1 - \kappa_n)^{\frac{1}{n}}$$

**Kitabeppu** (2011):

Lower estimate for  $\kappa(x, y)$  under  $\text{CD}(K, N)$

**Veysseire** (2012):  $m$ -sym Markov process

$$\bar{\kappa}(x, y) := \overline{\lim}_{t \rightarrow 0} \frac{1}{t} \left( 1 - \frac{d_{W_1}(P_t(x, \cdot), P_t(y, \cdot))}{d(x, y)} \right) \geq \kappa \in \mathbb{R}$$

$$\Rightarrow d_{W_1}(P_t(x, \cdot), P_t(y, \cdot)) \leq e^{-\kappa t} d(x, y),$$

$$m(E) < \infty, \kappa \leq \frac{\mathcal{E}(f)}{\|f - \langle m, f \rangle\|_2^2} \text{ if } \kappa > 0.$$

**Ex** 4.1 (Sym. simple random walk on  $\mathbb{Z}^n$ )

$$E := \mathbb{Z}^n,$$

$$d_{\mathbb{Z}^n}(x, y) := \sum_{i=1}^n |x_i - y_i|: x, y \in \mathbb{Z}^n:$$

$$d_{\mathbb{R}^n}(x, y) := \left( \sum_{i=1}^n |x_i - y_i|^2 \right)^{\frac{1}{2}}: x, y \in \mathbb{Z}^n$$

X: symmetric simple random walk on  $\mathbb{Z}^n$ .

$$P(x, dy) := \frac{1}{2n} \sum_{|x-z|=1, z \in \mathbb{Z}^n} \delta_z(dy).$$

$\implies \kappa(x, y) = 0$  w.r.t. either of  $d_{\mathbb{Z}^n}$  or  $d_{\mathbb{R}^n}$ .

**Ex** 4.2 (RW on locally finite graph)

**Jost-Liu** (2011):

$G = (V, E)$ : a locally finite graph

$d_x$ : degree at vertex  $x \in V$

$x \sim y \stackrel{\text{def}}{\iff} xy \in E$

$P(x, dz) := \frac{1}{d_x} \sum_{x \sim y} \delta_y(dz)$

$$\kappa(x, y) \geq -2 \left( 1 - \frac{1}{d_x} - \frac{1}{d_y} \right)_+$$

Equality holds if  $G = (V, E)$  is a tree.



**Ex** 4.3 (RW on Riemannian mfd)

$E = M$ :  $C^\infty$  compl.  $N$ -dim Riem mfd.

$\varepsilon > 0$ .  $m = \text{vol}$ : volume measure.

$X$ :  $\varepsilon$ -step Random walk on  $E$  defined by

$$P_x(dy) = \frac{1}{m(B_\varepsilon(x))} \mathbf{1}_{B_\varepsilon(x)}(y) m(dy).$$

**Ollivier(09)**

$$\implies \kappa(x, y) = \frac{\varepsilon^2 \text{Ric}(v, v)}{2(N+2)} + O(\varepsilon^3 + \varepsilon^2 d(x, y))$$

for  $v \in U_x M$  and  $y \in \exp_x tv$  with  $t =$

$d(x, y)$  small enough.

**Ex** 4.4 (Circle graph)

$G = (V, E)$ : a circle graph of size  $N$ ;

$V := \{x_i\}_{i=1}^N$ : vertices,

$E := \{x_i x_{i+1}\}_{i=1}^N$  ( $x_{N+i} = x_i$  ( $i \in \mathbb{N}$ )): edges,

$d_x(G) = 2$  for  $x \in V$ : degree at  $x \in V$ ,

$P_{x_i}(dy) := \frac{1}{2}\delta_{x_{i-1}}(dy) + \frac{1}{2}\delta_{x_{i+1}}(dy)$ .

$\kappa(x, y) = 0$  for  $(x, y) \in V \times V \setminus \mathbf{diag}$ ,

$\kappa_n(x, y) \geq 0$  for  $(x, y) \in V \times V \setminus \mathbf{diag}$ ,

$X$  (hence  $X^n$ ) is  $m$ -symmetric w.r.t.

$$m(dy) := \frac{1}{N} \sum_{i=1}^N \delta_{x_i}(dy).$$

We take  $N = 5$ .

