On spectral bounds for symmetric Markov chains with coarse Ricci curvatures

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1 Aim

Under the coarse Ricci curvature lower bound,

(1) Upper estimate of (non-linear) spectral radius

(2) Lower estimate of (non-linear) spectral gap

(3) Strong L^p-Liouville property for P-harmonic maps

2 Plan of talk

(1) Wasserstein distance (Historical Remark)

(2) Coarse Ricci curvature

(3) CAT(0)-space, 2-uniformly convex space

(4) Main Theorems

3 Wasserstein space

Def 3.1 (Wasserstein distance)

(E, d): Polish space, $p \in [1, \infty[$.

$$\mathcal{P}^p(E) := \{\mu \in \mathcal{P}(E) \mid \int_E d^p(\cdot, {}^{\exists/\forall}x_0) d\mu < \infty\},$$

For $\mu, \nu \in \mathcal{P}^p(E)$,

$$d_{W_p}(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \left(\int_{E \times E} d^p(x, y) \pi(dxdy) \right)^{1/p}$$

: p-Wasserstein distance.

Rem 3.1

(1) d_{W_1} is nothing but the Kantorovich-Rubinste

distance. d_{W_p} was (re)discovered by various authors independently:

Gini ('14): d_{W_1} on discrete prob. on $\mathbb{R}$.

Kantorovich ('42): d_{W_1} on prob. on cpt sp

Salvemini ('43): For discrete $\mu, \nu \in \mathcal{P}(E)$,

Dall'Aglio ('56): For general $\mu, \nu \in \mathcal{P}^p(E)$,

$d_{W_p}(\mu,\nu)^p = \int_0^1 |F_\mu^{-1}(t) - F_\nu^{-1}(t)|^p dt.$

Fréchet ('57): metric properties of d_{W_p}.

Kantorovich–Rubinshtein ('58):

$$d_{W_1}(\mu,\nu) = \sup_{f:1\text{-Lip}} \left(\int_E f\,d\mu - \int_E f\,d\nu\right)$$

Vasershtein ('69):

$$d_{W_1}(\mu,\nu) := \inf_{X\sim\mu, Y\sim\nu} \mathbf{E}[d(X,Y)]$$

Dobrushin ('70) named '**Vasershtein** distance'

Mallows ('72): d_{W_2} in statistical context

Tanaka ('73): d_{W_2}, Boltzmann equation

Bickel–Freedman ('80): d_{W_2} was named as Mallows metric

(2) In English literatures, the German spelling 'Wasserstein'[1] is used (attributed to the name 'Vasershtein' being of Germanic origin).

[1] Vaserstein himself uses the terminology 'Wasserstein distance' in http://www.math.psu.edu/vstein/

(E, d): Polish space, $\mathcal{E} = \mathcal{B}(E)$: Borel field. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

$\mathrm{X} = (\Omega, X_n, \mathcal{F}_n, \mathcal{F}_\infty, \mathrm{P}_x)_{x \in E}$:

conservative Markov chain on $(E, \mathcal{E})$.

$\Omega := E^{\mathbb{N}_0}$: set of all E-valued sequences

$\omega = \{\omega(n)\}_{n \in \mathbb{N}_0}$. $X_n(\omega) := \omega(n)$, $n \in \mathbb{N}_0$.

$P(x, dy) := \mathrm{P}_x(X_1 \in dy),\ x \in E$:

transition kernel of X:

$P(x, dy)$ satisfies the following:

(P1) For each $x \in E$, $P(x, \cdot) \in \mathcal{P}(E)$.

(P2) For each $A \in \mathcal{E}$, $P(\cdot, A) \in \mathcal{E}$.

Further we impose the following:

(P3) For each $x \in E$, $P(x, \cdot) \in \mathcal{P}^1(E)$.

We set $P_x(A) := P(x, A)$, $A \in \mathcal{E}$ and $Pf(x) := \int_E f(y) P_x(dy) = \mathrm{E}_x[f(X_1)]$. For the given Markov chain $\mathbf{X}$ as above and a fixed $n \in \mathbb{N}$, a Markov chain $\mathbf{X}^n = (\Omega, X_k^n, \mathfrak{F}_k^n, \mathfrak{F}_\infty^n, \mathrm{P}_x^n)_{x \in E}$ with state space (E, d) defined by the transition kernel $P^n(x, dy)$ is called an *n-step Markov chain*.