3-step Markov chain  $X^3$  is associated with

$G^3 := (V^3, E^3)$  defined by  $V^3 := V$  and

$E^3 := \{x_i x_j \mid 1 \leq i, j \leq 5 \text{ with } i \neq j\}$ .

The transition kernel  $P_x^3(dy)$  is given by

$$P_{x_i}^3 = \frac{1}{8} \delta_{x_{i-2}} + \frac{3}{8} \delta_{x_{i-1}} + \frac{3}{8} \delta_{x_{i+1}} + \frac{1}{8} \delta_{x_{i+2}}.$$

$$d_x(G^3) = 4.$$

The 3-step coarse Ricci curvature  $\kappa_3(x, y)$  for  $xy \in E^3$  can be estimated by use of **Bauer-Jost-Liu** (2011).

$$\kappa_3(x_i, x_{i+1}) = \frac{3}{8}, \quad \frac{5}{8} \leq \kappa_3(x_i, x_{i+2}) \leq \frac{7}{8}.$$

Therefore,  $\kappa_3(x, y) \geq \frac{3}{8}$  for all  $(x, y) \in V \times V \setminus \text{diag}$ .

## 5 CAT(0)-space, 2-unif. convex sp

Def 5.1 (CAT(0)-space)

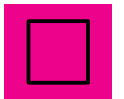
$(Y, d_Y)$ : CAT(0)-space  $\iff$  For  $\forall z, x, y \in Y$ ,  $\exists \gamma : [0, 1] \rightarrow Y$  with  $\gamma_0 = x$ ,  $\gamma_1 = y$  s.t. for  $t \in [0, 1]$

$$d_Y^2(z, \gamma_t) \leq (1 - t)d_Y^2(z, x) + td_Y^2(z, y) - t(1 - t)d_Y^2(x, y).$$

**C**artan-**A**lexandrov-**T**oponogov

## Ex 5.1 (Examples of CAT(0)-spaces)

- Hadamard manifold; simply connected smooth compl Riem mfd with NPC.
- products of CAT(0)-sp ● Hilbert space
- convex subset of CAT(0)-space
- Tree ● Euclidean Buildings
- CAT(0)-space valued  $L^2$ -maps



**Def** 5.2 (2-Uniformly Convex Space)

$(Y, d)$ : *2-uniformly convex with  $k > 0$*

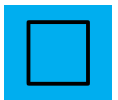
$\stackrel{\text{def}}{\iff} (Y, d)$ : geodesic space &  $\forall x, y, z \in$

$Y$ ,  $\forall \gamma := (\gamma_t)_{t \in [0,1]}$ : min. geo. in  $Y$  from

$x$  to  $y$  &  $\forall t \in [0, 1]$ ,

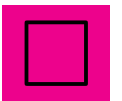
$$d^2(z, \gamma_t) \leq (1 - t)d^2(z, x) + td^2(z, y) - \frac{k}{2}t(1 - t)d^2(x, y).$$

$z = \gamma_t \implies k \in ]0, 2]$ .



## Ex 5.2 (Examples of 2-Unif. Conv. Spaces)

- Convex subset of a 2-uniformly convex space.
- CAT(0)-space
- CAT(1)-space with  $\text{diam} < \frac{\pi}{2}$  is 2-uniformly convex (see Ohta (2007))
- $L^2$ -maps into a CAT(1)-sp. with  $\text{diam} < \frac{\pi}{2}$





$(Y, d_Y)$ : complete 2-unif, convex space

$\gamma, \eta (\subset Y)$ : minimal geodesic segments

$$\gamma \perp_p \eta \stackrel{\text{def}}{\iff} p \in \gamma \cap \eta,$$

$$d_Y(x, p) \leq d_Y(x, y) \quad \forall x \in \gamma, y \in \eta.$$

$$(B): \gamma \perp_p \eta \iff \eta \perp_p \gamma.$$

### Ex 5.3 (Examples satisfying (B))

- complete CAT(0)-space.
- complete CAT(1)-sp with  $\text{diam} < \pi/2$ .