Def 4.1 (Ollivier (2009))

The *coarse Ricci curvature* $\kappa(x, y)$ along (xy) for $x \neq y$ is defined by

$$\kappa(x, y) := 1 - \frac{d_{W_1}(P_x, P_y)}{d(x, y)} (\leq 1)$$

and $\kappa := \inf\{\kappa(x, y) \mid (x, y) \in E^2 \setminus \mathsf{diag}\}$ is said to be the *lower bound of the coarse Ricci curvature.* $\kappa \in [-\infty, 1]$.

The *n-step coarse Ricci curvature* $\kappa_n(x,y)$ of X along (xy) is defined to be

$$\kappa_n(x,y) := 1 - \frac{d_{W_1}(P_x^n, P_y^n)}{d(x,y)}$$

and $\kappa_n := \inf\{\kappa_n(x,y) \mid (x,y) \in E^2 \backslash \mathsf{diag}\}$ is its lower bound. $\kappa_n(x,y)$ is nothing but the coarse Ricci curvature for X^n and $\kappa_1(x,y) = \kappa(x,y)$ for $(x,y) \in E^2 \setminus \mathsf{diag}$. Note that $\kappa_n \geq 1-(1-\kappa)^n$ holds.

Recent works on coarse Ricci curvature:

Lin-Yau (2010): locally finite graphs

$$\kappa(x,y) \geq -2\Big(1 - \frac{1}{d_x} - \frac{1}{d_y}\Big)$$

Lin-Lu-Yau (2011): New def for $\kappa(x,y)$.

Jost-Liu (2011): locally finite graphs

$$\kappa(x,y) \geq -2\Big(1 - \frac{1}{d_x} - \frac{1}{d_y}\Big)_+$$

Bauer-Jost-Liu (2011): graphs with loops

$$1-(1-\kappa_n)^{\frac{1}{n}} \leq \lambda_1 \leq \cdots \leq \lambda_{N-1} \leq 1+(1-\kappa_n)^{\frac{1}{n}}$$

Kitabeppu (2011):

Lower estimate for $\kappa(x,y)$ under $\mathrm{CD}(K,N)$

Veysseire (2012): m-sym Markov process

$$\overline{\kappa}(x,y) := \varlimsup_{t\to 0} \frac{1}{t}\left(1 - \frac{d_{W_1}(P_t(x,\cdot), P_t(y,\cdot))}{d(x,y)}\right) \geq \kappa \in \mathbb{R}$$

$$\Rightarrow d_{W_1}(P_t(x,\cdot), P_t(y,\cdot)) \leq e^{-\kappa t} d(x,y),$$

$$m(E) < \infty, \kappa \leq \frac{\mathcal{E}(f)}{\|f - \langle m, f\rangle\|_2^2} \text{ if } \kappa > 0.$$

Ex 4.1 (Sym. simple random walk on $\mathbb{Z}^n$)

$\boldsymbol{E} := \mathbb{Z}^n$,

$d_{\mathbb{Z}^n}(x, y) := \sum_{i=1}^n |x_i - y_i|$: $x, y \in \mathbb{Z}^n$:

$d_{\mathbb{R}^n}(x, y) := \left(\sum_{i=1}^n |x_i - y_i|^2\right)^{\frac{1}{2}}$: $x, y \in \mathbb{Z}^n$

X: symmetric simple random walk on $\mathbb{Z}^n$.

$$P(x, dy) := \frac{1}{2n} \sum_{|x-z|=1, z \in \mathbb{Z}^n} \delta_z(dy).$$

$\Longrightarrow \kappa(x, y) = 0$ w.r.t. either of $d_{\mathbb{Z}^n}$ or $d_{\mathbb{R}^n}$.

Ex 4.2 (RW on locally finite graph)

Jost-Liu (2011):

$G = (V, E)$: a locally finite graph

d_x: degree at vertex $x \in V$

$x \sim y \overset{\text{def}}{\Longleftrightarrow} xy \in E$

$P(x, dz) := \frac{1}{d_x} \sum_{x \sim y} \delta_y(dz)$

$$\kappa(x, y) \geq -2\left(1 - \frac{1}{d_x} - \frac{1}{d_y}\right)_+$$

Equality holds if $G = (V, E)$ is a tree.

Ex 4.3 (RW on Riemannian mfd)

$E = M$: C^∞ compl. N-dim Riem mfd.

$\varepsilon > 0$. $m = \mathrm{vol}$: volume measure.

X: ε-step Random walk on E defined by

$$P_x(dy) = \frac{1}{m(B_\varepsilon(x))} 1_{B_\varepsilon(x)}(y) m(dy).$$

Ollivier(09)

$$\Longrightarrow \kappa(x,y) = \frac{\varepsilon^2 \mathrm{Ric}(v,v)}{2(N+2)} + O(\varepsilon^3 + \varepsilon^2 d(x,y))$$

for $v \in U_x M$ and $y \in \exp_x tv$ with $t = d(x,y)$ small enough.

Ex 4.4 (Circle graph)

$G=(V,E)$: a circle graph of size N;

$V:=\{x_i\}_{i=1}^N$: vertices,

$E:=\{x_ix_{i+1}\}_{i=1}^N$ $(x_{N+i}=x_i(i\in\mathbb{N}))$: edges,

$d_x(G)=2$ for $x\in V$: degree at $x\in V$,

$P_{x_i}(dy):=\frac{1}{2}\delta_{x_{i-1}}(dy)+\frac{1}{2}\delta_{x_{i+1}}(dy)$.

$\kappa(x,y)=0$ for $(x,y)\in V\times V\setminus\mathbf{diag}$,

$\kappa_n(x,y)\geq 0$ for $(x,y)\in V\times V\setminus\mathbf{diag}$, X (hence X^n) is m-symmetric w.r.t. $m(dy) := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}(dy)$.

We take $N = 5$.

3-step Markov chain X^3 is associated with $G^3 := (V^3, E^3)$ defined by $V^3 := V$ and $E^3 := \{x_i x_j \mid 1 \leq i, j \leq 5 \text{ with } i \neq j\}$.

The transition kernel $P_x^3(dy)$ is given by

$$P_{x_i}^3 = \frac{1}{8}\delta_{x_{i-2}} + \frac{3}{8}\delta_{x_{i-1}} + \frac{3}{8}\delta_{x_{i+1}} + \frac{1}{8}\delta_{x_{i+2}}.$$

$d_x(G^3) = 4.$

The 3-step coarse Ricci curvature $\kappa_3(x, y)$ for $xy \in E^3$ can be estimated by use of Bauer-Jost-Liu (2011).

$$\kappa_3(x_i, x_{i+1}) = \frac{3}{8}, \qquad \frac{5}{8} \leq \kappa_3(x_i, x_{i+2}) \leq \frac{7}{8}.$$

Therefore, $\kappa_3(x, y) \geq \frac{3}{8}$ for all $(x, y) \in V \times V \setminus \mathbf{diag}$.

5 CAT(0)-space, 2-unif. convex sp

Def 5.1 (CAT(0)-space)

(Y, d_Y): CAT(0)-space $\Longleftrightarrow$ For $\forall z, x, y \in Y$, $\exists \gamma : [0,1] \to Y$ with $\gamma_0 = x$, $\gamma_1 = y$ s.t. for $t \in [0,1]$

$$d_Y^2(z, \gamma_t) \le (1-t) d_Y^2(z, x) + t d_Y^2(z, y) - t(1-t) d_Y^2(x, y).$$

Cartan-Alexandrov-Toponogov

Ex **5.1 (Examples of CAT(0)-spaces)**

- **Hadamard manifold; simply connected smooth compl Riem mfd with NPC.**
- **products of CAT(0)-sp** • **Hilbert space**
- **convex subset of CAT(0)-space**
- **Tree** • **Euclidean Buildings**
- **CAT(0)-space valued L^2-maps** □

Def 5.2 (2-Uniformly Convex Space)

(Y,d): ***2-uniformly convex with $k>0$***

$\overset{\text{def}}{\Longleftrightarrow}$ (Y,d): geodesic space & $\forall x,y,z\in Y$, $\forall\gamma:=(\gamma_t)_{t\in[0,1]}$: min. geo. in Y from x to y & $\forall t\in[0,1]$,

$$d^2(z,\gamma_t)\le(1-t)d^2(z,x)+td^2(z,y) - \frac{k}{2}t(1-t)d^2(x,y).$$

$z=\gamma_t \Longrightarrow k\in]0,2]$. □

Ex 5.2 (Examples of 2-Unif. Conv. Spaces)