**Def 5.3 (Barycenter)**

$(Y, d_Y)$ : complete sep. 2-unif. convex space

$\mu \in \mathcal{P}^1(Y) \Rightarrow b(\mu)$ :  $\exists_1$  unique minimizer  
(independent of  $w \in Y$ ) of

$$z \mapsto \int_Y (d_Y^2(z, y) - d_Y^2(w, y)) \mu(dy).$$

We call  $b(\mu)$  the barycenter of  $\mu$ .

**Lem** 5.1 (Jensen's inequality, **K.** (2010))

$(Y, d_Y)$ : complete sep. 2-unif. convex space.  $\mu \in \mathcal{P}^1(Y)$ .

(B):  $\gamma \perp_p \eta \leftrightarrow \eta \perp_p \gamma$ .

Then for any l.s.c. convex func  $\varphi$  on  $Y$

$$\varphi(b(\mu)) \leq \int_Y \varphi(x) \mu(dx).$$

**Ass** 5.1

$m \in \mathcal{P}^1(E)$ ,  $\text{supp}[m] = E$ ,  $p \geq 1$ ,

$X$ :  $m$ -sym Markov chain on  $E$  with (P3),

$(Y, d_Y)$ : compl sep. 2-unif. convex space,

(B):  $\gamma \perp_p \eta \iff \eta \perp_p \gamma$ ,

(CG): Convex Geometry:

$\exists \Phi : Y^2 \rightarrow \mathbb{R}$  convex s.t.

$C^{-1}d_Y \leq \Phi \leq Cd_Y$  on  $Y \times Y$  for  $C > 0$ .

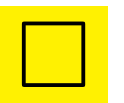
$L^p(E, Y, m)$ : space of  $L^p$ -maps,

$L^p(E, Y; m) := \{u : E \rightarrow Y \text{ m'ble map} \mid$   
 $\int_E d_Y^p(u(x), o) m(dx) < \infty \exists / \forall o \in Y\} / \sim,$

$d_{L^p}(u, v)^p := \int_E d_Y^p(u(x), v(x)) m(dx),$

$C_p^p := \int_E \int_E d^p(x, y) m(dx) m(dy) \leq \infty$

$(C_p < \infty \Leftrightarrow m \in \mathcal{P}^p(E)).$



**Def 5.4**  $u \in S(E, Y) \stackrel{\text{def}}{\iff} \#(\text{Im}(u)) < \infty.$

$u \in \text{Lip}(E, Y) \stackrel{\text{def}}{\iff} \text{Lip}(u) := \sup_{x \neq y} \frac{d_Y(u(x), u(y))}{d(x, y)} < \infty.$

$m \in \mathcal{P}^p(E) \Rightarrow \text{Lip}(E, Y) \subset L^p(E, Y; m)$

$u \in S(E, Y) \cup \text{Lip}(E, Y) \Rightarrow u_* P_x \in \mathcal{P}^1(Y)$

$\Rightarrow Pu(x) := b(u_* P_x).$

**Thm 5.1**  $S(E, Y) \stackrel{\text{dense}}{\hookrightarrow} L^p(E, Y; m)$  and

$\text{Lip}(E, Y) \stackrel{\text{dense}}{\hookrightarrow} L^p(E, Y; m)$  if  $m \in \mathcal{P}^p(E).$

**Lem 5.2**  $\kappa_n \in \mathbb{R}, u \in \text{Lip}(E, Y) \Rightarrow$

$\text{Lip}(P^n u) \leq C^2(1 - \kappa_n)\text{Lip}(u).$

**Pf.**  $d_Y(P^n u(x), P^n u(y))$

$$\leq C \Phi(b(u_{\#} P_x^n), b(u_{\#} P_y^n)) \stackrel{\text{(Jensen)}}{\leq} C \int_{Y \times Y} \Phi d\pi$$

$$\leq C^2 \int_{Y \times Y} d_Y d\pi = C^2 \int_{E \times E} d_Y(u(p), u(q)) d\pi_0(p, q)$$