- **Convex subset of a 2-uniformly convex space.**
- **CAT(0)-space**
- **CAT(1)-space with diam$< \frac{\pi}{2}$ is 2-uniformly convex (see Ohta (2007))**
- **L^2-maps into a CAT(1)-sp. with diam$< \frac{\pi}{2}$**

□

(Y, d_Y): complete 2-unif, convex space

γ, $\eta (\subset Y)$: minimal geodesic segments

$$\gamma \perp_p \eta \overset{\text{def}}{\Longleftrightarrow} p \in \gamma \cap \eta,$$

$$d_Y(x, p) \leq d_Y(x, y) \ \forall x \in \gamma, y \in \eta.$$

(B): $\gamma \perp_p \eta \leftrightarrow \eta \perp_p \gamma$.

Ex 5.3 (Examples satisfying (B))

- complete CAT(0)-space.
- complete CAT(1)-sp with diam$< \pi/2$.

Def 5.3 (Barycenter)

(Y, d_Y): complete sep. 2-unif. convex space

$\mu \in \mathcal{P}^1(Y) \Rightarrow b(\mu)$: $^{\exists_1}$unique minimizer (independent of $w \in Y$) of

$$z \mapsto \int_Y (d_Y^2(z,y) - d_Y^2(w,y))\mu(dy).$$

We call $b(\mu)$ the barycenter of μ.

Lem 5.1 (Jensen's inequality, K. (2010))

(Y, d_Y): complete sep. 2-unif. convex space. $\mu \in \mathcal{P}^1(Y)$.

(B): $\gamma \perp_p \eta \leftrightarrow \eta \perp_p \gamma$.

Then for any l.s.c. convex func φ on Y

$$\varphi(b(\mu)) \leq \int_Y \varphi(x)\mu(dx).$$

Ass **5.1**

$m \in \mathcal{P}^1(E)$, $\mathrm{supp}[m] = E$, $p \geq 1$,

X: m-sym Markov chain on E with (P3),

(Y, d_Y): compl sep. 2-unif. convex space,

(B): $\gamma \perp_p \eta \leftrightarrow \eta \perp_p \gamma$,

(CG): Convex Geometry:

$\exists \Phi : Y^2 \to \mathbb{R}$ convex s.t.

$C^{-1} d_Y \leq \Phi \leq C d_Y$ on $Y \times Y$ for $C > 0$.

$\boldsymbol{L^p(E, Y, m)}$: **space of** $\boldsymbol{L^p}$**-maps,**

$$\boldsymbol{L^p(E, Y; m) := \{u : E \to Y \text{ m'ble map } | \int_E d_{\mathbf{Y}}^p(u(x), o)m(dx) < \infty \exists/\forall o \in Y\}/ \overset{m}{\sim},}$$

$$\boldsymbol{d_{L^p}(u, v)^p := \int_E d_{\mathbf{Y}}^p(u(x), v(x))m(dx),}$$

$$\boldsymbol{C_p^p := \int_E \int_E d^p(x, y)m(dx)m(dy) \leq \infty}$$

$$\boldsymbol{(C_p < \infty \Leftrightarrow m \in \mathcal{P}^p(E)).}$$

□

Def 5.4 $u \in S(E,Y) \overset{\text{def}}{\Leftrightarrow} \sharp(\mathrm{Im}(u)) < \infty.$

$$u \in \mathrm{Lip}(E,Y) \overset{\text{def}}{\Leftrightarrow} \mathrm{Lip}(u) := \sup_{x \neq y} \frac{d_Y(u(x),u(y))}{d(x,y)} < \infty.$$

$$m \in \mathcal{P}^p(E) \Rightarrow \mathrm{Lip}(E,Y) \subset L^p(E,Y;m)$$

$$u \in S(E,Y) \cup \mathrm{Lip}(E,Y) \Rightarrow u_* P_x \in \mathcal{P}^1(Y)$$

$$\Rightarrow Pu(x) := b(u_* P_x).$$

Thm 5.1 $S(E,Y) \overset{\text{dense}}{\hookrightarrow} L^p(E,Y;m)$ *and*

$\mathrm{Lip}(E,Y) \overset{\text{dense}}{\hookrightarrow} L^p(E,Y;m)$ *if* $m \in \mathcal{P}^p(E)$.