$$(\pi := (u \times u)_{\#} \pi_0 \in \Pi(u_{\#} P_x^n, u_{\#} P_y^n))$$

$$\leq C^2 \text{Lip}(u) \int_{E \times E} d(p, q) d\pi_0(p, q)$$

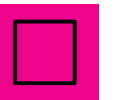
$$\leq C^2 \text{Lip}(u) d_{W_1}(P_x^n, P_y^n) (\pi_0 \in \Pi(P_x^n, P_y^n) : \text{opt})$$

$$\leq C^2 \text{Lip}(u) (1 - \kappa_n) d(x, y).$$



**Def 5.5** For  $u \in L^p(E, Y; m)$ , we define  $Pu := \lim_k Pu_k \in L^p(E, Y; m)$  by approximating seq  $\{u_k\} \subset S(E, Y)$  to  $u$ ,

$$\begin{aligned}
 d_{L^p}(Pu_l, Pu_k)^p &= \int_E d_Y^p(Pu_l(x), Pu_k(x)) m(dx) \\
 &\leq C^p \int_E \Phi^p(Pu_l, Pu_k) dm \\
 &\stackrel{\text{(Jensen)}}{\leq} C^p \int_E P\Phi^p(u_l, u_k) dm \\
 &\leq C^{2p} d_{L^p}(u_l, u_k)^p \rightarrow 0
 \end{aligned}$$





## 6 Results

Def 6.1 (Variance) For  $u \in L^p(E, Y; m)$ ,

$$\text{Var}_m^p(u) := \inf_{z \in Y} \int_E d_Y^p(u(x), z) m(dx),$$

$$\overline{\text{Var}}_m^p(u) := \frac{1}{2} \int_E \int_E d_Y^p(u(x), u(y)) m(dx) m(dy)$$

If  $p = 2$  and  $Y = H$ : Hilbert sp., for  $f, g \in$

$L^2(E, H; m)$ , we write  $\text{Var}_m(f)$ ,  $\overline{\text{Var}}_m(f)$

$$\text{Cov}_m(f, g) := \frac{1}{2} \int_{E^2} \langle f(x) - f(y), g(x) - g(y) \rangle_H m^2(dx dy).$$

**Def 6.2 (Energy of Maps)**

For  $u \in L^p(E, Y; m)$ ,

$$E^p(u) := \frac{1}{2} \int_E \int_E d_Y^p(u(y), u(x)) P(x, dy) m(dx)$$

: *p-energy* of  $u$  with respect to  $X$  and

$$E_*^p(u) := \frac{1}{2} \int_E d_Y^p(Pu(x), u(x)) m(dx) = \frac{1}{2} d_{L^p}^p(Pu, u)$$

: *quasi p-energy* of  $u$  with respect to  $X$ .

When  $p = 2$ , we simply write  $E(u) := E^2(u)$  (resp.  $E_*(u) := E_*^2(u)$ ). We use

$$\left\{ \begin{array}{l} D(E^p) := \{u \in L^p(E, Y; m) \mid E^p(u) < \infty\} \\ E^p(u) := \frac{1}{2} \int_E \int_E d_Y^p(u(y), u(x)) P_x(dy) m(dx), \end{array} \right.$$

When  $Y = H$ , we use the symbol  $\mathcal{E}$  instead of  $E$  for the (2-)energy on  $L^2(E, H; m)$

and for  $f, g \in D(\mathcal{E})$  we set

$$\mathcal{E}(f, g) := \frac{1}{2} \int_{E \times E} \langle f(y) - f(x), g(y) - g(x) \rangle_H P_x(dy) m(dx)$$

**Lem** 6.1 (Contraction on  $L^p(E, Y; m)/\{\text{const}\}$ )

For  $u \in L^p(E, Y; m)$  and  $\ell \in \mathbb{N}$ ,

$$\text{Var}_m^p(P^\ell u) \leq C^{2p} \text{Var}_m^p(u),$$

$$\overline{\text{Var}}_m^p(P^\ell u) \leq C^{2p} \overline{\text{Var}}_m^p(u)$$

and for  $u \in L^2(E, Y; m)$

$$\text{Var}_m(Pu) \leq \text{Var}_m(u), \quad \overline{\text{Var}}_m(Pu) \leq \overline{\text{Var}}_m(u).$$

$$\text{Pf. } \Phi^p(P^\ell u(x), z) \stackrel{(\text{Jensen})}{\leq} P^\ell \Phi^p(u, z)(x).$$

$$\Rightarrow d_{L^p}(P^\ell u, z)^p \leq C^{2p} d_{L^p}(u, z)^p. \quad \square$$

**Thm 6.1 (Kokubo-K (2012))**

Suppose  $\kappa_n \in \mathbb{R}$  for  $\exists n \in \mathbb{N}$  and  $m \in \mathcal{P}^p(E)$ .

$$\lim_{\ell \rightarrow \infty} \left( \sup_{u \in L^p(E, Y; m)} \frac{\text{Var}_m^p(P^\ell u)}{\text{Var}_m^p(u)} \right)^{\frac{1}{p\ell}} \leq (1 - \kappa_n)^{\frac{1}{n}} \wedge 1,$$

$$\lim_{\ell \rightarrow \infty} \left( \sup_{u \in L^p(E, Y; m)} \frac{\overline{\text{Var}}_m^p(P^\ell u)}{\overline{\text{Var}}_m^p(u)} \right)^{\frac{1}{p\ell}} \leq (1 - \kappa_n)^{\frac{1}{n}} \wedge 1.$$

*L.H.S.* = “Spectral radius of  $P$  on

$L^p(E, Y; m) / \{\text{const}\}$ ”

**Rem 6.1**  $a_\ell := \left( \sup_{u \in L^p(E, Y; m)} \frac{\text{Var}_m^p(P^\ell u)}{\text{Var}_m^p(u)} \right)^{\frac{1}{p}}$

$$\implies a_{i+j} \leq a_i a_j \quad \forall i, j \in \mathbb{N}.$$

$$\implies \exists \lim_{\ell \rightarrow \infty} a_\ell^{\frac{1}{\ell}} = \inf_{i \in \mathbb{N}} a_i^{\frac{1}{i}} = \lim_{\ell \rightarrow \infty} a_{n\ell}^{\frac{1}{n\ell}}$$

**Pf.** of **Thm 6.1**.

$$\begin{aligned} \text{Var}_m^p(P^{n\ell} u) &\leq 2 \overline{\text{Var}}_m^p(P^{n\ell} u) \\ &\leq 2 \text{Lip}(u)^p (1 - \kappa_n)^{p\ell} C_p^p \end{aligned}$$

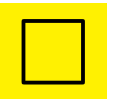
for any  $u \in \text{Lip}(E, Y)$ .

$\text{Lip}(E, Y)$  is dense in  $L^p(E, Y; m)$ .

$$\left( \sup_{u \in L^p(E, Y; m)} \frac{\text{Var}_m^p(P^{nl}u)}{\text{Var}_m^p(u)} \right)^{\frac{1}{pnl}} = \left( \sup_{u \in \text{Lip}(E, Y)} \frac{\text{Var}_m^p(P^{nl}u)}{\text{Var}_m^p(u)} \right)^{\frac{1}{pnl}}$$

$$\leq \sup_{\eta > 0} \left( \sup_{\substack{u \in \text{Lip}(E, Y) \\ \text{Var}_m^p(u) \geq 2\eta^p \text{Lip}(u)^p C_p^p}} \frac{\text{Var}_m^p(P^{nl}u)}{\text{Var}_m^p(u)} \right)^{\frac{1}{pnl}}$$

$$\leq \frac{(1 - \kappa_n)^{\frac{1}{n}}}{\eta^{1/nl}} + \varepsilon \xrightarrow{(\ell \rightarrow \infty)} (1 - \kappa_n)^{\frac{1}{n}} + \varepsilon.$$



**Cor** 6.1 (LSR of  $P$  on  $L^2(E, H; m)/\{\text{const}\}$ )

*Suppose  $\kappa_n \in \mathbb{R}$  for  $\exists n \in \mathbb{N}$ ,  $m \in \mathcal{P}^2(E)$*

*and  $Y = H$ . Then, for such  $\kappa_n \in \mathbb{R}$  we*

*have*

$$\lim_{\ell \rightarrow \infty} \left( \sup_{f \in L^2(E, H; m)} \frac{\text{Var}_m(P^\ell f)}{\text{Var}_m(f)} \right)^{\frac{1}{2\ell}} \leq (1 - \kappa_n)^{\frac{1}{n}} \wedge 1.$$