Lem 5.2 $\kappa_n \in \mathbb{R}$, $u \in \mathrm{Lip}(E,Y) \Rightarrow$

$$\mathrm{Lip}(P^n u) \leq C^2(1-\kappa_n)\mathrm{Lip}(u).$$

Pf. $d_Y(P^n u(x), P^n u(y))$

$$\leq C\Phi(b(u_\sharp P_x^n), b(u_\sharp P_y^n)) \overset{\text{(Jensen)}}{\leq} C \int_{Y\times Y} \Phi \, d\pi$$

$$\leq C^2 \int_{Y\times Y} d_Y \, d\pi = C^2 \int_{E\times E} d_Y(u(p), u(q)) \, d\pi_0(p,q)$$

$$(\ \pi := (u\times u)_\sharp \pi_0 \in \Pi(u_\sharp P_x^n, u_\sharp P_y^n))$$

$$\leq C^2 \mathrm{Lip}(u) \int_{E\times E} d(p,q) \, d\pi_0(p,q)$$

$$\leq C^2 \mathrm{Lip}(u) d_{W_1}(P_x^n, P_y^n) \ (\pi_0 \in \Pi(P_x^n, P_y^n) : \mathrm{op}$$

$$\leq C^2 \mathrm{Lip}(u)(1-\kappa_n) d(x,y).$$

$\square$

Def 5.5 For $u \in L^p(E, Y; m)$, we define $Pu := \lim_k Pu_k \in L^p(E, Y; m)$ by approximating seq $\{u_k\} \subset S(E, Y)$ to u,

$$
\begin{aligned}
d_{L^p}(Pu_l, Pu_k)^p &= \int_E d_Y^p(Pu_l(x), Pu_k(x))m(dx) \\
&\leq C^p \int_E \Phi^p(Pu_l, Pu_k)dm \\
&\overset{\text{(Jensen)}}{\leq} C^p \int_E P\Phi^p(u_l, u_k)dm \\
&\leq C^{2p} d_{L^p}(u_l, u_k)^p \longrightarrow 0
\end{aligned}
$$

□

6 Results

Def 6.1 (Variance) For $u \in L^p(E, Y; m)$,

$$\mathrm{Var}_m^p(u) := \inf_{z \in Y} \int_E d_{\mathbf{Y}}^p(u(x), z) m(dx),$$

$$\overline{\mathrm{Var}}_m^p(u) := \frac{1}{2} \int_E \int_E d_{\mathbf{Y}}^p(u(x), u(y)) m(dx) m(dy)$$

If $p = 2$ and $Y = H$: Hilbert sp., for $f, g \in L^2(E, H; m)$, we write $\mathrm{Var}_m(f)$, $\overline{\mathrm{Var}}_m(f)$

$$\mathrm{Cov}_m(f, g) := \frac{1}{2} \int_{E^2} \langle f(x) - f(y), g(x) - g(y) \rangle_H m^2(dx dy).$$

Def 6.2 (Energy of Maps)

For $u \in L^p(E, Y; m)$,

$$E^p(u) := \frac{1}{2}\int_E\int_E d_Y^p(u(y), u(x))P(x, dy)m(dx)$$

: *p-energy* of u with respect to X and

$$E_*^p(u) := \frac{1}{2}\int_E d_Y^p(Pu(x), u(x))m(dx) = \frac{1}{2}d_{L^p}^p(Pu, u)$$

:*quasi p-energy* of u with respect to X.

When $p=2$, we simply write $E(u):=E^2(u)$ (resp. $E_*(u):=E_*^2(u)$). We use

$$\begin{cases} D(E^p):=\{u\in L^p(E,Y;m)\mid E^p(u)<\infty\} \\ E^p(u):=\frac{1}{2}\int_E\int_E d_Y^p(u(y),u(x))P_x(dy)m(dx), \end{cases}$$

When $Y=H$, we use the symbol $\mathcal{E}$ instead of E for the (2-)energy on $L^2(E,H;m)$ and for $f,g\in D(\mathcal{E})$ we set

$$\mathcal{E}(f,g):=\frac{1}{2}\int_{E\times E}\langle f(y)-f(x),g(y)-g(x)\rangle_H P_x(dy)m(dx)$$

Lem 6.1 (Contraction on $L^p(E,Y;m)/\{\mathrm{const}\}$)