*Consequently,  $P$  is a  $(1 - \kappa_n)^{\frac{1}{n}}$ -contraction*

*operator on  $L^2(E, H; m)/\{\text{const}\}$  for such*

*an  $n \in \mathbb{N}$ .*



*In particular, for  $f \in L^2(E, H; m) / \{\text{const}\}$   
the following hold:*

$$\begin{aligned}\text{Var}_m(Pf) &\leq ((1 - \kappa_n)^{\frac{2}{n}} \wedge 1) \text{Var}_m(f), \\ |\text{Cov}_m(Pf, f)| &\leq ((1 - \kappa_n)^{\frac{1}{n}} \wedge 1) \text{Var}_m(f).\end{aligned}$$

Thm 6.2 (Poincaré ineq., Kokubo-K (2012))

Suppose  $\kappa_n \in \mathbb{R}$  for  $\exists n \in \mathbb{N}$ ,  $m \in \mathcal{P}^2(E)$

and  $Y = H$ . Then, for  $f \in L^2(E, H; m)$

and such  $\kappa_n$

$$(1 - (1 - \kappa_n)^{\frac{2}{n}} \wedge 1) \text{Var}_m(f) \leq \int_E \text{Var}_{P_x}(f) m(dx),$$

$$1 - (1 - \kappa_n)^{\frac{1}{n}} \wedge 1 \leq \frac{\mathcal{E}(f)}{\text{Var}_m(f)} \leq 1 + (1 - \kappa_n)^{\frac{1}{n}} \wedge 1.$$

If  $\kappa_n > 0$ , we have

$$\begin{aligned}
 0 < 1 - (1 - \kappa_n)^{\frac{1}{n}} &\leq \inf_{f \in L^2(E, H; m)} \frac{\mathcal{E}(f)}{\text{Var}_m(f)} \\
 &\leq \sup_{f \in L^2(E, H; m)} \frac{\mathcal{E}(f)}{\text{Var}_m(f)} \\
 &\leq 1 + (1 - \kappa_n)^{\frac{1}{n}} < 2.
 \end{aligned}$$

$$\begin{aligned}
 \kappa > 0 \Rightarrow 0 < \kappa &\leq \inf_{f \in L^2(E, H; m)} \frac{\mathcal{E}(f)}{\text{Var}_m(f)} \text{ (Ollivier 09)} \\
 &\leq \sup_{f \in L^2(E, H; m)} \frac{\mathcal{E}(f)}{\text{Var}_m(f)} \leq 2 - \kappa < 2.
 \end{aligned}$$

**Thm** 6.3 (Strong  $L^p$ -Liouville property)

**Kokubo-K** (2012): Suppose  $\kappa_n > 0$  for

$\exists n \in \mathbb{N}. \forall u \in L^p(E, Y; m),$

$Pu = u$   $m$ -a.e. on  $E \implies u \equiv c$   $m$ -a.e.

**Pf.**  $u = Pu$   $m$ -a.e. &  $\overline{\text{Var}}_m^p(u) \neq 0 \implies$

$1 \leq 1 - (1 - \kappa_n)^{\frac{1}{n}}$  contradicts  $\kappa_n > 0$ .  $\square$

**Cor** 6.2 (Ergodicity)  $\kappa_n > 0$  for  $\exists n \in \mathbb{N}$

$P1_A = 1_A$   $m$ -a.e.  $\implies m(A) = 0$  or  $m(A^c) = 0$ .

**Thm 6.4** (Poincaré inequality for maps)

**Kokubo-K (2012)**: Suppose  $\kappa_n > 0$  for

$\exists n \in \mathbb{N}, m \in \mathcal{P}^2(E)$ . For  $\forall \varepsilon < 1 - (1 -$

$\kappa_n)^{1/n} \wedge 1, \exists \ell_0 = \ell(\varepsilon, E, d, m, X, Y) \in \mathbb{N}$

s.t.

$$\inf_{u \in L^2(E, Y; m)} \frac{E(u)}{\text{Var}_m(u)} \geq \frac{(1 - (1 - \kappa_n)^{1/n} \wedge 1 - \varepsilon)^2}{4C^2 \ell_0^2}$$

**Prop** 6.1 (**Kokubo-K** (2012))

$X$ :  $m$ -symmetric Markov chain on  $(E, d)$ .