For $u \in L^p(E,Y;m)$ and $\ell \in \mathbb{N}$,

$$\mathrm{Var}_m^p(P^\ell u) \leq C^{2p}\mathrm{Var}_m^p(u),$$

$$\overline{\mathrm{Var}}_m^p(P^\ell u) \leq C^{2p}\overline{\mathrm{Var}}_m^p(u)$$

and for $u \in L^2(E,Y;m)$

$$\mathrm{Var}_m(Pu) \leq \mathrm{Var}_m(u), \quad \overline{\mathrm{Var}}_m(Pu) \leq \overline{\mathrm{Var}}_m(u).$$

Pf. $\Phi^p(P^\ell u(x), z) \overset{\text{(Jensen)}}{\leq} P^\ell \Phi^p(u,z)(x)$.

$\Rightarrow d_{L^p}(P^\ell u, z)^p \leq C^{2p} d_{L^p}(u,z)^p$. □

Thm 6.1 (Kokubo-K (2012))

Suppose $\kappa_n \in \mathbb{R}$ *for* $\exists n \in \mathbb{N}$ *and* $m \in \mathcal{P}^p(E)$.

$$\lim_{\ell\to\infty}\left(\sup_{u\in L^p(E,Y;m)}\frac{\mathrm{Var}_m^p(P^\ell u)}{\mathrm{Var}_m^p(u)}\right)^{\frac{1}{p\ell}} \le (1-\kappa_n)^{\frac{1}{n}} \wedge 1,$$

$$\lim_{\ell\to\infty}\left(\sup_{u\in L^p(E,Y;m)}\frac{\overline{\mathrm{Var}}_m^p(P^\ell u)}{\overline{\mathrm{Var}}_m^p(u)}\right)^{\frac{1}{p\ell}} \le (1-\kappa_n)^{\frac{1}{n}} \wedge 1.$$

L.H.S.= "*Spectral radius of* P *on*

$L^p(E,Y;m)/\{\mathrm{const}\}$"

Rem 6.1 $a_\ell := \left(\sup_{u\in L^p(E,Y;m)} \frac{\mathrm{Var}_m^p(P^\ell u)}{\mathrm{Var}_m^p(u)}\right)^{\frac{1}{p}}$

$\Longrightarrow a_{i+j} \le a_i a_j \ \forall i,j \in \mathbb{N}.$

$\Longrightarrow \exists \lim_{\ell\to\infty} a_\ell^{\frac{1}{\ell}} = \inf_{i\in\mathbb{N}} a_i^{\frac{1}{i}} = \lim_{\ell\to\infty} a_{n\ell}^{\frac{1}{n\ell}}$

Pf. of **Thm** 6.1.

$$\begin{aligned}\mathrm{Var}_m^p(P^{n\ell}u) &\le 2\overline{\mathrm{Var}}_m^p(P^{n\ell}u)\\ &\le 2\mathrm{Lip}(u)^p(1-\kappa_n)^{p\ell}C_p^p\end{aligned}$$

for any $u \in \mathrm{Lip}(E,Y)$.

$\mathrm{Lip}(E,Y)$ is dense in $L^p(E,Y;m)$.

$$\left(\sup_{u\in L^p(E,Y;m)}\frac{\mathrm{Var}_m^p(P^{n\ell}u)}{\mathrm{Var}_m^p(u)}\right)^{\frac{1}{pn\ell}}=\left(\sup_{u\in\mathrm{Lip}(E,Y)}\frac{\mathrm{Var}_m^p(P^{n\ell}u)}{\mathrm{Var}_m^p(u)}\right)^{\frac{1}{pn\ell}}$$

$$\leq\sup_{\eta>0}\left(\sup_{\substack{u\in\mathrm{Lip}(E,Y)\\ \mathrm{Var}_m^p(u)\geq 2\eta^p\mathrm{Lip}(u)^pC_p^p}}\frac{\mathrm{Var}_m^p(P^{n\ell}u)}{\mathrm{Var}_m^p(u)}\right)^{\frac{1}{pn\ell}}$$

$$\leq\frac{(1-\kappa_n)^{\frac{1}{n}}}{\eta^{1/n\ell}}+\varepsilon\overset{(\ell\to\infty)}{\longrightarrow}(1-\kappa_n)^{\frac{1}{n}}+\varepsilon.$$

□

Cor 6.1 (LSR of P on $L^2(E, H; m)/\{\mathrm{const}\}$)

Suppose $\kappa_n \in \mathbb{R}$ for $\exists n \in \mathbb{N}$, $m \in \mathcal{P}^2(E)$ and $Y = H$. Then, for such $\kappa_n \in \mathbb{R}$ we have

$$\lim_{\ell \to \infty} \left(\sup_{f \in L^2(E,H;m)} \frac{\mathrm{Var}_m(P^\ell f)}{\mathrm{Var}_m(f)} \right)^{\frac{1}{2\ell}} \leq (1-\kappa_n)^{\frac{1}{n}} \wedge 1.$$

Consequently, P is a $(1-\kappa_n)^{\frac{1}{n}}$-contraction operator on $L^2(E, H; m)/\{\mathrm{const}\}$ for such an $n \in \mathbb{N}$.

In particular, for $f \in L^2(E, H; m)/\{\mathrm{const}$

the following hold:

$$\mathrm{Var}_m(Pf) \le ((1-\kappa_n)^{\frac{2}{n}} \wedge 1)\mathrm{Var}_m(f),$$
$$|\mathrm{Cov}_m(Pf, f)| \le ((1-\kappa_n)^{\frac{1}{n}} \wedge 1)\mathrm{Var}_m(f).$$

Thm **6.2 (Poincaré ineq., Kokubo-K (2012))**

Suppose $\boldsymbol{\kappa_n} \in \mathbb{R}$ **for** $\exists \boldsymbol{n} \in \mathbb{N}$, $\boldsymbol{m} \in \mathcal{P}^2(\boldsymbol{E})$ **and** $\boldsymbol{Y} = \boldsymbol{H}$. **Then, for** $\boldsymbol{f} \in \boldsymbol{L}^2(\boldsymbol{E}, \boldsymbol{H}; \boldsymbol{m})$ **and such** $\boldsymbol{\kappa_n}$

$$(1-(1-\kappa_n)^{\frac{2}{n}} \wedge 1)\mathrm{Var}_m(f) \leq \int_E \mathrm{Var}_{P_x}(f) m(dx),$$

$$1-(1-\kappa_n)^{\frac{1}{n}} \wedge 1 \leq \frac{\mathcal{E}(f)}{\mathrm{Var}_m(f)} \leq 1+(1-\kappa_n)^{\frac{1}{n}} \wedge 1.$$

If $\boldsymbol{\kappa_n > 0}$, we have

$$
\begin{aligned}
0 < 1-(1-\kappa_n)^{\frac{1}{n}} &\le \inf_{f\in L^2(E,H;m)} \frac{\mathcal{E}(f)}{\mathrm{Var}_m(f)} \\
&\le \sup_{f\in L^2(E,H;m)} \frac{\mathcal{E}(f)}{\mathrm{Var}_m(f)} \\
&\le 1+(1-\kappa_n)^{\frac{1}{n}} < 2.
\end{aligned}
$$

$$
\begin{aligned}
\kappa > 0 \Rightarrow 0 < \kappa &\le \inf_{f\in L^2(E,H;m)} \frac{\mathcal{E}(f)}{\mathrm{Var}_m(f)} \text{(Ollivier 09)} \\
&\le \sup_{f\in L^2(E,H;m)} \frac{\mathcal{E}(f)}{\mathrm{Var}_m(f)} \le 2-\kappa < 2.
\end{aligned}
$$

Thm **6.3 (Strong L^p-Liouville property)**

Kokubo-K **(2012): Suppose** $\kappa_n > 0$ **for** $\exists n \in \mathbb{N}$. $\forall u \in L^p(E, Y; m)$,

$Pu = u$ m-**a.e. on** $E \Longrightarrow u \equiv c$ m-**a.e.**

Pf. $u = Pu$ m-a.e.& $\overline{\mathrm{Var}}_m^p(u) \neq 0 \Rightarrow$
$1 \leq 1 - (1 - \kappa_n)^{\frac{1}{n}}$ **contradicts** $\kappa_n > 0$. □

Cor **6.2 (Ergodicity)** $\kappa_n > 0$ *for* $\exists n \in \mathbb{N}$
$P1_A = 1_A$ *m-a.e.* $\Rightarrow m(A) = 0$ *or* $m(A^c) = 0$.