$(Y, d_Y)$ : complete 2-unif. convex space

For a measurable map  $u : E \rightarrow Y$ ,

$$E_*(u) \leq 4E(u),$$

$$\sqrt{\text{Var}_m(u)} \leq \sqrt{\text{Var}_m(Pu)} + \sqrt{2E_*(u)}.$$

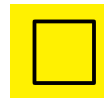
Here

$$E_*(u) := \frac{1}{2} \int_E d_Y^2(Pu(x), u(x)) m(dx).$$

**Pf.** of **Thm** 6.4. We show for the case  $\kappa \in \mathbb{R}$ . By applying **Prop** 6.1 repeatedly, we have

$$\begin{aligned} \sqrt{\text{Var}_m(u)} &\leq \sum_{i=0}^{\ell_0-1} \sqrt{E_*[P^i u]} + \sqrt{\text{Var}_m(P^i u)} \\ &\leq \sum_{i=0}^{\ell_0-1} \sqrt{E_*[P^i u]} + \sqrt{\text{Var}_m(u)} (1 - \kappa + \varepsilon) \end{aligned}$$

$$E_*[P^i u] \leq C^2 E_*[u] \leq 4C^2 E[u].$$



**Thank you for your  
attention!**

**Vielen Dank für Ihre  
Aufmerksamkeit!**



## 7 Estimates without $\kappa(x, y) \geq \kappa > 0$

**Def** 7.1 (Wang's invariant)

$X$ :  $m$ -sym. Markov chain on  $(E, d)$ .

Set  $G := (E, d, m, X)$ . For  $G$  and complete 2-unif. convex  $(Y, d_Y)$ ,

$$\lambda_1^W(G, Y) := \inf_{u \in L^2(E, Y; m)} \frac{E(u)}{\text{Var}_m(u)}.$$

When  $Y = \mathbb{R}$ , we set  $\lambda_1(G) := \lambda_1^W(G, \mathbb{R})$ .

**Thm** 6.4 says  $\lambda_1^W(G, Y) > 0$  for  $\kappa > 0$ .

**Def 7.2** (Izeki-Nayatani invariant)

$(Y, d_Y)$ : complete 2-unif. convex with

(B).  $\delta(Y)$  defined below is called *Izeki-*

*Nayatani invariant* if

$$\delta(Y) := \sup_{\nu \in \mathcal{P}^*(Y)} \delta(Y, \nu),$$

$$\delta(Y, \nu) := \inf \delta(Y, H, \nu),$$

$H$  : Hilbert space  
with  $\dim(H) = \infty$

$$\delta(Y, H, \nu) := \inf_{\substack{\phi \in 1\text{-Lip}(\text{supp}[\nu], H) \\ \|\phi\|_H = d_Y(b(\nu), \cdot)}} \frac{\left\| \int_Y \phi d\nu \right\|_H^2}{\int_Y \|\phi\|_H^2 d\nu}, \quad \nu \in \mathcal{P}^*(Y)$$

Thm 7.1 (Kokubo-K (2012))

$X$  :  $m$ -symm. Markov chain on  $(E, d)$ .

$(Y, d_Y)$  : 2-unif. convex space satisfying

(B). Then

$$(1 - \delta(Y))\lambda_1(G) \leq \lambda_1^W(G, Y) \leq \lambda_1(G).$$

Rem 7.1

- (1) **Thm** 7.1 was firstly proved by Izeki-Nayatani for finite graph  $G$  and any CAT(0)-space.
- (2)  $\exists$  CAT(0)-space  $(Y, d_Y)$  s.t.  $\delta(Y) = 1$  by T. Kondo, Math Z.(2012)
- (3) However, for finite graph  $G$  and CAT(0)  $Y$ ,  $\lambda_1^W(G, Y) \geq \frac{1}{|V|} \lambda_1(G)$  by Izeki-Kondo-Nayatani (private communication).