Thm 6.4 (Poincaré inequality for maps)

Kokubo-K (2012): Suppose $\kappa_n > 0$ for $\exists n \in \mathbb{N}$, $m \in \mathcal{P}^2(E)$. For $\forall \varepsilon < 1 - (1 - \kappa_n)^{1/n} \wedge 1$, $\exists \ell_0 = \ell(\varepsilon, E, d, m, X, Y) \in \mathbb{N}$ s.t.

$$\inf_{u \in L^2(E,Y;m)} \frac{E(u)}{\mathrm{Var}_m(u)} \geq \frac{(1 - (1 - \kappa_n)^{1/n} \wedge 1 - \varepsilon)^2}{4C^2 \ell_0^2}$$

Prop 6.1 (Kokubo-K (2012))

X: m-symmetric Markov chain on (E, d).

(Y, d_Y): complete 2-unif. convex space

For a measurable map $u : E \to Y$,

$$E_*(u) \le 4E(u),$$

$$\sqrt{\mathrm{Var}_m(u)} \le \sqrt{\mathrm{Var}_m(Pu)} + \sqrt{2E_*(u)}.$$

Here

$$E_*(u) := \frac{1}{2}\int_E d_Y^2(Pu(x), u(x))m(dx).$$

Pf. **of** **Thm** **6.4.** **We show for the case** $\kappa \in \mathbb{R}$. **By applying** **Prop** **6.1 repeatedly, we have**

$$\sqrt{\mathrm{Var}_m(u)} \leq \sum_{i=0}^{\ell_o - 1} \sqrt{E_*[P^i u] + \sqrt{\mathrm{Var}_m(P^i u)}}$$

$$\leq \sum_{i=0}^{\ell_0 - 1} \sqrt{E_*[P^i u] + \sqrt{\mathrm{Var}_m(u)}(1 - \kappa + \varepsilon)}$$

$$E_*[P^i u] \leq C^2 E_*[u] \leq 4C^2 E[u].$$

$\square$

Thank you for your attention!

Vielen Dank für Ihre Aufmerksamkeit!

7 Estimates without $\kappa(x, y) \geq \kappa > 0$

Def 7.1 (Wang's invariant)

X: m-sym. Markov chain on (E, d).

Set $G := (E, d, m, \mathrm{X})$. For G and complete 2-unif. convex (Y, d_Y),

$$\lambda_1^W(G, Y) := \inf_{u \in L^2(E, Y; m)} \frac{E(u)}{\mathrm{Var}_m(u)}.$$

When $Y = \mathbb{R}$, we set $\lambda_1(G) := \lambda_1^W(G, \mathbb{R})$.

Thm 6.4 says $\lambda_1^W(G, Y) > 0$ for $\kappa > 0$.

Def 7.2 (Izeki-Nayatani invariant)

(Y, d_Y): complete 2-unif. convex with (B). $\delta(Y)$ defined below is called *Iseki-Nayatani invariant* if

$\delta(Y) := \sup_{\nu \in \mathcal{P}^*(Y)} \delta(Y, \nu)$,

$$\delta(Y, \nu) := \inf_{\substack{H\,:\,\text{Hilbert space} \\ \text{with } \dim(H) = \infty}} \delta(Y, H, \nu),$$

$$\delta(Y, H, \nu) := \inf_{\substack{\phi \in \text{1-Lip}(\text{supp}[\nu], H) \\ \|\phi\|_H = d_Y(b(\nu), \cdot)}} \frac{\left\| \int_Y \phi \, d\nu \right\|_H^2}{\int_Y \|\phi\|_H^2 d\nu}, \ \nu \in \mathcal{P}^*(Y)$$

Thm **7.1 (Kokubo-K (2012))**

X : m-symm. Markov chain on (E, d).

(Y, d_Y) : 2-unif. convex space satisfying (B). Then

$$(1-\delta(Y))\lambda_1(G) \le \lambda_1^W(G, Y) \le \lambda_1(G).$$

Rem 7.1

(1) **Thm 7.1 was firstly proved by Izeki-Nayatan**
for finite graph G and any CAT(0)-space.

(2) $^{\exists}$**CAT(0)-space (Y, d_Y) s.t. $\delta(Y) = 1$ by**
T. Kondo, Math Z.(2012)

(3) **However, for finite graph G and CAT(0)**
Y, $\lambda_1^W(G, Y) \geq \frac{1}{|V|}\lambda_1(G)$ by Izeki-Kondo-
Nayatani (private communication).